Matroid union
Graphic? Binary? Neither?

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Abstract: There is a conjecture that if the union (also called sum) of graphic matroids is not graphic then it is nonbinary. Some special cases have been proved only, for example if several copies of the same graphic matroid are given. If there are two matroids and the first one can either be represented by a graph with two points, or is the direct sum of a circuit and some loops, then a necessary and sufficient condition is known for the other matroid to ensure the graphicity of the union and the above conjecture holds for these cases. We prove the sufficiency of this condition for the graphicity of the union of two arbitrary graphic matroids. Then we present a weaker necessary condition which is of similar character. Finally we suggest a more general framework of the study of such questions by introducing matroid classes formed by those graphic (or arbitrary) matroids whose union with any graphic (or arbitrary) matroid is graphic (or either graphic or nonbinary).

Keywords: matroid theory, graphic matroid, union of matroids

1 Introduction

Graphic matroids form one of the most significant classes in matroid theory. When introducing matroids, Whitney concentrated on relations to graphs. The definition of some basic operations like deletion, contraction and direct sum were straightforward generalizations of the respective concepts in graph theory. Most matroid classes, for example those of binary, regular or graphic matroids, are closed with respect to these operations. This is not the case for the union. The union of two graphic matroids can be non-graphic.

The first paper studying the graphicity of the union of graphic matroids was probably that of Lovász and Recski [2], they examined the case if several copies of the same graphic matroid are given.

Another possible approach is to fix a graph G_0 and characterize those graphs G where the union of their cycle matroids $M(G_0) \vee M(G)$ is graphic. (Observe that we may clearly disregard the cases if G_0 consists of loops only, or if it contains coloops.) As a byproduct of some studies on the application of matroids in electric network analysis, this characterization has been performed for the case if G_0 consists of loops and a single circuit of length two only, see the first graph of Figure 1. (In view of the above observation this is the simplest nontrivial choice of G_0.)

Theorem 1 *[4] Let A and B be the cycle matroids of the graphs shown in Figure 1 on ground sets $E_A = \{1, 2, ..., n\}$ and $E_B = \{1, 2, i, j, k\}$, respectively. Let M be an arbitrary graphic matroid on E_A. Then the union $A \vee M$ is graphic if and only if B is not a minor of M with any triplet i, j, k.*

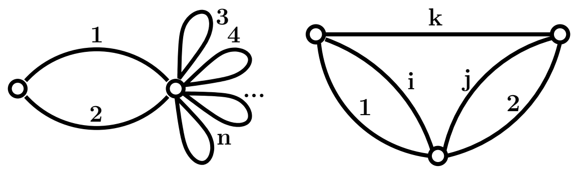

Figure 1: A graphic representation of A (left) and B (right)

Conjecture 2 *Recski [7] conjectured some thirty years ago that if the union of two graphic matroids is not graphic then it is nonbinary.*

This is known to be true if the two graphic matroids are identical or if one of them is A as given in Theorem 1 – these results follow in a straightforward way from [2] and from [4], respectively.

In a previous paper [1] we extended the result of Theorem 1 if G_0 either consists of loops and two points joined by n parallel edges or if it consists of loops and a single circuit of length n. We proved that deciding whether $M(G_0) \vee M(G)$ is graphic can be performed in polynomial time if G_0 is one of these two matroids. Our results also implied that the above conjecture is true if one of these two types of graphs play the role of G_0.

Observe that the first graph of Figure 1, representing A, has only two non-loop edges (1 and 2), while the second graph, representing B, has the property that the complement of the set $\{1, 2\}$ of non-loop edges of A contains both a circuit and a spanning tree. This property turned out to be crucial if we consider a larger set of non-loop edges which are either all parallel or all serial.

2 Reduction steps

While during our study of the union of the two graphic matroids $M_1 = M(G_0)$ and $M_2 = M(G)$ the former one had a very special structure in [1], in the present section, we formulate some reduction steps for arbitrary graphic matroids M_1 and M_2 on the same ground set.
Throughout M_1 and M_2 will be graphic matroids on the same ground set E. We shall refer to them as *addends*. It is well known that if a matroid is graphic then so are all of its submatroids and minors. Hence if a matroid has a non-graphic minor then the matroid is not graphic.

Definition 3 *We call some non-coloop edges of a matroid serial if they belong to exactly the same circuits.*

Definition 4 *Let $L(M)$ and $NL(M)$denote the set of loops and non-loops, respectively, in the matroid M.*

The following lemmata contain the main opportunities when we can simplify our addend matroids. Since they refer to graphic matroids only, we can use graph theoretical terminology. Throughout, $M \setminus X$ and M/X will denote deletion and contraction, respectively, of the set X in a matroid M, while $X - Y$ will denote the difference of the sets X and Y. We shall write $Y \cup x$, $Y - x$, $M \setminus x$ and M/x instead of $Y \cup \{x\}$, $Y - \{x\}$, $M \setminus \{x\}$ and $M/\{x\}$, respectively.

2.1 The earlier steps

Lemmata 5 through 11 were proved in [1] and they will be useful for our new results as well.

Lemma 5 *Let X and Y denote the set of coloops in M_1 and in M_2, respectively. The union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus (X \cup Y)) \vee (M_2 \setminus (X \cup Y))$ is graphic.*

Recall that a matroid is connected if it does not arise as the direct sum of two smaller matroids. If M is not connected and X is the ground set of a connected component of M then $M/X = M \setminus X$.

Lemma 6 *If the ground set of a connected component X of the matroid M_1 is a subset of $L(M_2)$ then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus X) \vee (M_2 \setminus X)$ is graphic.*

Recall that the cycle matroid of a loopless graph with no isolated vertices is connected if and only if the graph is 2-vertex-connected.

