Convex and Exact Games with Non-transferable Utility

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Abstract

We generalize exactness to games with non-transferable utility (NTU). A game is exact if for each coalition there is a core allocation on the boundary of its payoff set.

Convex games with transferable utility are well-known to be exact. We consider five generalizations of convexity in the NTU setting. We show that each of ordinal, coalition merge, individual merge and marginal convexity can be unified under NTU exactness. We provide an example of a cardinally convex game which is not NTU exact.

Finally, we relate the classes of Π-balanced, totally Π-balanced, NTU exact, totally NTU exact, ordinally convex, cardinally convex, coalition merge convex, individual merge convex and marginal convex games to one another.

Keywords: NTU Games, Exact Games, Convex Games

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1 Introduction

Convex cooperative games with transferable utility (TU) introduced by Shapley (1971) arise from a wide range of applications. Airport games (Littlechild and Owen, 1973),

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bankruptcy games (Aumann and Maschler, 1985), sequencing games (Curiel, Pederzoli, and Tijs, 1989) and standard tree games (Granot, Maschler, Owen, and Zhu, 1996) are all convex. Recently, Pulido and Sánchez-Soriano (2009) studied convex games with a coalitional structure.

Convex TU games are exact (Schmeidler, 1972). A game is exact if for each coalition there is a core allocation such that the coalition only gets its stand-alone value. Calleja, Borm, and Hendrickx (2005) show that the class of multi-issue allocation games coincides with the class of non-negative exact games. Csóka, Herings, and Kóczy (2009) demonstrate that the class of exact games equals the class of risk allocation games with no aggregate uncertainty. Branzei, Tijs, and Zarzuelo (2009) use exactness as one of the properties characterizing convex multi-choice games. Casas-Méndez, García-Jurado, van den Nouweland, and Vázquez-Brage (2003) show that if you take any exact game and a coalitional structure, then the resulting coalitional game will be quasi balanced. Quasi balancedness is a requirement under which their proposed solution concept, the coalitonal τ-value can be defined.

Although transferable utility has proved itself to be a very valuable workhorse, it is a restrictive assumption, and generalizations of convexity and exactness to the non-transferable utility case are highly desired. Vilkov (1977) and Sharkey (1981) have extended convexity to games with non-transferable utility (NTU) to define ordinal and cardinal convexity, respectively. Hendrickx, Borm, and Timmer (2002) analyze coalition merge convexity, individual merge convexity, and marginal convexity in an NTU setting. The aforementioned five classes of NTU convex games do not coincide in general. The only general result (restated in this paper as Theorem 2.11) is that coalition merge convexity implies individual merge convexity, and individual merge convexity implies marginal convexity.

In this paper we generalize exactness to the NTU setting. An NTU game is exact if for each coalition there is a core element on the boundary of its payoff set, meaning that this coalition does not necessarily benefit from the gains of forming the grand coalition in an allocation which is robust against all coalitional deviations. We show that each of ordinal, coalition merge, individual merge, and marginal convexity implies NTU exactness. We provide an example of a cardinally convex game which is not NTU exact.

The structure of the paper is as follows. We start with the notation and the necessary definitions for TU and NTU games. In Section 3 we define NTU exactness and from this perspective analyze the five classes of NTU convex games. In Section 4 we conclude by relating the various classes of NTU games to one another.

2 Notation, Definitions, Existing Results

Let $N = \{1, \ldots, n\}$ denote the finite *set of players*, $2^N = \{C \mid C \subseteq N\}$ is the power set of N, $\mathcal{N} = 2^N \setminus \{\emptyset\}$ is the collection of *coalitions*, the non-empty subsets of N. Let $\mathbb{R}$ denote the set of all real numbers. $\mathbb{R}^N$ is the n-dimensional Euclidean space generated by the set of players. An element of $\mathbb{R}^N$ is denoted by a vector $x = (x_i)_{i \in N}$. For a coalition $C \in \mathcal{N}$, let $x^C = (x_i)_{i \in C}$ denote the restriction of x on C. For $x, y \in \mathbb{R}^N$, $y \geq x$ denotes $y_i \geq x_i$

for all $i \in N$, and $y \gg x$ denotes $y_i > x_i$ for all $i \in N$.

For a set $A \subseteq \mathbb{R}^N$, the symbols cl A, ∂A and int A denote, respectively, the closure, the boundary and the interior of A. For $x \in \mathbb{R}^N$, $x \in \text{cl } A$ if there exists a sequence $(x^k)_{k\in\mathbb{N}}$ with $x^k \in A$ for all $k \in \mathbb{N}$ and $(x^k)_{k\in\mathbb{N}} \to x$; $x \in \partial A$ if $x \in \text{cl } A \cap \text{cl } (\mathbb{R}^N \setminus A)$; and $x \in \text{int } A$ if $x \in A \setminus \partial A$.

2.1 Transferable Utility Games

A value function $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$ gives rise to a *cooperative game with transferable utility* (*TU game*, for short) (N, v). Let Γ^{TU} denote the set of TU games with player set N. A utility *allocation* is a vector $x \in \mathbb{R}^N$, where x_i is the payoff of player $i \in N$. For a coalition $C \in \mathcal{N}$, let $x(C) = \sum_{i\in C} x_i$. An allocation $x \in \mathbb{R}^N$ is called *efficient* if $x(N) = v(N)$, *individually rational* if $x_i \geq v(\{i\})$ for all $i \in N$, and *coalitionally rational* if $x(C) \geq v(C)$ for all $C \in \mathcal{N}$. The *core* is the set of efficient and coalitionally rational allocations.

Shapley (1971) and Schmeidler (1972) introduce *exact* TU games.

