ON THE SPECTRUM OF A RANDOM GRAPH

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The spectrum of a graph is the spectrum of its adjacency matrix.

In the present paper we deal with the asymptotic behaviour of the spectrum of a random graph. We will show that the first eigenvalue is of order \( n \), and then there is a sudden drop: the second largest eigenvalue is \( O\left(n^{\frac{1}{2}+\epsilon}\right) \).

**Proposition 1.** Let \( A \) be an \( n \times n \) symmetric \((0,1)\) matrix, in which the density of ones is \( d \) (i.e. the number of ones is \( n^2d \)). Then the largest eigenvalue \( \lambda_1 \) of \( A \) satisfies the inequality

\[
n d \leq \lambda_1 \leq n\sqrt{d}.
\]

**Proof.** Let \( e = (1, \ldots, 1) \) be the \( n \)-dimensional all-one vector. Then

\[
\lambda_1 \geq \frac{(e,Ae)}{(e,e)} = \frac{n^2d}{n}.
\]

If \( \lambda_i \) are the eigenvalues of the matrix, then

\[
\sum_{i=1}^{n} \lambda_i^2 = n^2d.
\]

Hence \( \lambda_1 \leq n\sqrt{d} \).

Q.E.D.
Proposition 2. Let \( A_n = (a_{ij}) \) be an \( n \times n \) symmetric matrix where \( a_{ii} = 0, \ a_{ij} \) for \( i > j \) are independent random variables. Suppose that \( P(a_{ij} = 1) = p, \ P(a_{ij} = 0) = q = 1 - p. \) If \( \lambda_1 = \lambda_1(n) \) is the largest eigenvalue of the matrix \( A_n, \) then \( \lim_{n \to \infty} \frac{\lambda_1}{n} = p \) is probability.

Proof. We know that

\[
\min_{1 < i < n} \sum_{j=1}^{n} a_{ij} < \lambda_1 < \max_{1 < i < n} \sum_{j=1}^{n} a_{ij}.
\]

(Perron – Frobenius theorem). Since for a given value of \( i \) the probability \( P \left( \left| \frac{1}{n} \sum_{j=1}^{n} a_{ij} - p \right| > \delta \right) \) is exponentially small,

\[
\lim_{n \to \infty} P \left( \max_{1 < i < n} \left| \frac{1}{n} \sum_{j=1}^{n} a_{ij} - p \right| > \delta \right) = 0
\]

Q.E.D.

Now we are interested in the behaviour of the other eigenvalues. The Wigner theorem – generalized by Ludwig Arnold – describes the overall behaviour of the eigenvalues.

Theorem (Wigner). Let \( A_n = (a_{ij}) \) be an \( n \times n \) symmetric matrix, where \( a_{ij} \) for \( i \geq j \) are independent random variables. Suppose that \( a_{ii} \) are identically distributed with the distribution function \( G \) and \( a_{ij} \) for \( i > j \) are identically distributed with the distribution function \( H. \) Assume that \( \int x^2 \, dH(x) = \sigma^2 < \infty. \) Denote the empirical distribution function of the eigenvalues of \( \frac{1}{\sqrt{n}} A_n \) by \( F_n(x) = \frac{1}{n} \sum_{\lambda_i < x \sqrt{n}} 1 \) where \( \lambda_i \) are the eigenvalues of \( A_n. \) Then, for arbitrary \( x \) we have

\[
\lim_{n \to \infty} F_n(x) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} f(x) \, dx \quad \text{in probability}
\]

where

\[
f(x) = \begin{cases} \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} & |x| < 2\sigma \\ 0 & \text{otherwise.} \end{cases}
\]
For the proof and generalizations see [2], [3] and [4].

This theorem describes only the behaviour of the majority of the eigenvalues, and allows many of them (in fact $o(n)$) to be as large as $n$. We show now that this is not the case and only the first eigenvalue of a graph is of order $n$, the rest is near $\sqrt{n}$. (Actually this happens with any symmetric random matrix if the expectation of the entries is non-zero.)

**Proposition 3.** Let $A_n$ be as in Proposition 2. Then for the second largest eigenvalue $\lambda_2 = \lambda_2(n)$ of $A_n$ we have $\lambda_2 = O\left(n^{1/2 + \epsilon}\right)$ in probability.

**Proof.** Let us consider the matrix $B_n = P_n A_n P_n$ where $P_n$ is the orthogonal projection parallel to the all-one vector $e_n = (1, \ldots, 1)$. By the Courant - Fischer theorem $\lambda_2 \leq \|B_n\|$ where $\|\cdot\|$ denotes the maximal eigenvalue of the matrix. It is easy to see that $B_n = (b_{ij})$, $b_{ij} = a_{ij} - d_i - d_j + d$ where $d_i$ and $d$ are the density of ones in the $i$-th row and in the whole matrix, respectively. We can write $B_n = C_n + R_n$ where $C_n = (c_{ij})$, $c_{ij} = a_{ij} - p$ and $R_n = (r_{ij})$, $r_{ij} = d_i + d_j - d - p$. So $\|B_n\| \leq \|C_n\| + \|R_n\| \leq \|C_n\| + \|R_n\|_2$, where $\|R_n\|_2^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij}^2$. We have

$$D^2(d_i + d_j - d) = \frac{1}{n^2} D^2\left(\sum_{k=1}^{n} \sum_{i=k+1}^{n} h_{ki}a_{kl}\right) \leq \frac{3pq}{n}$$

where $D^2(\cdot)$ stands for variance. Hence the expectation $E \|R_n\|_2^2 \leq n^2 \frac{3pq}{n}$. The Markov inequality implies that the probability that the maximal eigenvalue of the matrix $R_n$ is larger then $K\sqrt{n}$ is $O\left(\frac{1}{K}\right)$. For the estimation of $\|C_n\|$ we use the following statement.

**Proposition 4.** Let $A_n$ be as in Wigner's theorem, and assume that the expectation of the entries is zero: $\int x dH(x) = 0$. If also all moments of $H$ are finite, then for any $\epsilon > 0$ we have for the maximal eigenvalue $\lambda = \lambda(n)$ of $A_n$ the relation $\lim P\left(|\lambda| > n^{1/2 + \epsilon}\right) = 0$. 

- 315 -
Proof. It can be seen from [5] and [1] that \( E\lambda^2 \leq E \sum_{i=1}^{n} \lambda_i^2 = O(n^{l+1}) \). Whence

\[
P\{|\lambda| > n^{\frac{1}{l^2 + e}}\} < n^{-(l+2le)}E\lambda^2 = O(n^{-(2le-1)}) = o(1)
\]

if \( l \) was chosen large enough. Q.E.D.

We remark that if we do not assume the existence of higher moments in Proposition 4, then the conclusion \( \lambda = o(n) \) in probability still holds.

REFERENCES


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