Lemma 7 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ in which $X \subset E$ determines a connected subgraph and $E - X$ has exactly two common vertices with X (call them a and b).*
Let M_1' be the cycle matroid of $G' := G(V, (E - X) \cup \{(a, b)\})$ and $M_2' := (M_2 \setminus X) \cup loop(a, b)$ (Here $loop(a, b)$ denotes a loop corresponding to the edge (a, b) in G').
If X is a subset of $L(M_2)$ then the union $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.

After these preliminaries we can define the reduction that will be the most important concept to reduce the infinite number of cases.

Definition 8 *We say that a pair M_1, M_2 is reduced if none of the above lemmata can help us to decrease the number of edges.*

Corollary 9 *Assume that the application of the previous lemmata to M_1 and M_2 leads to a reduced pair of matroids M_1', M_2'. Then $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.*

Proposition 10 *Assume that M_1 and M_2 are given by their graphs G_1 and G_2, respectively. Then we can perform the reduction of these matroids in polynomial time.*

Lemma 11 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ and E_0 is the edge set of a 2-connected component X of G which has only one edge x from $NL(M_2)$. Then the union $M_1 \vee M_2$ is graphic if and only if $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$ is graphic.*

2.2 The new steps

We can formalize some lemmata similar to the aforementioned ones. The main observation is that we can reduce the addends if there are some edges which are in special relation in both matroids, namely if two edges are parallel, serial, one of them is a loop, or they don't have a common circuit.

Lemma 12 *If two parallel edges x and y of M_1 are serial in M_2 then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus x) \vee (M_2/x)$ is graphic.*

Proof: Let N denote the union $(M_1 \setminus x) \vee (M_2/x)$. One can easily see that $M_1 \vee M_2$ can be obtained from N by a series extension $\{x, y\}$ of the element y. Since series extension cannot change graphicity or non-graphicity, this proves the assertion. $\Box$

Lemma 13 *If two serial edges x and y of M_1 are serial in M_2 as well then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus \{x, y\}) \vee (M_2 \setminus \{x, y\})$ is graphic.*

Proof: x and y will be coloops in the union, so they don't influence the graphicity of the union. $\Box$

Observe that the case if two serial edges of M_1 are loops in M_2 has been covered by Lemma 7.

Lemma 14 *Suppose that x and y are serial edges in M_1 and they are not contained in any common circuit of M_2. Assume x is not a loop of M_2. Let $M_1' = M_1/x$ and relabel y to z. Let M_2' be obtained from M_2 as shown in Figure 2. Then $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.*

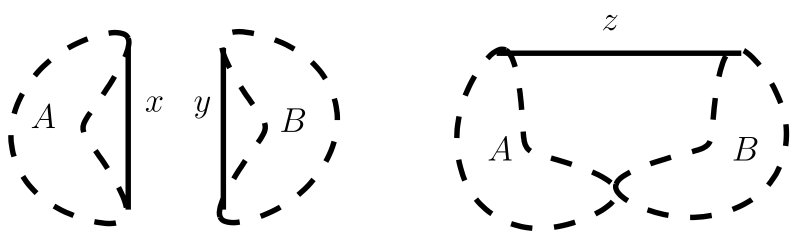

Figure 2: The structure of M_2 and M_2'

Proof: If there exists a circuit C_x in the union $M_1 \vee M_2$ which contains x but not y then $C_x - x$ is independent in the union, that is, $C_x - x = I_1 \cup I_2$ where I_j is independent in M_j. This is a contradiction since $I_1 \cup x$ will also be independent in M_1 (since x and y were serial), hence C_x would be independent in the union.
So x and y are both coloops in the union, or they are serial. In both cases the graphicity of the union is equivalent to the graphicity of $M_1' \vee M_2'$. (Moreover $M_1 \vee M_2$ can easily be constructed from $M_1' \vee M_2'$.)
$\square$

The last two reductions below are different from the previous ones because they state that if some conditons don't hold then the union is nonbinary, but if the conditions hold then we can make an equivalent reduction. This means that these lemmata can not help us to give a necessary and sufficient condition to the graphicity of the union of graphic matroids, but can help when thinking about a possible minimal counterexample of Conjecture 2.

Definition 15 *We will call a partition S_1, S_2 of a set S a* **good partition** *if S_i is independent in M_i for $i = 1, 2$.*

Observe that exactly the independent sets of the union $M_1 \vee M_2$ have good partitions.

Lemma 16 *Let two parallel edges x and y of M_1 be parallel in M_2 too. If x and y are coloops or serial in the union then there exists a subscript $k \in \{1, 2\}$ so that $M_1 \vee M_2$ is graphic if and only if $(M_k/x) \vee (M_{3-k} \setminus x)$ is graphic. If they are neither serial nor coloops then the union is nonbinary.*