Definition 2.1. A TU game (N, v) is *exact* if for each $C \in 2^N$ there exists a core allocation x such that $x(C) = v(C)$.

Let $\Gamma^{\text{TU}}_{\text{e}}$ denote the class of exact TU games with player set N. *Convex* TU games (Shapley, 1971) can be defined and characterized as follows.

Definition 2.2. A TU game (N, v) is *convex* if it satisfies the following three equivalent conditions:

$$\forall S, T \in 2^N\colon\ v(S) + v(T) \leq v(S \cup T) + v(S \cap T), \tag{1}$$

$$\forall U \in 2^N; \forall S \subsetneq T \subseteq N \setminus U\colon\ v(S \cup U) - v(S) \leq v(T \cup U) - v(T), \tag{2}$$

$$\forall i \in N; \forall S \subsetneq T \subseteq N \setminus \{i\}\colon\ v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T). \tag{3}$$

Let $\Gamma^{\text{TU}}_{\text{c}}$ denote the class of convex TU games with player set N.

A *permutation* of the players in N is a bijection $\sigma : \{1, \ldots, n\} \to N$, where $\sigma(i)$ denotes which player in N is at position i, and $\sigma^{-1}(i)$ denotes the position of player i. Let Σ^N denote the set of all permutations on N. For a permutation $\sigma \in \Sigma^N$, $P^\sigma_i = \{j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$ denotes the coalition of players which precede i with respect to the order σ. In a permutation $\sigma \in \Sigma^N$, $m^\sigma_i(v) = v(P^\sigma_i \cup \{i\}) - v(P^\sigma_i)$ denotes the *marginal contribution* of player i to the preceding players, and $m^\sigma(v) = (m^\sigma_1(v), m^\sigma_2(v), \ldots, m^\sigma_n(v))$ is the *vector of marginal contributions*. Shapley (1971) and Ichiishi (1981) characterize convex TU games as follows.

Theorem 2.3. *The TU game (N, v) is convex if and only if $m^\sigma(v)$ belongs to the core of (N, v) for all permutations $\sigma \in \Sigma^N$.*

Theorem 2.3 directly implies the following theorem.

Theorem 2.4. *If a TU game (N, v) is convex, then it is exact, $\Gamma_{\mathrm{c}}^{\mathrm{TU}} \subseteq \Gamma_{\mathrm{e}}^{\mathrm{TU}}$.*

For a TU game (N, v) and a coalition $C \in \mathcal{N}$ the *subgame* (C, v^C) is obtained by restricting v to subsets of C. Following Biswas, Parthasarathy, Potters, and Voorneveld (1999), we define totally exact TU games.

Definition 2.5. A TU game (N, v) is *totally exact* if for every $C \in \mathcal{N}$ its subgame (C, v^C) is exact.

Let $\Gamma_{\mathrm{te}}^{\mathrm{TU}}$ denote the class of totally exact TU games with player set N. Biswas, Parthasarathy, Potters, and Voorneveld (1999) show the following theorem.

Theorem 2.6. *A TU game is totally exact if and only if it is convex, that is $\Gamma_{\mathrm{te}}^{\mathrm{TU}} = \Gamma_{\mathrm{c}}^{\mathrm{TU}}$.*

2.2 Non-transferable Utility Games

A cooperative game with non-transferable utility (*NTU game*, for short) (N, V) is a family of sets $V = (V(S))_{S \in 2^N}$ satisfying the following assumptions:

$$V(\emptyset) = \emptyset, \tag{4}$$
$$V(S) = V_{\mathrm{p}}(S) \times \mathbb{R}^{N \setminus S}, \text{ where } V_{\mathrm{p}}(S) \subseteq \mathbb{R}^S, \text{ for all } S \in \mathcal{N}, \tag{5}$$
$$0^N \in V(S) \text{ for all } S \in \mathcal{N}, \tag{6}$$
$$V(N) \text{ is closed}, \tag{7}$$
$$\text{if } x \in V(S),\ y \in \mathbb{R}^N,\ y^S \leq x^S, \text{ then } y \in V(S) \text{ (known as comprehensiveness)}, \tag{8}$$
$$\text{the sets } V_{\mathrm{p}}^{+}(S) = \mathbb{R}_{+}^{S} \cap V_{\mathrm{p}}(S) \text{ are bounded for all } S \in \mathcal{N}. \tag{9}$$

Let Γ^{NTU} denote the set of NTU games with player set N.

The *core* $C(V)$ of an NTU game $(N, V) \in \Gamma^{\mathrm{NTU}}$ consists of those elements $x \in V(N)$ for which it holds that there exist no $S \in \mathcal{N}$ and $y \in V(S)$ such that $x^S \ll y^S$, which by comprehensiveness is equivalent to $x \notin \operatorname{int} V(S)$ for any $S \in \mathcal{N}$. Therefore,

$$C(V) = V(N) \setminus \bigcup_{S \in \mathcal{N}} \operatorname{int} V(S). \tag{10}$$

Predtetchinski and Herings (2004) define Π-balancedness, which is a necessary and sufficient condition for the core in a non-transferable utility game to be non-empty. Let $\Gamma_{\Pi-\mathrm{b}}^{\mathrm{NTU}}$ denote the class of Π-balanced NTU games with player set N.

For an NTU game (N, V) and a coalition $S \in \mathcal{N}$ a *subgame* (S, V^S) is obtained by restricting V to subsets of S. It holds that $V^S(T) \subseteq \mathbb{R}^S$ for all $T \subseteq S$. We define $V^S(S) = \operatorname{cl} V_{\mathrm{p}}(S)$ to have a closed payoff set for the grand coalition in the subgame. Let $\Gamma_{\mathrm{t}-\Pi-\mathrm{b}}^{\mathrm{NTU}}$ denote the class of totally Π-balanced NTU games with player set N, the class of games with a non-empty core in each subgame.

