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ON THE SPECTRUM OF A RANDOM GRAPH

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The spectrum of a graph is the spectrum of its adjacency matrix.

In the present paper we deal with the asymptotic behaviour of the spectrum of a random graph. We will show that the first eigenvalue is of order n, and then there is a sudden drop: the second largest eigenvalue is $O\left(n^{\frac{1}{2}+\epsilon}\right)$.

Proposition 1. *Let* A *be an* $n \times n$ *symmetric* $(0, 1)$ *matrix, in which the density of ones is* d *(i.e. the number of ones is* $n^2 d$*). Then the largest eigenvalue* λ_1 *of* A *satisfies the inequality*

$$nd \leqslant \lambda_1 \leqslant n\sqrt{d}.$$

Proof. Let $e = (1, \ldots, 1)$ be the n-dimensional all-one vector. Then $\lambda_1 \geqslant \frac{(e, Ae)}{(e, e)} = \frac{n^2 d}{n}$. If λ_i are the eigenvalues of the matrix, then $\sum_{i=1}^{n} \lambda_i^2 = n^2 d$. Hence $\lambda_1 \leqslant n\sqrt{d}$.

Q.E.D.

Proposition 2. *Let* $A_n = (a_{ij})$ *be an* $n \times n$ *symmetric matrix where* $a_{ii} \equiv 0$, a_{ij} *for* $i > j$ *are independent random variables. Suppose that* $P(a_{ij} = 1) = p$, $P(a_{ij} = 0) = q = 1 - p$. *If* $\lambda_1 = \lambda_1(n)$ *is the largest eigenvalue of the matrix* A_n, *then* $\lim_{n \to \infty} \frac{\lambda_1}{n} = p$ *is probability.*

Proof. We know that

$$\min_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} a_{ij} \leqslant \lambda_1 \leqslant \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} a_{ij}.$$

(Perron – Frobenius theorem). Since for a given value of i the probability $P\left(\left|\frac{1}{n} \sum_{j=1}^{n} a_{ij} - p\right| > \delta\right)$ is exponentially small,

$$\lim_{n \to \infty} P\left(\max_{1 \leqslant i \leqslant n} \left|\frac{1}{n} \sum_{j=1}^{n} a_{ij} - p\right| > \delta\right) = 0$$

Q.E.D.

Now we are interested in the behaviour of the other eigenvalues. The Wigner theorem – generalized by Ludwig Arnold – describes the overall behaviour of the eigenvalues.

Theorem (Wigner). *Let* $A_n = (a_{ij})$ *be an* $n \times n$ *symmetric matrix, where* a_{ij} *for* $i \geqslant j$ *are independent random variables. Suppose that* a_{ii} *are identically distributed with the distribution function* G *and* a_{ij} *for* $i > j$ *are identically distributed with the distribution function* H. *Assume that* $\int x^2 \, dH(x) = \sigma^2 < \infty$. *Denote the empirical distribution function of the eigenvalues of* $\frac{1}{\sqrt{n}} A_n$ *by* $F_n(x) = \frac{1}{n} \sum_{\lambda_i < x\sqrt{n}} 1$ *where* λ_i *are the eigenvalues of* A_n. *Then, for arbitrary* x *we have*

$$\lim_{n \to \infty} F_n(x) = \int_{-\infty}^{x} f(x)\, dx \qquad \text{in probability}$$

where

$$f(x) = \begin{cases} \dfrac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} & |x| < 2\sigma \\ 0 & \text{otherwise.} \end{cases}$$

For the proof and generalizations see [2], [3] and [4].

This theorem describes only the behaviour of the majority of the eigenvalues, and allows many of them (in fact $o(n)$) to be as large as n. We show now that this is not the case and only the first eigenvalue of a graph is of order n, the rest is near $\sqrt{n}$. (Actually this happens with any symmetric random matrix if the expectation of the entries is non-zero.)

Proposition 3. *Let A_n be as in Proposition 2. Then for the second largest eigenvalue $\lambda_2 = \lambda_2(n)$ of A_n we have $\lambda_2 = O\left(n^{\frac{1}{2}+\epsilon}\right)$ in probability.*

Proof. Let us consider the matrix $B_n = P_n A_n P_n$ where P_n is the orthogonal projection parallel to the all-one vector $e_n = (1, \dots, 1)$. By the Courant – Fischer theorem $\lambda_2 \leqslant \|B_n\|$ where $\|\cdot\|$ denotes the maximal eigenvalue of the matrix. It is easy to see that $B_n = (b_{ij})$, $b_{ij} = a_{ij} - d_i - d_j + d$ where d_i and d are the density of ones in the i-th row and in the whole matrix, respectively. We can write $B_n = C_n + R_n$ where $C_n = (c_{ij})$, $c_{ij} = a_{ij} - p$ and $R_n = (r_{ij})$, $r_{ij} = d_i + d_j - d - p$. So $\|B_n\| \leqslant \|C_n\| + \|R_n\| \leqslant \|C_n\| + \|R_n\|_2$, where $\|R_n\|_2^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij}^2$. We have

$$D^2(d_i + d_j - d) = \frac{1}{n^4} D^2\left(\sum_{k=1}^{n} \sum_{l=k+1}^{n} h_{kl} a_{kl}\right) \leqslant \frac{3pq}{n}$$

where $D^2(\cdot)$ stands for variance. Hence the expectation $E\|R_n\|_2^2 \leqslant n^2 \frac{3pq}{n}$. The Markov inequality implies that the probability that the maximal eigenvalue of the matrix R_n is larger then $K\sqrt{n}$ is $O\left(\frac{1}{K}\right)$. For the estimation of $\|C_n\|$ we use the following statement.

Proposition 4. *Let A_n be as in Wigner's theorem, and assume that the expectation of the entries is zero: $\int x\,dH(x) = 0$. If also all moments of H are finite, then for any $\epsilon > 0$ we have for the maximal eigenvalue $\lambda = \lambda(n)$ of A_n the relation $\lim P\left(|\lambda| > n^{\frac{1}{2}+\epsilon}\right) = 0$.*

Proof. It can be seen from [5] and [1] that $E\lambda^{2l} \leqslant E \sum_{i=1}^{n} \lambda_i^{2l} = O(n^{l+1})$. Whence

$$P\left(|\lambda| > n^{\frac{1}{2}+\epsilon}\right) < n^{-(l+2l\epsilon)} E\lambda^{2l} = O(n^{-(2l\epsilon - 1)}) = o(1)$$

if l was chosen large enough.

Q.E.D.

We remark that if we do not assume the existence of higher moments in Proposition 4, then the conclusion $\lambda = o(n)$ in probability still holds.

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