Proof: **Case 1:** If x and y are neither serial nor coloops then there exists a circuit C_x which contains x but not y. Then $(C_x - x) \cup y$ will be a circuit too (because of the symmetric role of x and y in both M_1 and M_2). In a binary matroid there must be a circuit in the symmetric difference of two other ones, but $\{x, y\}$ can not be dependent in the union so it means that the union is not binary.
Case 2: On the other hand if x and y are both coloops in the union then $(M_1 \vee M_2) \setminus x$ equals the modified union $(M_k/x) \vee (M_{3-k} \setminus x)$ for both values $k = 1, 2$.
Case 3: Finally suppose that x and y are serial in the union. We claim that there exists a $k \in \{1, 2\}$ so that every independent set S of the union with $x, y \notin S$ has such a good partition $S = S_1 \cup S_2$ where $S_k \cup x$ is also independent in M_k, leading to a good partition of $S \cup x$.
Indirectly suppose that there exist two independent sets P, Q in $M_1 \vee M_2$ so that $x, y \notin P, Q$, $P_1 \cup x$ is dependent in M_1 for every good partition $P = P_1 \cup P_2$ (let us call this *Property 1*) and $Q_2 \cup x$ is dependent in M_2 for every good partition $Q = Q_1 \cup Q_2$ (let us call this *Property 2*). Observe that an independent set of $M_1 \vee M_2$ avoiding x and y cannot have both Property 1 and Property 2 because x and y are serial in $M_1 \vee M_2$.
Choose P and Q such that $|P \cap Q|$ is maximum. If $P \cup b$, for $b \in Q - P$, is an independent set of $M_1 \vee M_2$, then $P \cup b$ has Property 1; a contradiction to the choice of P. Hence P spans Q in $M_1 \vee M_2$. If $(P - a) \cup b$ is independent in $M_1 \vee M_2$, for $b \in Q - P$ and $a \in P - Q$, then, by the choice of P, $(P - a) \cup b$ does not have Property 1. Therefore $(P - a) \cup b$ has Property 2.

Claim 17 *Q is dependent in the union.*

PROOF: P does not have Property 2, hence there is a good partition $P = P_1 \cup P_2$ so that $P_2 \cup x$ is independent in M_2. Let b denote a fixed element of $Q - P$. Construct $Z_1 \subseteq P_1$ and $Z_2 \subseteq P_2$ in the following way. Put an edge e into Z_i if there exists an ordered sequence of edges $< a_0, a_1, ..., a_k >$ so that $a_0 = b$, $a_k = e$ and the following property is true:
For all $j \leq k$: $(P_1 - A(i,j,k)) \cup B(i,j,k)$ is independent in M_1, and $(P_2 - B(i,j,k)) \cup A(i,j,k) \cup x$ is independent in M_2. Here $A(i,j,k)$ and $B(i,j,k)$ denote the elements of the sequence with the same parity as $i+j+k$ preceeding a_j, and with different parity, respectively. Formally, $A(i,j,k) = \cup_{t=0}^{t \leq \frac{j-1}{2}} a_{2t+1}$ if $i+j+k$ is odd, $A(i,j,k) = \cup_{t=0}^{t \leq \frac{j}{2}} a_{2t}$ if $i+j+k$ is even and $B(i,j,k) = \cup_{t=0}^{t \leq j} a_t - A(i,j,k)$.
We will call such an edge sequence an *alternating sequence*. Observe that the modifed version of P_1 or P_2 along an alternating sequence (by the proper A and B) have the same closure as the original ones in M_1 or M_2, respectively.
Claim 17 follows from the fact that $Z = Z_1 \cup Z_2$ is a subset of Q but $Z \cup b$ is dependent in the union.
In order to prove $Z \subseteq Q$ suppose the contrary. Let the last element a_k of an alternating sequence $< a_0, ..., a_k >$ be not in Q but suppose that $a_i \in Q$ for all $i < k$. Then $P' = (P - a_k) \cup b$ were independent in the union but would not have Property 2 contradicting to the definition of P.
We claim that $Z \cup b$ is dependent in the union because $r_1(Z_1) = r_1(Z \cup b)$ and $r_2(Z_2) = r_2(Z \cup b)$. It is easy to see that the edges of the unique path between the end vertices of b from both P_1 (in a graph of M_1) and P_2 (in a graph of M_2) are elements of Z. Observe that exactly these edges will have a corresponding alternating sequence with $k = 1$. This means that b is spanned by Z_1 in M_1 and by Z_2 in M_2. This argument remains true even if b happens to be a loop in M_i for $i = 1$ or 2 since then b is obviously spanned by any subset Z_i.
Indirectly suppose that there exists an edge e in Z_i which is not spanned by Z_{3-i} in M_{3-i}. Consider the alternating sequence $< a_0, ..., a_k = e >$. We can suppose that a_i $(i \leq k)$ is in Q. We know that P_{3-i} spans e in M_{3-i} (otherwise $P \cup b$ would be independent in the union). This means that the modified version of P_{3-i} along any alternating sequence will also span e in M_{3-i}.
If an edge f is in the unique path between the two end vertices of e in $(P_{3-i} - B(0,0,k)) \cup A(0,0,k)$ in a graph of M_{3-i}, then either f has an alternating sequence $< a_0, ..., a_k, f >$ or $f \in A(k)$. The first case means that we have to put f into Z_{3-i}. The second one means that f is one of the edges from the alternating sequence of e with same parity of subscript as k, which means that f is spanned by Z_{3-i} in M_{3-i} (because e is the first in that sequence which is not spanned). These together give that there is a path between the end vertices of e in M_{3-i} which consists of edges of Z_{3-i} and edges which are spanned by Z_{3-i}, so e is spanned by Z_{3-i} in M_{3-i} what is a contradiction.
This proves Claim 17. $\Box$

Our indirect assumption contained that Q is independent, so we get a contradiction. This means that there exists a $k \in \{1, 2\}$ so that every independent set of the union without x and y has a good partition A_1, A_2 so that $A_k \cup x$ is independent in M_k. That way if $k = 1$ then $(M_1/x) \vee (M_2 \setminus x) = (M_1 \vee M_2)/x$ or if $k = 2$ then $(M_1 \setminus x) \vee (M_2/x) = (M_1 \vee M_2)/x$. $\Box$

Lemma 18 *Let x and y be two parallel edges of M_1 and suppose that x is a loop, but y is not a loop in M_2. Let x and y be coloops or serial in the union. Then the union is graphic if and only if $(M_1/x) \vee (M_2 \setminus x)$ is graphic. On the other hand, if they are neither serial nor coloops then the union is not binary.*

Recall that the case if both x and y are loops in M_2 has been covered by Lemma 7.
PROOF: If there exists a circuit C in the union so that $x \in C$ but $y \notin C$ then $(C - x) \cup y$ is also a circuit.