There are various classifications of NTU games. For surveys see Peleg and Sudhölter (2003) or Ichiishi (1993). We will only give those definitions that we use later in the paper. NTU convex games have been defined in five ways.

Definition 2.7. (Vilkov, 1977) An NTU game (N, V) is *ordinally convex* if for all $S, T \in \mathcal{N}$ we have $V(S) \cap V(T) \subseteq V(S \cap T) \cup V(S \cup T)$.

Let $\Gamma^{\text{NTU}}_{\text{oc}}$ denote the class of ordinally convex NTU games with player set N. Ordinal convexity has numerous applications. Peleg (1984) transforms a social choice situation with a convex effectivity function into an NTU game which is ordinally convex. Demange (1987) provides two examples: a model of public goods and a production economy with increasing returns to scale; Masuzawa (2003) adds N-person prisoners' dilemma games and oligopoly models to this class.

For $S \in \mathcal{N}$ let $V^\circ(S) = \{x \in V(S) \mid x_i = 0 \text{ for all } i \in N \setminus S\}$ and let $V^\circ(\emptyset) = 0^N$. Note that $V^\circ(S) = V_{\text{p}}(S) \times \left\{0^{N\setminus S}\right\}$, for all $S \in \mathcal{N}$.

Definition 2.8. (Sharkey, 1981) An NTU game (N, V) is *cardinally convex* if for all $S, T \in \mathcal{N}$ we have $V^\circ(S) + V^\circ(T) \subseteq V^\circ(S \cap T) + V^\circ(S \cup T)$.

Let $\Gamma^{\text{NTU}}_{\text{cc}}$ denote the class of cardinally convex NTU games with player set N.

Hendrickx, Borm, and Timmer (2002) introduce the following three marginalistic interpretations of NTU convexity.

Equation (2) in Definition 2.2 of convexity for TU games states that for any coalition U, the marginal contribution of U to a coalition is at least equal to U's contribution to a smaller coalition. The same idea in the NTU setting is formulated as coalition merge convexity.[1] Let $\Gamma^{\text{NTU}}_{\text{cmc}}$ denote the class of coalition merge convex NTU games with player set N.

Equation (3) in Definition 2.2 of convexity for TU games says that for any player i, the marginal contribution of i to some coalition is at least equal to i's contribution to a smaller coalition. The analogous concept in the NTU setting is called individual merge convexity.[2] Let $\Gamma^{\text{NTU}}_{\text{imc}}$ denote the class of individual merge convex NTU games with player set N.

We now define the vector of marginal contributions for an NTU game.

Definition 2.9. Consider an NTU game (N, V) and a permutation $\sigma \in \Sigma^N$. The *vector of marginal contributions* $M^\sigma(V)$ is defined by

$$\begin{aligned} M^\sigma_{\sigma(j)}(V) = \sup\{y_{\sigma(j)} | y \in V(\{\sigma(1), \ldots, \sigma(j)\}),& \\ \forall i \in \{1, \ldots, j-1\} : y_{\sigma(i)} \geq M^\sigma_{\sigma(i)}(V)\}& \end{aligned}$$

for all $j \in \{1, \ldots, n\}$.[3]

Theorem 2.3 suggests the following convexity notion for NTU games.

Definition 2.10. An NTU game (N, V) is *marginal convex* if for all $\sigma \in \Sigma^N$ we have $M^\sigma(V) \in C(V)$.

[1] For the definition of coalition merge convexity, we refer to the electronic supplement.
[2] For the definition of individual merge convexity, we refer to the electronic supplement.
[3] We use the convention $\sup(\emptyset) = -\infty$.

Let $\Gamma^{\text{NTU}}_{\text{mc}}$ denote the class of marginal convex NTU games with player set N.

The five notions of NTU convexity are not equivalent in general. Hendrickx, Borm, and Timmer (2002) show that ordinal and cardinal convexity are not related to each other and to the other three types of convexity. They also provide the following theorem on the relation of the last three convexity notions.

Theorem 2.11. *If an NTU game (N, V) is coalition merge convex, then it is individual merge convex, that is $\Gamma^{\text{NTU}}_{\text{cmc}} \subseteq \Gamma^{\text{NTU}}_{\text{imc}}$. If an NTU game (N, V) is individual merge convex, then it is marginal convex, that is $\Gamma^{\text{NTU}}_{\text{imc}} \subseteq \Gamma^{\text{NTU}}_{\text{mc}}$.*

Since our definition of the NTU game is slightly different from the one of Hendrickx, Borm, and Timmer (2002), we provide a proof of Theorem 2.11 in the electronic supplement.

To illustrate the subtle differences between the various notions of NTU convexity, consider the following example of an ordinally convex NTU game which is neither cardinally, nor marginal, thus by Theorem 2.11 nor individual merge, nor coalition merge convex.

