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A Burgess-like subconvex bound for twisted L-functions

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Abstract. Let g be a cuspidal newform (holomorphic or Maass) of arbitrary level and nebentypus, χ a primitive character of conductor q, and s a point on the critical line $\Re s = \frac{1}{2}$. It is proved that

$$L(g \otimes \chi, s) \ll_{\varepsilon, g, s} q^{1/2-(1/8)(1-2\theta)+\varepsilon},$$

where $\varepsilon > 0$ is arbitrary and $\theta = \frac{7}{64}$ is the current known approximation towards the Ramanujan–Petersson conjecture (which would allow $\theta = 0$); moreover, the dependence on s and all the parameters of g is polynomial. This result is an analog of Burgess' classical subconvex bound for Dirichlet L-functions. In Appendix 2 the above result is combined with a theorem of Waldspurger and the adelic calculations of Baruch–Mao to yield an improved uniform upper bound for the Fourier coefficients of holomorphic half-integral weight cusp forms.

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1 Introduction

Let g be a general cusp form, that is,

(1.1)
- a holomorphic form of integral weight $k_g \geq 1$, level D, and nebentypus χ_g,
- or a Maass form of weight 0 or 1 (without loss of generality), level D, and nebentypus χ_g having spectral parameter $t_g = \left(\lambda - \frac{1}{4}\right)^{1/2}$, where λ is the Laplacian eigenvalue.

We suppose that g is "new" in the sense of Atkin-Lehner theory; in particular, g is an eigenform of the Hecke operators T_n, $n \geq 1$, and we will denote by $\lambda_g(n)$ its n-th Hecke eigenvalue. We shall be concerned with the twist $g \otimes \chi$, where χ is a primitive

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character to a large modulus q. This is again a newform (of level dividing Dq^2), and its L-function $L(g \otimes \chi, s)$ equals up to finitely many Euler factors $\sum_n \lambda_g(n)\chi(n)n^{-s}$; to be precise, we have

$$L(g \otimes \chi, s) = \sum_{n \geq 1} \frac{\lambda_{g\otimes\chi}(n)}{n^s}$$

$$= \prod_{p \nmid (q,D)} \left(1 - \frac{\lambda_g(p)\chi(p)}{p^s} + \frac{\chi^2 \cdot \chi_g(p)}{p^{2s}}\right)^{-1} \prod_{p \mid (q,D)} \left(1 - \frac{\lambda_{g\otimes\chi}(p)}{p^s} + \frac{\chi_{g\otimes\chi}(p)}{p^{2s}}\right)^{-1}.$$

In this paper we are interested in upper bounds for $L(g \otimes \chi, s)$ when s is on the critical line $\Re s = \frac{1}{2}$.

In some respects, the weight k_g of a holomorphic form and the spectral parameter t_g of a Maass form behave similarly, the reason being that if g is holomorphic of weight k, then $y^{k/2}g(z)$ is Maass having spectral parameter $t = i\frac{k-1}{2}$. For a uniform notation let us therefore define the infinity type $\tilde{\mu}_g$ of g as

$$\tilde{\mu}_g := \begin{cases} \frac{k_g - 1}{2} & \text{if } g \text{ is holomorphic of weight } k_g \\ t_g & \text{if } g \text{ is a Maass form of weight 0 or 1 and} \\ & \text{Laplacian eigenvalue } \frac{1}{4} + t_g^2, \end{cases}$$

and let us write $\mu_g := 1 + |\tilde{\mu}_g|$. Then the general convexity bound gives

$$L(g \otimes \chi, s) \ll_\varepsilon (|s|\mu_g q D)^\varepsilon (|s|\mu_g)^{1/2} D^{1/4} q^{1/2}$$

for $\Re s = \frac{1}{2}$ and for any $\varepsilon > 0$ which, however, is often not sufficient for applications. In particular, it is of interest to break convexity in the q-aspect while keeping a polynomial control in the remaining parameters $|s|$, μ_g, D.