This is because for every $\alpha \in C - x$ we know that $(C - \alpha) \cup y$ is independent, since if something in the union is independent with x, it means that we chose x from M_1 and in M_1 the role of x and y are exactly the same (they are parallel). In a binary matroid there must be a circuit in the symmetric difference of two other ones, but $\{x, y\}$ can not be dependent in the union because x, y is a good partition, so it means

that the union is not binary.
On the other hand if x and y are both coloops in the union then $(M_1 \vee M_2) \setminus x = (M_1/x) \vee (M_2 \setminus x)$.
Finally suppose that x and y are serial in the union, this means that x is a coloop in $(M_1 \vee M_2) \setminus y$. Then every independent set S in the union has a good partition S_1, S_2 so that $S_1 \cup x$ is independent in M_1. That way $(M_1/x) \vee (M_2 \setminus x) = (M_1 \vee M_2)/x$. □

3 Sufficient condition

In a previous paper [1] we proved the following two theorems:

Theorem 19 *If G_1 consists of loops and a single circuit of length n ($n \geq 2$) and $M(G_2)$ is an arbitrary graphic matroid on the same ground set then their union is graphic if and only if for the reduced pair M_1', M_2' either $NL(M_1)$ contains a cut set in G_2' or $M_2' \setminus NL(M_1)$ is the free matroid.*

Theorem 20 *If G_1 consists of loops and two points joined by n ($n \geq 2$) parallel edges and $M(G_2)$ is an arbitrary graphic matroid on the same ground set then their union is graphic if and only if for the reduced pair M_1', M_2' no 2-connected component of G_2' has two non-serial edges a and b from $NL(M_1)$ so that $M_2' \setminus \{a, b\}$ is not the free matroid..*

Now let both matroids be arbitrary. The following theorems in this section will show that these conditions can be formalized together to a sufficient but no longer necessary condition for the graphicity of the union.

Theorem 21 *Let M_1, M_2 be two matroids defined on the same ground set E. Then $M_1 \vee M_2$ is graphic if for every circuit C in M_1 either $r_2(E - C) < r_2(E)$ or $r_2(E - C) = |E - C|$ holds.*

PROOF: We shall apply the following observation:

Proposition 22 *If there exists an edge $\alpha \in E$ so that $E - \{\alpha\}$ is independent in a matroid M then M is graphic.*

PROOF: If E is independent as well then M is the free matroid which is the cycle matroid of a tree. Otherwise E contains a unique circuit C hence M is the cycle matroid of a graph composed of a circuit (formed by the edges of C) and some coloops (corresponding to the edges of $E - C$). □

Lemma 23 *If M_1 has a circuit C such that the set $E - C$ is independent in M_2 then $M_1 \vee M_2$ is graphic.*

PROOF: For every element α of C the set $C - \{\alpha\}$ is independent in M_1 and $E - C$ is independent in M_2. This means $E - \{\alpha\}$ is independent in the union, hence $M_1 \vee M_2$ is graphic by Proposition 22. □

We consider the cases according to the circuits of M_1:

1. If there exists a circuit C of M_1 so that $r_2(E - C) = |E - C|$ then $M_1 \vee M_2$ is graphic by Lemma 23.

2. Let $C_1, C_2, ..., C_k$ be the circuits of M_1. The only remaining case is that $r_2(E - C_i) < r_2(E)$ holds for every i. This means that every base of M_2 intersects every circuit C_i. Let $X \subseteq E$ be a base of M_2 then $E - X$ must be independent in M_1 (since it can not contain a circuit). This means that $X \cup (E - X) = E$ is independent in the union $M_1 \vee M_2$ so the union is the free matroid.

In summary, the union contains at most one circuit. □

$U_{0,2} \vee U_{0,2}$ is the simplest example to show that this condition is not necessary.

If the requirements of Theorem 21 are prescribed for circuits of length at least two only, then a slightly weaker condition will still suffice.

Theorem 24 *Assume that M_2 is graphic. Then $M_1 \vee M_2$ is graphic if for every circuit C of length at least two in M_1 either $r_2(E-C) < r_2(E)$ or $r_2(E-C) = |E-C|$.*

PROOF: We follow the same line of thought as in Theorem 21.

1. If there exists a circuit C of M_1 so that $r_2(E-C) = |E-C|$ then $M_1 \vee M_2$ is graphic by Lemma 23.

2. Suppose now that $r_2(E-C) < |E-C|$ for every circuit C, $|C| > 1$ of M_1 and let γ be a non-coloop element. Let $C_1, C_2, ..., C_k$ be the circuits of M_1 containing γ (we may suppose that $k > 0$). Now $r_2(E-C_i) < r_2(E)$ holds for every i, hence every base of M_2 intersects every circuit C_i. Let $X \subseteq E - \{\gamma\}$ be an independent set in $M_1 \vee M_2$ and let X_1, X_2 be a good partition of X. If $X_1 \cup \{\gamma\}$ is dependent in M_1 it must contain a unique circuit C_1. X_2 is independent in M_2 so it is a subset of a base B. Then $B \cap C_1$ is not empty, let a denote one of its elements. $X_1 \cup \{\gamma\} - \{a\}$ is independent in M_1 since C_1 is the only circuit in $X_1 \cup \{\gamma\}$ and $a \in C_1$. $X_2 \cup \{a\}$ is independent in M_2 (it is a subset of B). This means $(X_1 \cup \{\gamma\} - \{a\}) \cup (X_2 \cup \{a\}) = X \cup \{\gamma\}$ is independent in the union $M_1 \vee M_2$. So every γ edge of this type will be coloop in the union.