Example 2.12. (Hendrickx, Borm, and Timmer, 2002, Example 4.1.) Consider the following NTU game with player set $N = \{1, 2, 3\}$. Let

$$\begin{aligned} V(\{i\}) &= \{x \in \mathbb{R}^3 \mid x_i \leq 0\} \text{ for all } i \in N, \\ V(\{1,2\}) &= \{x \in \mathbb{R}^3 \mid x_1 \leq 0, x_2 \leq 2\}, \\ V(\{1,3\}) &= \{x \in \mathbb{R}^3 \mid x_1 + x_3 \leq 1\}, \\ V(\{2,3\}) &= \{x \in \mathbb{R}^3 \mid x_2, x_3 \leq 0\}, \\ V(N) &= \{x \in \mathbb{R}^3 \mid \sum_{i \in N} x_i \leq 2\}. \end{aligned}$$

To show that (N, V) is ordinally convex, let $S, T \in \mathcal{N}$ and let $x \in V(S) \cap V(T)$. If $S \subseteq T$, $T \subseteq S$ or $S \cap T = \emptyset$, then ordinal convexity is easy to check. If $S = \{1, 2\}$ and $T = \{1, 3\}$, then $x_1 \leq 0$ and thus $x \in V(S \cap T)$. Otherwise, $\sum_{i \in N} x_i \leq 2$, thus $x \in V(S \cup T)$.

Cardinal convexity of (N, V) fails, since $(0, 2, 0) \in V^\circ(\{1, 2\})$ and $(0, 0, 1) \in V^\circ(\{1, 3\})$, but $(0, 2, 0) + (0, 0, 1) = (0, 2, 1) \notin V^\circ(\{1\}) + V^\circ(N)$.

Marginal convexity of (N, V) is also not satisfied, since the vector of marginal contributions corresponding to $\sigma = (1, 2, 3)$, $M^\sigma(V) = (0, 2, 0)$ does not belong to the core: coalition $\{1, 3\}$ blocks it. Therefore, by Theorem 2.11, (N, V) is neither individual merge, nor coalition merge convex.

We will continue Example 2.12 in Examples 3.3 and 3.6.

3 Exact NTU Games

Theorem 2.4 claims that convex TU games are exact. In this section we generalize exactness to the NTU setting and analyze the relationship of NTU exactness and the various notions of NTU convexity.

Definition 3.1. An NTU game (N, V) is *NTU exact* if for each $S \in \mathcal{N}$ there exists a core allocation $x \in C(V)$ such that $x \in \partial V(S)$.

Let $\Gamma_{\mathrm{e}}^{\mathrm{NTU}}$ denote the class of exact NTU games with player set N. Every TU game (N, v) with $v(S) \geq 0$ for all $S \in \mathcal{N}$ gives rise to an NTU game (N, V) by defining $V(S) = \{x \in \mathbb{R}^N \mid x(S) \leq v(S)\}$ for all $S \in \mathcal{N}$. Note that Assumptions (4)-(9) are satisfied by (N, V). It is a straightforward exercise to verify the following theorem.

Theorem 3.2. *A TU game (N, v) is exact if and only if the corresponding NTU game (N, V) is NTU exact.*

Note that if an NTU game (N, V) is NTU exact, then each of its subgames has a core element, since by definition for each $S \in \mathcal{N}$ there exists a core allocation $x \in C(V)$ such that $x \in \partial V(S)$, and x cannot be blocked in the subgame (S, V^S) either. Thus exact NTU games are a subset of totally Π-balanced games, $\Gamma_{\mathrm{e}}^{\mathrm{NTU}} \subseteq \Gamma_{\mathrm{t-\Pi-b}}^{\mathrm{NTU}}$.

Next, we check whether the ordinally convex NTU game in Example 2.12 is NTU exact.

Example 3.3. (Example 2.12 continued.) The NTU game (N, V) in Example 2.12 is NTU exact, since $(0, 0, 2)$ is a core element on the boundary of $V(\{1\}), V(\{2\})$, and $V(\{1, 2\})$; $(2, 0, 0)$ is a core element on the boundary of $V(\{2\}), V(\{3\})$, and $V(\{2, 3\})$; and $(1, 1, 0)$ is a core element on the boundary of $V(\{1, 3\})$.

If for all $S \in \mathcal{N}$ all core elements of the subgame (S, V^S) could be extended to the core of the original game by an appropriate choice for the elements outside S, then NTU exactness would follow immediately from ordinal convexity, since core elements of (S, V^S) are on the boundary of $V(S)$. Example 3.3 shows that NTU exactness of an ordinally convex NTU game cannot be demonstrated in this way. The core of the subgame related to coalition $\{1, 2\}$ is $\{x \in \mathbb{R}^2 \mid x_1 = 0,\ 0 \leq x_2 \leq 2\}$. Note that only some elements in this core can be extended to the core of the original game: $\{x \in \mathbb{R}^2 \mid x_1 = 0,\ 0 \leq x_2 \leq 1\}$, since if $y_1 = 0,\ 1 < y_2 \leq 2,\ y_3 = 2 - y_2$, then coalition $\{1, 3\}$ blocks allocation y in the original game.

Peleg (1986) gives the following sufficient condition under which certain core elements of a subgame in an ordinally convex NTU game can be extended to the core of the original game.

Theorem 3.4. *(Peleg, 1986, Corollary 2.10). Let (N, V) be an ordinally convex game. Let $T \in \mathcal{N} \setminus \{N\},\ z \in V^T(T)$ such that $z \in C(V^T)$ and for all $R \subseteq T, R \neq T, z \notin \mathrm{cl}\ V^T(R)$. Then there exists an allocation $x \in C(V)$ such that $x^T = z$.*

In Example 3.3 let $T = \{1, 2\}$ and take any $z \in C(V^T)$. Since $z_1 = 0$, we have that $z \in \mathrm{cl}\ V^T(\{1\})$, hence Theorem 3.4 cannot be used to show that ordinally convex NTU games are exact.