The first breakthrough was obtained by Duke–Friedlander–Iwaniec [DFI93]. If g is holomorphic of level $D = 1$ they proved the subconvex exponent

$$(1.2) \qquad \frac{1}{2} - \frac{1}{22},$$

using the δ-symbol method. In the case of a general holomorphic cusp form of weight at least 2, Bykovskiĭ [By96] derived, by a different method, the stronger subconvex exponent

$$(1.3) \qquad \frac{1}{2} - \frac{1}{8}$$

as long as $(D, q) = 1$. While it is unclear whether and to what extent Bykovskiĭ's method carries over to the general case (1.1), the second and third author independently used the strategy from [DFI93] to break convexity also in the Maass case [H03a, H03b, M04a]. As a notable feature of [H03a], a very flexible variant of the δ-symbol method due to Jutila [J92, J96, J99] was introduced into the argument.

Sarnak [S01] recently developed a new method using relatively deep spectral analysis and in particular estimates for triple products of automorphic forms. Although not stated explicitly, his method yields

$$\frac{1}{2}-\frac{1-2\theta}{14+4\theta} \tag{1.4}$$

when g is holomorphic; see also [Co03, CoPSS] for an explicit version in the more general context of holomorphic modular forms over totally real fields. This is stronger than (1.2), but weaker than (1.3). Here and henceforth, $\theta \geq 0$ denotes any *admissible* constant, by which we mean that the following approximation to the Generalized Ramanujan–Petersson conjecture is satisfied:

Hypothesis H_θ. *For any cuspidal automorphic representation π on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with local Hecke parameters $\alpha_\pi^{(1)}(p)$, $\alpha_\pi^{(2)}(p)$ for $p < \infty$ and $\mu_\pi^{(1)}(\infty)$, $\mu_\pi^{(2)}(\infty)$, one has the bounds*

$$|\alpha_\pi^{(j)}(p)| \leq p^\theta, \quad j = 1, 2$$

$$(\textit{resp. } |\Re\mu_\pi^{(j)}(\infty)| \leq \theta,\ j = 1, 2)$$

provided π_p (resp. π_∞) is unramified.

Currently, the best admissible constant is $\theta = \frac{7}{64}$ as follows from the work of Shahidi, Kim–Shahidi, Kim and Kim–Sarnak [KS02, K03, KS03].

Eventually, Sarnak's method and the bound (1.4) can be generalized to arbitrary g, but this requires very delicate arguments from the theory of automorphic representations. However, these difficulties can be avoided, and in this paper we combine various ideas from [Bl04b, HM04b] to obtain the stronger exponent

$$\frac{1}{2}-\frac{1-2\theta}{8}, \tag{1.5}$$

valid in the general case (1.1). Precisely, we have

Theorem 1. *Let g be a cuspidal automorphic newform (i.e., either a holomorphic form or a Maass cusp form) as in (1.1), and let χ be a primitive character of conductor q. For any $\varepsilon > 0$ and for $\Re s = \frac{1}{2}$ one has*

$$L(g \otimes \chi, s) \ll_\varepsilon (|s|\mu_g D q)^\varepsilon |s|^A \mu_g^B D^C q^{1/2-(1/8)(1-2\theta)}, \tag{1.6}$$

where

$$A := \frac{31 + 4\theta}{16}, \quad B := \frac{75 + 12\theta}{16}, \quad C := \frac{9}{16},$$

and $0 \leq \theta < \frac{1}{2}$ *is any real number such that Hypothesis* H_θ *is satisfied. Since* $\theta = \frac{7}{64}$ *is currently admissible, the current subconvex exponent equals*

$$\frac{1}{2} - \frac{25}{256} = \frac{103}{256} = 0.40234375.$$

Remark 1.1. Unlike all previous bounds, our estimate is explicit in all the parameters of g which turns out to be useful for applications. The numerical values of A, B, C can be improved with more careful estimates.

Remark 1.2. The bound (1.6) should be compared with Burgess' bound [B63] for Dirichlet L-functions. Indeed, the square of $L(\chi, s)$ can be interpreted as the twisted L-function $L(E \otimes \chi, s)$, where E denotes the (derivative of the) standard weight zero Eisenstein series of level 1 and Laplacian eigenvalue $\frac{1}{4}$. In this context, Burgess' bound (in its hybrid version by Heath-Brown [HB78]) is written as

$$L(E \otimes \chi, s) \ll_\varepsilon (|s|q)^\varepsilon |s|^{1/2} q^{1/2 - 1/8}.$$

Thus the bound of Theorem 1 is the cuspidal analog of Burgess' result under the Ramanujan–Petersson conjecture (i.e., $\theta = 0$).