We have to study the loops of M_1.
We may suppose that there is no circuit C of M_1 with $r_2(E-C) = |E-C|$ (see Case 1). Observe that the edges which are only contained by circuits as in the second case can not ruin the graphicity of the union, since they will be coloops. This way we can simply delete all edges like that from both matroids and the union will be graphic if and only if the union of the original matroids is graphic. The initial condition changes to the requirement that there can be only loops in M_1'. This means that the union $M_1' \vee M_2' = M_2'$ namely $M_1 \vee M_2 = M_2' \oplus \{\text{coloops}\}$. $\square$

Now $(U_{1,2} \oplus U_{0,1}) \vee U_{0,3}$ is the simplest example to show that this condition is still not necessary.

In order to obtain further, gradually weaker conditions which will still suffice, first we may form a symmetric version of Theorem 24, that is, the union is graphic if the circuits of one of the matroids satisfy the rank requirements in the other matroid. However, $(U_{1,2} \oplus U_{0,2}) \vee (U_{0,2} \oplus U_{1,2})$ is the simplest example to show that this condition is still not necessary (the loops of the first matroid are the parallel edges in the second matroid).

Next it is enough to require this property to a reduced pair of matroids only. However $(U_{1,3} \oplus U_{0,3}) \vee (U_{0,2} \oplus U_{0,2} \oplus U_{0,2})$, where every component of the second matroid has exactly one loop from the first matroid shows that even this condition is not necessary.
It is easy to see that Lemma 14 eliminates this case because there exist serial edges in M_2 so that one is a loop in M_1. The following example shows that even with all these extensions, and with the help of Lemmata 12, 13 and 14 the property is not necessary for the graphicity of the union.

Example 25 *Let M_1 be the matroid which is the direct sum of three paralell edges a, b, c and $M(K_4)$ where $1, 2, 3$ are three edges incident to a common vertex with other endpoints P, Q, R, respectively, $f_1 = (P, Q)$, $f_2 = (Q, R)$ and $f_3 = (R, P)$.*
Let M_2 be the matroid which is the direct sum of a three long circuit $1, 2, 3$, two parallel edges f_1, f_2, three parallel edges a, b, f_3 and a loop c.
The union will have a circuit a, b, c and coloops hence graphic. However a, b is a length two circuit in M_1 so that $M_2 \setminus \{a, b\}$ contains a spanning tree and a circuit too. Nevertheless M_1, M_2 is a reduced pair. This means this is a counterexample for the necessity of the property.

In fact in Theorem 20 where one of the matroids consists of parallel edges and loops, we stated this property in a slightly different way, there the circuit C for which $M_2 \setminus C$ consists spanning tree and circuit too were in one component of M_2. Observe that a and b are in the same component of M_2 in Example 25 hence that remains a counterexample for the necessity even if we add this condition.

However if we use Lemmata 16 and 18 then this example can be reduced too. In fact either of the two will do, because a and b are parallel in both matroids (Lemma 16), on the other hand c is a loop in M_2 and c is parallel to a in M_1 (Lemma 18). Recall that Lemmata 16 and 18 are not about equivalent reduction (just in the case where the union is binary), so we can no longer speak about necessity of the extended version of the conditions.

4 Necessary condition

In this section we show a necessary condition for the binarity of the union of two graphic matroids. This condition is formalized in a similar way to the sufficient condition in the previous section. Unfortunately they are not exactly the same so there remains a gap which consists of those cases where there might exist a counterexample for Conjecture 2 (a pair of graphic matroids which have a nongraphic but binary union). This is the main motivation of the lemmata in Section 2. They imply that a possible minimal counterexample must have some special properties.

Theorem 26 *Let M_1 and M_2 be graphic matroids. If all of the following conditions hold then the union $M_1 \vee M_2$ is not binary.*

1. *$\exists X_i$ dependent sets in M_i for both $i = 1$ and 2*
2. *$X_1 \cap X_2 = \emptyset$*
3. *$\exists$ a circuit C_i of M_i in X_i so that $|C_i| \geq 2$*
4. *$r_i(X_i) = r_i(X_1 \cup X_2)$*
5. *There are two distinct elements $a, b \in C_1 \cup C_2$ such that for $i \in \{1, 2\}$:*
   - *if $a \in C_i$ and $b \in C_{3-i}$ then a and b are in the same component in both matroids*
   - *if $a, b \in C_i$ then there exists $X'_{3-i} \subset X_{3-i}$ so that if we contract X'_{3-i} in M_{3-i} then a and b are distinct diagonals of C_{3-i}*

Condition 1 is obviously implied by Condition 3, it is mentioned separately for the simplification of the discussion below. Observe that the two minimal examples of graphic pairs which have nonbinary union motivate the last condition.