To proceed, we define the notion of a *reduced game* for the case where one player leaves the grand coalition. This notion of reduced game originates from Greenberg (1985),

Definition 3.5. Take any NTU game (N, V), $n \geq 2$, and a player $i \in N$. Define:

$$\begin{aligned} M =& N \setminus \{i\},\ m = n - 1, \\ \alpha_i =& \sup\{x_i \mid x \in V(\{i\})\}, \\ W(S) =& \{x \in \mathbb{R}^M \mid \exists \beta > \alpha_i \text{ such that } (x, \beta) \in V(S \cup \{i\})\},\ S \subseteq M. \\ P(S) =& V_{\mathrm{p}}(S) \times \mathbb{R}^{M \setminus S},\ S \subseteq M. \end{aligned}$$

Then, the *reduced game* (M, U) is given by:

$$U(S) = \begin{cases} \{x \in \mathbb{R}^M \mid (x, \alpha_i) \in V(N)\}, & \text{for } S = M, \\ \emptyset, & \text{for } S = \emptyset, \\ W(S) \cup P(S), & \text{otherwise.} \end{cases}$$

The definition of the reduced game is illustrated in the following example.

Example 3.6. (Example 2.12 continued.) If player 3 leaves the grand coalition in Example 2.12, then the derived reduced game looks as follows. $U(\{1, 2\}) = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \leq 2\}$, $U(\emptyset) = \emptyset$. Moreover, $W(\{1\}) = \{x \in \mathbb{R}^2 \mid x_1 < 1\}$, $W(\{2\}) = \emptyset$, $P(\{1\}) = \{x \in \mathbb{R}^2 \mid x_1 \leq 0\}$ and $P(\{2\}) = \{x \in \mathbb{R}^2 \mid x_2 \leq 0\}$ imply that $U(\{1\}) = \{x \in \mathbb{R}^2 \mid x_1 < 1\}$ and $U(\{2\}) = \{x \in \mathbb{R}^2 \mid x_2 \leq 0\}$.

Note that the reduced game is not zero normalized and $U(\{1\})$ is open. Moreover, *all* the core elements of the reduced game $\{x \in \mathbb{R}^2 \mid x_1 + x_2 = 2,\ 1 \leq x_1 \leq 2,\ 0 \leq x_2 \leq 1\}$ can be extended to a core element of the original game by setting $x_3 = \alpha_3 = 0$.

In general, a reduced game is not always an NTU game. However, Greenberg (1985) shows the following lemma about reduced games of ordinally convex NTU games.

Lemma 3.7. *(Greenberg, 1985) Consider an ordinally convex NTU game (N, V). Then the reduced game (M, U) is an ordinally convex NTU game.*

In his proof Greenberg (1985) considers the setting when $V(S) \subseteq \mathbb{R}^N_+$ instead of $V(S) \subseteq \mathbb{R}^N$, for all $S \in \mathcal{N}$, but due to Assumptions (6) and (8) all the arguments can be carried over to our setting.

We show the following theorem.

Theorem 3.8. *If an NTU game (N, V) is ordinally convex, then it is NTU exact, that is $\Gamma^{\mathrm{NTU}}_{\mathrm{oc}} \subseteq \Gamma^{\mathrm{NTU}}_{\mathrm{e}}$.*

Proof. The proof proceeds by induction on the cardinality of N.

Let $n = 1$. If an NTU game (N, V) is ordinally convex, then it is NTU exact, since $\max\{x \mid x \in V(N)\}$ is well defined, is on the boundary of $V(N)$ and belongs to the core.

Assume that the theorem holds for any game with less than n players. We will show that it also holds for n players.

Let (N, V) be an ordinally convex NTU game with $n \geq 2$ players. Consider some coalition $S \subseteq N$. We show that there exists $y \in C(V)$ such that $y \in \partial V(S)$ and thereby prove that (N, V) is NTU exact.

Let $i \in S$ be arbitrarily chosen and let $M = N \setminus \{i\}$. Lemma 3.7 and the induction hypothesis imply that the reduced game (M, U) is NTU exact. Then let $x \in C(U)$ be such that $x \in \partial U(S \setminus \{i\})$ if $S \neq \{i\}$, and let $x \in C(U)$ be arbitraryly chosen otherwise. Moreover, let $y \in \mathbb{R}^N$ be defined by $y^M = x$ and $y^i = \alpha_i$. Then, in Step I we show that $y \in C(V)$, in Step II we establish that $y \in \partial V(S)$.

Step I, $y \in C(V)$
Since $x \in C(U)$ by definition $x \in U(M)$, that is $y \in V(N)$.

Case 1: First, we show that y cannot be blocked by any coalition $T \subsetneq N$. Suppose to the contrary that there exist $\beta > \alpha_i$, $z \gg x$ and $T \subsetneq N$ such that $(z, \beta) \in V(T)$. We consider two subcases: $T = M$ or $T \neq M$.

Case 1a: $T = M$. Then $(z, \beta) \in V(M)$ and by comprehensiveness for all $\epsilon > 0$ we have that $(z, \alpha_i - \epsilon) \in V(M)$. Also, for all $\epsilon > 0$ we have that $(z, \alpha_i - \epsilon) \in V(\{i\})$ by the definition of α_i. Ordinal convexity implies that $V(M) \cap V(\{i\}) \subseteq V(N)$, thus for all $\epsilon > 0$ we have that $(z, \alpha_i - \epsilon) \in V(N)$. Since $V(N)$ is closed, $(z, \alpha_i) \in V(N)$, implying that $z \in U(M)$, contradicting $x \in C(U)$.

Case 1b: $T \neq M$. If $i \notin T$, then $z \in P(T)$ and hence T would block x in (M, U), contradicting $x \in C(U)$. If $i \in T$, then $T \setminus \{i\} \neq \emptyset$, since $\beta > \alpha_i$ implies $(z, \beta) \notin V(\{i\})$. Therefore, $z \in W(T \setminus \{i\})$, again contradicting $x \in C(U)$.

