On the sum of graphic matroids

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Abstract

There is a conjecture that if the sum of graphic matroids is not graphic then it is nonbinary.

Some special cases have been proved only, for example if several copies of the same graphic matroid are given.

If there are two matroids and the first one can be drawn as a graph with two points, then a necessary and sufficient condition is given for the other matroid to ensure the graphicity of the sum.

Hence the conjecture holds for this special case.

1 Introduction

Graphic matroids form one of the most significant classes in matroid theory. When introducing matroids, Whitney concentrated on relations to graphs. The definition of some basic operations like deletion, contraction and direct sum were straightforward generalizations of the respective concepts in graph theory. Most matroid classes, for example those of binary, regular or graphic matroids, are closed with respect to these operations. This is not the case for the sum. The sum of two graphic matroids can be nongraphic.

The purpose of our work is to study the graphicity of the sum of graphic matroids. The first paper in this area was that of Lovász and Recski: they examined the case if several copies of the same graphic matroid are given [1]. Then Recski conjectured thirty years ago that if the sum of graphic matroids is not graphic then it is nonbinary [5]. He also studied the case if we fix one simple graphic matroid and take its sum with every possible graphic matroid. His main result is the following theorem.

Theorem 1 *[2] Let $A = M(E, I_1)$ where $E = \{1, 2, 3...n\}$, $I_1 = \{\emptyset, \{1\}, \{2\}\}$, let $M = M(E, I_2)$ an arbitrary graphic matroid, and $B = M(E_3, I_3)$ where $E_3 = \{1, 2, i, j, k\}$*

($3 \le i, j, k \le n$ differing) and $I_3 = \{\{1\}, \{2\}, \{i\}, \{j\}, \{k\}, \{1, 2\}, \{1, j\}, \{1, k\}, \{2, i\}, \{2, k\}, \{i, j\}, \{i, k\}, \{j, k\}\}$, see Figure 1.
Then the sum $A \vee M$ is graphic if and only if B is not a minor of M with any triplet i, j, k.

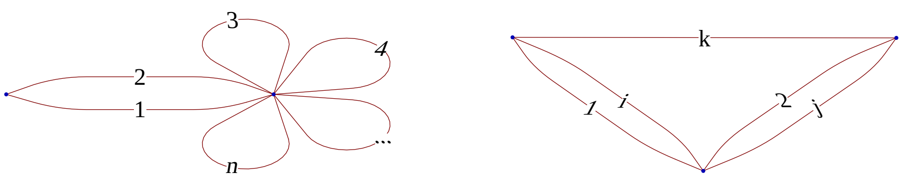

Figure 1: A graphic representation of A (left) and B (right)

We shall use Tutte's theorem which is fundamental in matroid theory:

Theorem 2
- *A matroid is binary if and only if it does not contain $U_{4,2}$ as a minor.*
- *A matroid is regular if and only if it does not contain $U_{4,2}$, F_7 and F_7^* as minors.*
- *A matroid is graphic if and only if it does not contain $U_{4,2}$, F_7, F_7^*, $M^*(K_5)$ and $M^*(K_{3,3})$ as minors.*
- *A matroid is the circuit matroid of a plane graph if and only if it does not contain $U_{4,2}$, F_7, F_7^*, $M^*(K_5)$, $M^*(K_{3,3})$, $M(K_5)$ and $M(K_{3,3})$ as minors.*

Forbidden minors will be of importance in our forthcoming results as well, although they will not appear in our final statement.

2 Results

In most of the cases we speak about graphic matroids so I will call the elements of the matroids edges.
Observe that all but two of the edges of the graph representing the matroid A of Theorem 1 are loops (see Figure 1 as well). Also observe that bridges in a matroid remain bridges of its sum with any other matroid. Hence, in order to generalize Theorem 1 I started to analyze the case when we have only three edges which are neither bridges nor loops. There are two types of matroids with this property, the one with a circuit of length three, and the other with three parallel edges. I found in both cases that there are some forbidden minors so that if any of them appears in the other matroid then the sum is not graphic,

while if the matroid doesn't contain any of them then the sum is graphic.

After these results [7] I started to work with the cases with n parallel edges or with circuits of length n (of course there may be many loops and bridges). After some disappointing results (that the n long circuit's cases surely can't lead to the same type of conditions what I wanted to prove in the other case) the case with n parallel edges lead to a very useful result.

For a transparent presentation of the theorems what will follow, we have to put up some definitions and prove some lemmata, which help us to reduce the infinite number of cases.

We study the sum of two graphic matroids M_1 and M_2. Throughout we shall refer to M_1 and M_2 as addends.