Proof: There are two cases. At first we study if $a, b \in C_1$, and then if $a \in C_1$ and $b \in C_2$. These two cases cover all the possibilities by the symmetries of the conditions.
Suppose that $a, b \in C_1$. We can extend X'_2 to X''_2 from C_2 so that if we contract X''_2 in M_2 then there remains only three edges α, β, γ from C_2 in M'_2 and a is parallel to α, b is parallel to β. Also we can contract a proper subset $P \subset X_1$ in M_1 so that a and b will be parallel in M'_1 and $r'_1(X_1 - P) = 1$.
Now examine the union contracted to $X''_2 \cup P \cup a$. We show that the elements b, α, β and γ form a $U_{2,4}$. Consider the rank of the set $X_1 \cup X_2$ in the union: $r_{union}(X_1 \cup X_2) \leq r_1(X_1 \cup X_2) + r_2(X_1 \cup X_2) = r_1(X_1) + r_2(X_2)$. We know that $r_2(X''_2) = r_2(X_2) - 2$ and $r_1(P \cup a) = r_1(X_1)$, this shows that $r'_{union}(\{b, \alpha, \beta, \gamma\}) \leq 2$. For $x_1 \in \{a, b\}$ and $A \in \{\{a, \beta\}, \{\alpha, \beta\}, \{a, \gamma\}, \{\alpha, \gamma\}, \{b, \alpha\}, \{\beta, \gamma\}\}$, $(x_1 \cup A) - a$ will be independent if $(x_1 \cup A)$ contains a, as the partition $(P \cup x_1) \cup (X''_2 \cup A)$ shows, where the first subset is independent in M_1 and the second is independent in M_2. Note that x_1 and A can be chosen so that $(x_1 \cup A) - a$ is equal to any 2-subset of $\{b, \alpha, \beta, \gamma\}$. Hence we have a $U_{2,4}$ minor in the union, thus it is not binary.

For the other case suppose that $a \in C_1$, $b \in C_2$. According to the fifth condition of the theorem there exist $X'_1 \subset X_1$ and $X'_2 \subset X_2$ so that if we contract X'_1 in M_1 then C_1 is contracted to a and an other element c, and b is parallel to them and if we contract X'_2 in M_2 then C_2 is contracted to b and an

other element d, and a is parallel to them. According to the second condition of the theorem c and d are different elements. Choose X_1' and X_2' to be maximal. Thus $r(M_1/X_1') = r(M_2/X_2') = 1$.
Now examine the union contracted to $X_1' \cup X_2'$. We show that the elements a, b, c and d form a $U_{2,4}$. Again $r_{union}(X_1 \cup X_2) \leq r_1(X_1 \cup X_2) + r_2(X_1 \cup X_2) = r_1(X_1) + r_2(X_2)$. Now $r_1(X_1') = r_1(X_1) - 1$ and $r_2(X_2') = r_2(X_2) - 1$ so $r'_{union}(\{a, b, c, d\}) \leq 2$. For $x_1 \in \{a, b, c\}$ and $x_2 \in \{a, b, d\}$, $x_1 \neq x_2$, $\{x_1, x_2\}$ will be independent, as the partition $(X_1' \cup x_1) \cup (X_2' \cup x_2)$ shows, where the first subset is independent in M_1 and the second is independent in M_2. Note that x_1 and x_2 can be chosen so that $\{x_1, x_2\}$ is equal to any 2-subset of $\{a, b, c, d\}$. Hence we have a $U_{2,4}$ minor in the union, thus it is not binary. □

While it is not quite apparent at first, the conditions of Theorem 26 are similar to those of the symmetric version of Theorem 24. If the sufficient condition is not met then circuits C_1, C_2 of length at least two must exist in the respective matroids, satisfying $r_{3-i}(E - C_i) = r_{3-i}(E)$ and $r_{3-i}(E - C_i) < r_{3-i}(E - C_i)$ for $i = 1, 2$. Only the fifth condition seems to be different but it just denies the degenerate cases (which did not come up in the earlier cases either).
As we already mentioned the minimal counterexample must be unreducible if exists.

5 New questions

In order to put Conjecture 2 into a more general framework, we formally define eight matroid classes as follows.
Let A be the set of those graphic matroids which give a graphic or non-binary union with any graphic matroid.
Let B be the set of those graphic matroids which give a graphic union with any graphic matroid.
Let C be the set of those graphic matroids which give a graphic or non-binary union with any matroid.
Let D be the set of those graphic matroids which give a graphic union with any matroid.
Let E be the set of those matroids which give a graphic or non-binary union with any graphic matroid.
Let F be the set of those matroids which give a graphic union with any graphic matroid.
Let G be the set of those matroids which give a graphic or non-binary union with any matroid.
Let H be the set of those matroids which give a graphic union with any matroid.
Observe that Conjecture 2 states that A is the set of all graphic matroids.

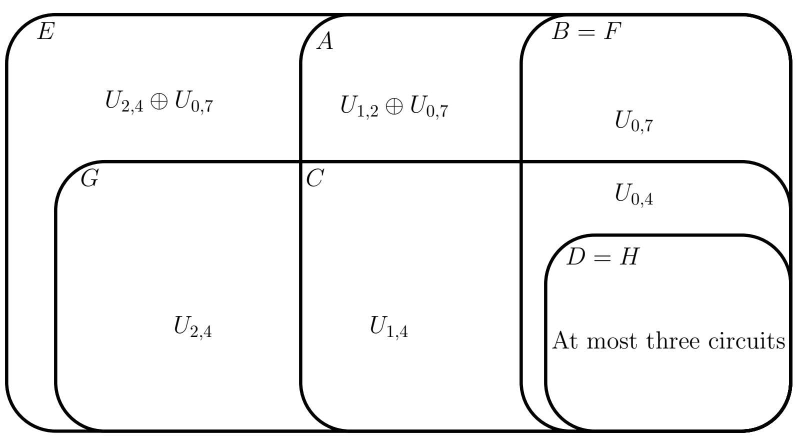

Figure 3: Examples for all nonempty subsets

Most of the relationships between the sets are trivial ($D \subseteq C \subseteq A$, $D \subseteq B \subseteq A$, $H \subseteq G \subseteq E$, $H \subseteq F \subseteq E$, $A \subseteq E$, $B \subseteq F$, $C \subseteq G$, $D \subseteq H$) see Figure 3. For $D = H$ recall that the union of M and $U_{0,k}$ is M so if the union is graphic then M is also graphic. Since $U_{0,k}$ is graphic $F = B$ follows similarly. $(A \cap G) - C$ is empty because if a matroid is in G but not in C then it is not graphic.