Case 2: Next, we show that y cannot be blocked by N either. Otherwise there exist $\beta > \alpha_i$, $z \gg x$ such that $(z, \beta) \in V(N)$. It follows using comprehensiveness that $(z, \alpha_i) \in V(N)$, implying that $(z, \alpha_i) \in U(M)$, again contradicting $x \in C(U)$. Thus $y \in C(V)$.
Note that the construction used shows that all core elements of the *reduced game* can be extended to core elements of the original game.

Step II. $y \in \partial V(S)$
Recall that i is a member of S. If $S = \{i\}$, then $y \in \partial V(\{i\})$ by the definition of α_i. If $S = N$, then $y \in C(V)$ by Step I, which implies that $y \in \partial V(N)$.
If $S \neq \{i\}$ and $S \neq N$, then $U(S \setminus \{i\}) = W(S \setminus \{i\}) \cup P(S \setminus \{i\})$ and $x \in \partial U(S \setminus \{i\})$. So

$$
\begin{aligned}
x \in \partial &\Big(W(S \setminus \{i\}) \cup P(S \setminus \{i\}) \Big) \\
&= \mathrm{cl}\, \Big(W(S \setminus \{i\}) \cup P(S \setminus \{i\}) \Big) \cap \mathrm{cl}\, \Big(\mathbb{R}^M \setminus (W(S \setminus \{i\}) \cup P(S \setminus \{i\})) \Big) \\
&= \Big(\mathrm{cl}\, W(S \setminus \{i\}) \cup \mathrm{cl}\, P(S \setminus \{i\}) \Big) \cap \mathrm{cl}\, \Big(\mathbb{R}^M \setminus (W(S \setminus \{i\}) \cup P(S \setminus \{i\})) \Big) \\
&= \Big(\partial W(S \setminus \{i\}) \setminus \mathrm{int}\, P(S \setminus \{i\}) \Big) \cup \Big(\partial P(S \setminus \{i\}) \setminus \mathrm{int}\, W(S \setminus \{i\}) \Big),
\end{aligned}
$$

which implies that there are two (not exclusive) cases:
$x \in \partial W(S \setminus \{i\}) \setminus \mathrm{int}\, P(S \setminus \{i\})$ or $x \in \partial P(S \setminus \{i\}) \setminus \mathrm{int}\, W(S \setminus \{i\})$.

Case 1: $x \in \partial W(S \setminus \{i\}) \setminus \text{int}\, P(S \setminus \{i\})$. Then, $x \in \partial W(S \setminus \{i\})$ implies $x \in \text{cl}\, W(S \setminus \{i\}) \cap \text{cl}\, (\mathbb{R}^M \setminus W(S \setminus \{i\}))$. Since $x \in \text{cl}\, W(S \setminus \{i\})$, there exists a sequence $(x^k)_{k \in \mathbb{N}}$ with $x^k \in W(S \setminus \{i\})$ for all $k \in \mathbb{N}$ and $(x^k)_{k \in \mathbb{N}} \to x$. Then, by the definition of $W(S \setminus \{i\})$, there exists a sequence $(\beta^k)_{k \in \mathbb{N}}$ with $\beta^k > \alpha_i$ and $(x^k, \beta^k) \in V(S)$ for all $k \in \mathbb{N}$. Due to comprehensiveness $(x^k, \alpha_i) \in V(S)$ for all $k \in \mathbb{N}$ as well, and the sequence $(x^k, \alpha_i)_{k \in \mathbb{N}}$ converges to (x, α_i), implying that $(x, \alpha_i) \in \text{cl}\, V(S)$. Since $x \in \text{cl}\, (\mathbb{R}^M \setminus W(S \setminus \{i\}))$ as well, there exists a sequence $(x^k)_{k \in \mathbb{N}}$ with $x^k \in \mathbb{R}^M \setminus W(S \setminus \{i\})$ for all $k \in \mathbb{N}$ and $(x^k)_{k \in \mathbb{N}} \to x$, that is for all $\beta > \alpha_i$ we have that $(x^k, \beta) \in \mathbb{R}^N \setminus V(S)$ for all $k \in \mathbb{N}$. In particular, $(x^k, \alpha_i + 1/(k+1)) \in \mathbb{R}^N \setminus V(S)$ for all $k \in \mathbb{N}$, and $(x^k, \alpha_i + 1/(k+1))_{k \in \mathbb{N}} \to (x, \alpha_i)$, implying that $(x, \alpha_i) \in \text{cl}\, (\mathbb{R}^N \setminus V(S))$. So $(x, \alpha_i) \in \text{cl}\, V(S) \cap \text{cl}\, (\mathbb{R}^N \setminus V(S))$, thus $y \in \partial V(S)$.

Case 2: $x \in \partial P(S \setminus \{i\}) \setminus \text{int}\, W(S \setminus \{i\})$. By ordinal convexity of (N, V) we have $V(S \setminus \{i\}) \cap V(\{i\}) \subseteq V(S)$, which together with $x \in \partial P(S \setminus \{i\})$ implies that there exists a sequence $(x^k, \alpha_i^k)_{k \in \mathbb{N}}$ with $(x^k, \alpha_i^k) \in V(S)$ for all $k \in \mathbb{N}$ and $(x^k, \alpha_i^k)_{k \in \mathbb{N}} \to (x, \alpha_i)$, so $(x, \alpha_i) \in \text{cl}\, V(S)$. Since $x \notin \text{int}\, W(S \setminus \{i\})$, for all $z \gg x$ and for all $\beta > \alpha_i$ we have $(z, \beta) \notin V(S)$. Thus there exists a sequence $(x^k, \alpha_i + 1/(k+1))_{k \in \mathbb{N}} \to (x, \alpha_i)$ such that $(x^k, \alpha_i + 1/(k+1)) \in \mathbb{R}^N \setminus V(S)$, implying that $(x, \alpha_i) \in \text{cl}\, (\mathbb{R}^N \setminus V(S))$. So $(x, \alpha_i) \in \text{cl}\, V(S) \cap \text{cl}\, (\mathbb{R}^N \setminus V(S))$, thus $y \in \partial V(S)$. □

Next, we provide an example of a cardinally convex game which is not NTU exact.