It is well known that if a matroid is graphic then so are all of its submatroids and minors. Hence if a matroid has a non graphic minor then the matroid can't be graphic.

Definition 3 *We call some edges of the matroid serial if they belong to exactly the same circuits.*

Definition 4 *We call an edge of M_1 essential if it is not a loop in M_2 and we call it irrelevant otherwise.*

Definition 5 *We call a submatroid of an addend devoid if it contains irrelevant edges only.*

The following lemmata contain the main opportunities when we want to simplify our addend matroids. It is important that these are valid for graphic matroids only, so I can use graph theoretical concepts.

Lemma 6 *Let X and Y denote the set of bridges in M_1 and in M_2 respectively.*
The sum $M_1 \vee M_2$ is graphic if and only if $M_1 \backslash (X \cup Y) \vee M_2 \backslash (X \cup Y)$ is graphic.

Lemma 7 *If a devoid submatroid X of M_1 is a connected component of it then the sum is graphic if and only if $(M_1 \backslash X) \vee (M_2 \backslash X)$ is graphic.*

Lemma 8 *M_1 is the circuit matroid of a graph $G(V, E)$ in which X is a connected set of edges and $E \backslash X$ has exactly two common vertices with X (call them a and b).*
Let M_1' be the circuit matroid of $G' := G(V, E \cup (a, b) \backslash X)$ and $M_2' := M_2 \backslash X \cup loop(a, b)$
(Here $loop(a, b)$ denotes a loop corresponding to the edge (a, b) in G').
If X is devoid then the sum $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.

Lemma 9 *M_1 is the circuit matroid of a graph $G(V, E)$ where X is a connected component of G. If X has only one essential edge x then the sum $M_1 \vee M_2$ is graphic if and only if $(M_1 \backslash X \cup loop(x)) \vee (M_2 \backslash (X \backslash x))$ is graphic.*

The next lemma will be less general, it is only for the cases with parallel edges and loops.

Lemma 10 *M_1 consist of n parallel edges and k loops. If two essential edges x and y in M_2 are serial then the sum $M_1 \vee M_2$ is graphic if and only if $(M_1 \backslash x) \vee M_2'$ is graphic where M_2' is defined as follows:*
Simply replace the edges x and y in M_2 by a single edge xy so that a set S containing xy is independent if and only if $S \backslash xy \cup \{x, y\}$ was independent. Then xy will play the role of y in $M_1 \backslash x$.

After these preliminaries we can define the reduction, what will be the most important concept to reduce the infinite number of cases.

Definition 11 *We want to know if the sum $M_1 \vee M_2$ is graphic. We call M_2 reduced if none of the lemmata above can help us to decrease the number of edges. (Recall that Lemma 8 can be applied for a special case only while the other four types of simplifications can be applied in any case.)*

Corollary 12 *M_1 and M_2 are graphic matroids. Application of the previous lemmata to M_2 leads to a reduced matroid M_2' while M_1 changes to M_1' what we shall call the pair of M_2'. Then $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.*

We are only one step away from our theorem, but to write it in a pretty way we have to define the following:

Definition 13 *Suppose that a matroid has at least three circuits C_1, C_2, C_3 so that each circuit C_i has at least one element a_i satisfying $a_i \notin \bigcup_{j \neq i} C_j$. (These edges will be called proper edges.) Such a matroid fulfils the three circuits property if at least one of the edges belongs to at least two circuits.*

Theorem 14 *Let M_1 be a matroid which consists of n parallel edges and k loops and let M_2' be the matroid reduced from M_2. Then $M_1 \vee M_2$ is non graphic if and only if M_2' fulfils the three-circuit property.*

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References

[1] L. Lovász and A. Recski, On the sum of matroids, *Acta Math. Acad. Sci. Hungar.,* **24**, *1973, 329-333.*

[2] A. Recski, On the sum of matroids II, *Proc.* 5*th British Combinatorial Conf. Aberdeen, 1975, 515-520.*

[3] A. Recski, Matroids – the Engineers' Revenge, in W J Cook, L Lovasz, J Vygen (eds.) *Research trends in combinatorial optimization,* Springer, 2008, 387-398.

[4] A. Recski, *Matroid Theory and its Applications in Electric Network Theory and in Statics,* Springer, Berlin, 1989.

[5] A. Recski, Some open problems of matroid theory, suggested by its applications, *Coll. Math. Soc. J. Bolyai* **40**, 1982, 311-325.

[6] W.T. Tutte, *Introduction to the Theory of Matroids,* American Elsevier, New York, 1971.

[7] Cs.Gy. Csehi, *Matroidelméleti vizsgálatok és alkalmazásaik,* Master's Thesis, Budapest University of Technology and Economics, 2012.