Definition 27 *A graph consisting of three internally disjoint paths (each of length at least one) between two points is called a θ graph.*

Theorem 28 *A matroid is in D if and only if it contains at most three circuits.*

PROOF: The condition that a matroid contains at most three circuits holds if and only if it contains only a θ graph in addition to coloops or it is the direct sum of at most three circuits and some coloops. Let M denote a matroid in D. Let M_2 denote the other matroid in the union. According to the reduction we can suppose that neither M nor M_2 contains any coloop.
For the if part of the proof we consider two cases.
Case 1: Let M be the cycle matroid of a θ graph. Let P, Q, R denote the three paths of the θ graph and $\alpha_i, \beta_i, \gamma_i$ be their elements, respectively.

- If $r(M_2) = 0$ then the union is isomorphic to M.
- If $r(M_2) = 1$ then we have the case as in Theorem 20, let $[n]$ denote the parallel edges in M_2. This means that the union is graphic if no two edges $a, b \in [n]$ exist such that $M \setminus \{a, b\}$ contains both a spanning tree and a circuit. If we pick two edges from the same path in the θ graph then no spanning tree remains, while if we pick two from different paths then no circuit remains. Thus the union is graphic.
- If $r(M_2) \geq 2$ and there is a base B in M_2 so that it contains at least one edge from two different paths of the θ graph then the union is the free matroid since we can pick B from M_2 and $E - B$ from M_1.
- Finally let $r(M_2) \geq 2$ and suppose that there is no base in M_2 containing at least one edge from two different paths of the θ graph. We claim that in this case all the elements in two paths of the θ graph of M are loops in M_2. This will suffice since then union $M \vee M_2$ is graphic, namely a circuit of these two paths of the θ graph with coloops for the third path. Indirectly suppose that $\alpha_1 \in P$ and $\beta_1 \in Q$ are non-loop elements of M_2. Let B_1 be a base of M_2 containing α_1 and, by the assumption, all of its further elements are in P. Since both B_1 and $\{\beta_1\}$ are independent and $|B_1| > 1$, there must exist an element $\alpha \in B_1$ so that $\{\alpha, \beta_1\}$ is also independent, hence it can be extended to a base of M_2, a contradiction.

Case 2: Let M be the direct sum of at most three circuits. Suppose that there are exactly three circuits in M. Let C_1, C_2, C_3 denote the three circuits of M and a_i, b_i, c_i be their elements, respectively.
Fortunately the cases where $r(M_2) \leq 1$ are the same as before. The only difference is that we have to pay attention to the fact that the two edges a, b which are parallel in M_2 so that $M \setminus \{a, b\}$ contains a spanning tree and a circuit, must be in the same component of M.
Again, we examine the cases according to the bases ($r(M_2) \geq 2$).

- If there is a base B in M_2 containing at least one edge from every circuit of M then B is independent in M_2 and $E - B$ is independent in M so the union is the free matroid.
- If there is no base in M_2 containing at least one edge from two distinct circuits of M then, just like in the last subcase of Case 1, only one of the three circuits of M has non-loop edges in M_2 so the union $M \vee M_2$ is graphic (two circuits remain the same as in M, while the elements of the third become coloops).
- If there is no base in M_2 containing at least one edge from every circuit of M, but there exist bases B_1, B_2, B_3 so that $B_1 \cap C_1 \neq \emptyset; B_1 \cap C_2 \neq \emptyset; B_2 \cap C_2 \neq \emptyset; B_2 \cap C_3 \neq \emptyset; B_3 \cap C_1 \neq \emptyset; B_3 \cap C_3 \neq \emptyset$, then the union $M \vee M_2$ will be graphic, namely a single circuit.
- If there is no base in M_2 containing at least one edge from every circuit of M, there exist bases B_1, B_2 so that $B_1 \cap C_i \neq \emptyset; B_1 \cap C_j \neq \emptyset; B_2 \cap C_j \neq \emptyset; B_2 \cap C_k \neq \emptyset$ ($\{i, j, k\} = \{1, 2, 3\}$) but does not exist a B_3 such that $B_3 \cap C_i \neq \emptyset; B_3 \cap C_k \neq \emptyset$, then the union $M \vee M_2$ will be a circuit of the edges of C_i and C_k and coloops for the edges of C_j, hence graphic.

- If there is no base in M_2 containing at least one edge from every circuit of M, but there exists a base B_1 so that $B_1 \cap C_i \neq \emptyset; B_1 \cap C_j \neq \emptyset$ and does not exists a base B_2 such that $B_2 \cap (C_i \cup C_j) \neq \emptyset; B_2 \cap C_k \neq \emptyset$ ($\{i, j, k\} = \{1, 2, 3\}$), then all the edges of C_k are loops in M_2. This means that the union $M \vee M_2$ will be a circuit of the edges of C_k and coloops for the edges of C_i and C_j, hence graphic.

On the other hand we shall show that if M contains more than three circuits, then its union with an appropriately chosen matroid will contain a $U_{2,4}$ minor.