Example 3.9. (A cardinally convex game which is not NTU exact). Consider the following NTU game with player set $N = \{1, 2, 3, 4\}$. Let

$$\begin{aligned}
V(\{i\}) &= \{x \in \mathbb{R}^4 \mid x_i \leq 0\}, \quad i \in N, \\
V(\{1,2\}) &= \{x \in \mathbb{R}^4 \mid x_1 + x_2 \leq 2\}, \\
V(\{1,3\}) &= \{x \in \mathbb{R}^4 \mid x_1, x_3 \leq 0\}, \\
V(\{1,4\}) &= \{x \in \mathbb{R}^4 \mid x_1, x_4 \leq 0\}, \\
V(\{2,3\}) &= \{x \in \mathbb{R}^4 \mid x_2, x_3 \leq 0\}, \\
V(\{2,4\}) &= \{x \in \mathbb{R}^4 \mid x_2, x_4 \leq 0\}, \\
V(\{3,4\}) &= \{x \in \mathbb{R}^4 \mid x_3, x_4 \leq 0\}, \\
V(\{1,2,3\}) &= \{x \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 \leq 4\}, \\
V(\{1,2,4\}) &= \{x \in \mathbb{R}^4 \mid x_1 + x_2 + x_4 \leq 4\}, \\
V(\{1,3,4\}) &= \{x \in \mathbb{R}^4 \mid x_1, x_3, x_4 \leq 0\}, \\
V(\{2,3,4\}) &= \{x \in \mathbb{R}^4 \mid x_2, x_3, x_4 \leq 0\}, \\
V(N) &= \{x \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 \leq 4,\ x_4 \leq 0\} \\
&\quad \cup \{x \in \mathbb{R}^4 | \ x_1 + x_2 + x_4 \leq 4,\ x_3 \leq 0\} \\
&\quad \cup \{x \in \mathbb{R}^4 | \ x_1 + x_2 + x_3 + x_4 \leq 6, x_1 \leq -1\}.
\end{aligned}$$

The game (N, V) above is cardinally convex, since

(i) $V^\circ(\{1,2,3\}) + V^\circ(\{1,2,4\}) \subseteq V^\circ(\{1,2\}) + V^\circ(N)$ using the third set in the definition of $V(N)$. Notice that to do so we make use of the fact that for $x \in V^\circ(N)$, x_1 and x_2 can be chosen to be negative in order to increase the values of x_3 and x_4.
(ii) For all other $S, T \in \mathcal{N}$ it is easy to verify that $V^\circ(S) + V^\circ(T) \subseteq V^\circ(S \cup T)$.

However, (N, V) is not NTU exact, since there is no core allocation on the boundary of $V(\{1,2\})$. To see that, assume that there is an allocation $x \in C(V)$ such that $x \in \partial V(\{1,2\})$. Since $x \in \partial V(\{1,2\})$, we have that $x_1 + x_2 = 2$. To have a core allocation, $x_3 \geq 2$ should hold to prevent blocking by coalition $\{1,2,3\}$ and $x_4 \geq 2$ should hold to prevent blocking by coalition $\{1,2,4\}$. Thus x should be in the third set in the definition of $V(N)$, requiring that $x_1 \leq -1$, which would be blocked by player 1.

By Theorem 2.11, to verify whether the marginalistic interpretations of NTU convexity imply NTU exactness, it is enough to analyze marginal convexity.

Theorem 3.10. *If an NTU game (N, V) is marginal convex, then it is NTU exact, that is $\Gamma^{\mathrm{NTU}}_{\mathrm{mc}} \subseteq \Gamma^{\mathrm{NTU}}_{\mathrm{e}}$.*

Proof. Consider a marginal convex NTU game (N, V), and a coalition $S \in \mathcal{N}$. For exactness we have to show that there is a core element on the boundary of $V(S)$. Let $\bar{\sigma}$ be a permutation such that $S \in \{\bar{\sigma}(1), \{\bar{\sigma}(1), \bar{\sigma}(2)\}, \{\bar{\sigma}(1), \bar{\sigma}(2), \bar{\sigma}(3)\}, \ldots, N\}$. Since (N, V) is marginal convex, we have that $M^{\bar{\sigma}}(V) \in C(V)$. By definition, $M^{\bar{\sigma}}(V)$ is on the boundary of $V(T)$ for all $T \in \{\bar{\sigma}(1), \{\bar{\sigma}(1), \bar{\sigma}(2)\}, \{\bar{\sigma}(1), \bar{\sigma}(2), \bar{\sigma}(3)\}, \ldots, N\}$, thus it is a core element on the boundary of $V(S)$ as well. □

Using Theorems 2.11 and 3.10 we have the following corollary.

Corollary 3.11. *Each of coalition merge convexity, individual merge convexity and marginal convexity implies exactness in the NTU setting, that is $\Gamma^{\mathrm{NTU}}_{\mathrm{cmc}} \subseteq \Gamma^{\mathrm{NTU}}_{\mathrm{imc}} \subseteq \Gamma^{\mathrm{NTU}}_{\mathrm{mc}} \subseteq \Gamma^{\mathrm{NTU}}_{\mathrm{e}}$.*

4 Conclusion

In this paper we have generalized exactness to games with non-transferable utility to get the class of NTU exact games ($\Gamma^{\mathrm{NTU}}_{\mathrm{e}}$). A game is NTU exact if for each coalition there is a core allocation on the boundary of its payoff set, meaning that this coalition does not necessarily benefit from the gains of forming the grand coalition in an allocation which is robust against all coalitional deviations. We have noted that NTU exact games are a subset of totally Π-balanced NTU games ($\Gamma^{\mathrm{NTU}}_{\mathrm{t-\Pi-b}}$), having a non-empty core in each of their subgames.

We have shown that the classes of ordinally convex ($\Gamma^{\mathrm{NTU}}_{\mathrm{oc}}$), coalition merge convex ($\Gamma^{\mathrm{NTU}}_{\mathrm{cmc}}$), individual merge convex ($\Gamma^{\mathrm{NTU}}_{\mathrm{imc}}$), and marginal convex ($\Gamma^{\mathrm{NTU}}_{\mathrm{mc}}$) NTU games are a subset of NTU exact games. Moreover, we have given an example of a cardinally convex game ($\Gamma^{\mathrm{NTU}}_{\mathrm{cc}}$) which is not NTU exact.

Hendrickx, Borm, and Timmer (2002) show that the aforementioned five classes of NTU convex games do not coincide for more than three players. The only general relationship between these five classes (Theorem 2.11) is that coalition merge convexity implies individual merge convexity ($\Gamma^{\text{NTU}}_{\text{cmc}} \subseteq \Gamma^{\text{NTU}}_{\text{imc}}$), and individual merge convexity implies marginal convexity ($\Gamma^{\text{NTU}}_{\text{imc}} \subseteq \Gamma^{\text{NTU}}_{\text{mc}}$).

Theorem 2.6 claims that the class of convex TU games coincides with the class of totally exact TU games. In the NTU setting we do not have such a theorem. Let $\Gamma^{\text{NTU}}_{\text{te}}$ denote the class of totally exact NTU games with player set N, being NTU exact in all of their subgames. Since an ordinally convex game is exact, and all subgames of an ordinally convex game are ordinally convex, we have that $\Gamma^{\text{NTU}}_{\text{oc}} \subseteq \Gamma^{\text{NTU}}_{\text{te}}$. For marginal convex games a similar argument leads to $\Gamma^{\text{NTU}}_{\text{mc}} \subseteq \Gamma^{\text{NTU}}_{\text{te}}$.

However, using our results it is easy to provide counterexamples where NTU total exactness implies none of the NTU convexity notions. For instance, the NTU game in Example 2.12 is ordinally convex, and as we argued that game is totally NTU exact. But it is neither cardinal, nor marginal, nor individual merge, nor coalition merge convex. So neither cardinal, nor marginal, nor individual merge, nor coalition merge convexity is implied by total NTU exactness in general. Hendrickx, Borm, and Timmer (2000) provide an example (Example 4.6 there) for an NTU game which is marginal convex but not ordinally convex. That example can be used to show that total NTU exactness does not imply ordinal convexity either.

We summarize the relationships between the various classes of NTU games for more than three players in Figure 1.

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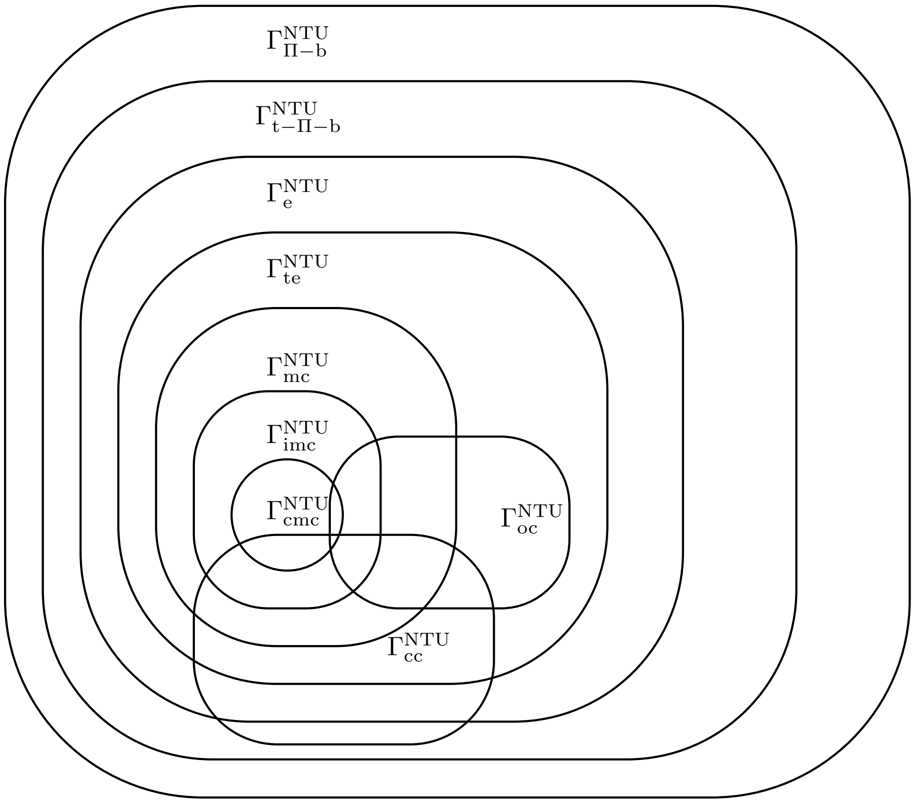

Figure 1: Subsets of Π-balanced games.

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