Lemma 29 *If a graphic matroid contains at least four circuits then it contains at least one of the following three minors: $U_{1,4}$, $U_{0,4}$, $U_{1,3} \oplus U_{0,1}$.*

Proof: We have already seen that a matroid M containing at least three circuits either contains three pairwise disjoint circuits or a θ-graph. In the former case the extension of three disjoint circuits with a fourth one either leads to a minor $U_{0,4}$ (if the fourth circuit is disjoint to the previous ones) or to $U_{1,3} \oplus U_{0,1}$ (if the fourth circuit intersects at least one of the old ones).
On the other hand, if M contains a θ-graph then the fourth circuit may be disjoint to it, leading to $U_{1,3} \oplus U_{0,1}$ or contributes to the θ-graph and we obtain a $U_{1,4}$ as a minor. □

For all the three cases we construct M_2 so that $M \vee M_2$ contains $U_{2,4}$, hence not graphic. For the set X of those edges which are not in the minor (M_{minor}) we can simply make loops in M_2, leading to $(M \vee M_2)/X = M_{minor} \vee (M_2 \setminus X)$.
Case 1: If M has a $U_{1,4}$ minor then let $M_2 \setminus X = U_{1,4}$. Then $(M \vee M_2)/X = U_{1,4} \vee U_{1,4} = U_{2,4}$, hence not graphic.
Case 2: If M has a $U_{0,4}$ minor then let $M_2 \setminus X = U_{2,4}$. Then $(M \vee M_2)/X = U_{0,4} \vee U_{2,4} = U_{2,4}$, hence not graphic.
Case 3: If M has a $U_{1,3} \oplus U_{0,1}$ minor then let $M_2 \setminus X = U_{1,4}$. Then $(M \vee M_2)/X = (U_{1,3} \oplus U_{0,1}) \vee U_{1,4} = U_{2,4}$, hence not graphic.
□

All the containments as indicated in Figure 3 are proper, as shown by the examples. The position of these examples are straightforward for all but one case, see Theorem 30 below.

Theorem 30 *The set $E - (G \cup A)$ is not empty, it contains the matroid $K = U_{2,4} \oplus U_{0,7}$.*

Proof: K is not graphic because it has a $U_{2,4}$ minor, hence it is not in A. $(U_{4,4} \oplus F_7) \vee K$ is not graphic but binary, hence K is not in G. To show that K is in E let M be an arbitrary graphic matroid. We have to prove that $M \vee K$ is either graphic or not binary. Just like in Corollary 9, we perform the possible reduction steps to obtain a reduced pair M', K' on the common underlying set E'.

Proposition 31 *If $NL(K')$ does not contain a cut set in M' then the union $M' \vee K'$ is not binary.*

Proof: If $NL(K')$ does not contain a cut set then $L(K')$ contains a base of M', so $(M' \vee K')/L(K') = U_{2,4}$
□

Proposition 32 *If $L(K')$ is independent in M' but $NL(K')$ contains a cut set X in M' then the union $M \vee K$ is graphic.*

Proof: Since X is a cut set there exists $x \in X$ so that $L(K') \cup \{x\}$ is independent in M'. Every two element subset of $NL(K')$ is independent in K' so there exist two distinct elements y, z in $NL(K') - \{x\}$

such that $L(K') \cup \{x, y, z\}$ is independent in the union. Now the statement follows from Proposition 22. □

Proposition 33 *If* $r_{M'}(E') - r_{M'}(E' - NL(K')) \geq 2$ *then* $M \vee K$ *is graphic.*

Proof: The above condition means that every base of M' contains at least two elements of $NL(K')$. In that case $M' \vee K' = (M' \setminus NL(K')) \oplus U_{4,4}$ and that is graphic, since M' is graphic. This means that the union $M \vee K$ is also graphic . □

Proposition 34 *If* $r_{M'}(E') - r_{M'}(E' - NL(K')) = 1$ *and* $L(K')$ *is not independent in* M' *then the union* $M \vee K$ *is not binary.*

Proof: Since M', K' is a reduced pair of matroids, a circuit C of M' in $L(K')$ must be spanned by the edge set $NL(K')$ and the length of C is at least three. Let X denote the cut set of M' in $NL(K')$ and x denote an element of it. Let $1, 2, 3$ be three different elements of C. Let Z denote the set $C \cup x - \{1, 2, 3\}$ extended by all the elements z of $L(K') - \{1, 2, 3\}$ for which it is true that $r_{M'}(Z \cup z) > r_{M'}(Z \cup \{1, 2, 3\})$ (in this way Z is independent in M'). Consider the matroid M'/Z, it is a graphic matroid which can be represented by a graph of three vertices, where $\{1, 2, 3\}$ forms a circuit, and there are parallel edges to at least two of 1,2 and 3. Suppose that a is parallel to 1, b is parallel to 2, and c is the third element of $NL(K')$ which remained. With this notation we show that $M' \vee K'/(\{b, c, x\} \cup Z) = U_{2,4}$. The rank of this matroid is trivially not larger than two. We have to show that every pair of edges is independent. $\{a, 1\}$ will be independent, as the partition $(Z \cup \{b, 1\}) \cup \{a, c\}$ shows, where the first subset is independent in M' and the second subset is independent in K'. For any other 2-subset P of $\{a, 1, 2, 3\}$, P will be independent, as the partition $(Z \cup P) \cup \{b, c\}$ shows, where the first subset is independent in M' and the second is independent in K'. We found a $U_{2,4}$ minor in a minor of $M \vee K$, hence it is not binary. □

The above propositions cover all the possible cases for M' and in every case the union is graphic or not binary, so K is in E. □

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