Remark 1.3. Under more restricted assumptions, the sharpest subconvex exponent for this problem is due to Conrey–Iwaniec [CI00], namely

$$L\left(g \otimes \chi, \frac{1}{2}\right) \ll_{\varepsilon, \mu_g} q^{1/3+\varepsilon}$$

for a quadratic character χ and for g either a cusp form of level 1 or $g(z) = E\left(z, \frac{1}{2} + it\right)$ the Eisenstein series of full level (here $t \in \mathbb{R}$); in the latter case, one has

$$L\left(g \otimes \chi, \frac{1}{2}\right) = \left| L\left(\chi, \frac{1}{2} + it\right) \right|^2,$$

so this bound is the exact analog of Weyl's $\frac{1}{6}$ bound for the Riemann zeta function. Note, however, that the argument leading to this bound uses crucially the positivity of the central value $L\left(g \otimes \chi, \frac{1}{2}\right)$ and is therefore limited to the case of χ a quadratic character, g a self-dual modular form and to the special value $s = \frac{1}{2}$.

Remark 1.4. It is a nice feature that our method permits a uniform treatment of all cusp forms on $\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Depending on the applications, Theorem 1 can be optimized with respect to various auxiliary parameters, and it can be used as an ingredient for several other subconvexity problems, some of which will be considered elsewhere. Here we want to focus on Rankin–Selberg L-functions. Let f and g be two cuspidal newforms. Then for s on the critical line one has the convexity bound

$$L(f \times g, s) \ll_{\varepsilon, \mu_f, g, s} q^{1/2+\varepsilon},$$

where q denotes the level of f. The problem of improving this estimate was solved in [KMV02, M04a, HM04a]. The hardest case is when the conductor of the nebentypus of f is large (if the nebentypus is primitive for instance). In this configuration, the subconvexity problem for twisted L-functions plays a key role. In [HM04a], we use the results of the present paper to obtain the following corollary:

Corollary. *There exist positive absolute constants $A, \delta > 0$ with the following property. For any two newforms f and g (holomorphic or Maass) of respective levels q, D and respective nebentypus χ_f, χ_g such that $\chi_f\chi_g$ is non-trivial, one has*

$$L(f \times g, s) \ll (|s|\mu_f \mu_g D)^A q^{1/2-\delta}$$

for $\Re s = \frac{1}{2}$. Assuming Hypothesis H_θ we can take

$$\delta = \frac{(1-2\theta)}{8} \frac{(1-2\theta)}{202} - \varepsilon,$$

so that at the current state of knowledge $\delta = \frac{1}{2648}$ is admissible.

Combining some of the methods of [HM04a, Bl04b] and of the present paper it is possible to reduce considerably the constant 202 above. We will return to this on another occasion.

Remark 1.5. Theorem 1 can be combined with the powerful results of Shimura and Waldspurger to improve on the known upper bounds for the Fourier coefficients of half-integral weight holomorphic or Maass cusp forms. The recent careful adelic calculations of Baruch–Mao enable one to derive these estimates with proper uniformity in all the parameters of the underlying cusp form. The details in the holomorphic case have been kindly worked out for us by Zhengyu Mao and have been included in this paper as Appendix 2 (see Theorem 6).

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2 Some general results

In this section, we indicate our normalization for the Fourier coefficients of modular forms, recall some of their properties and state various results from the spectral theory of automorphic forms; for more background and references, we refer to [DFI02] and to [HM04a].

2.1. Fourier coefficients. We will follow the notation of [DFI02] to large extent. A Maass cusp form g of weight $k \geq 0$ and Laplacian eigenvalue $\left(\frac{1}{2}+it\right)\left(\frac{1}{2}-it\right)$ admits an expansion

$$g(z) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \rho_g(n) W_{(n/|n|)(k/2),it}(4\pi|n|y)e(nx)$$

in terms of the Whittaker function $W_{\alpha,it}(y)$. Note that $W_{\alpha,it}(y) \sim y^{\alpha}e^{-y/2}$ for $y \to +\infty$. For an Eisenstein series $E_{\mathfrak{a}}\left(z,\frac{1}{2}+it\right)$ attached to some cusp $\mathfrak{a}$ of $\Gamma_0(D)$, we have a Fourier expansion of the type

$$E_{\mathfrak{a}}\left(z,\frac{1}{2}+it\right) = \delta_{\mathfrak{a}=\infty}y^{1/2+it} + \varphi_{\mathfrak{a}}\left(\frac{1}{2}+it\right)y^{1/2-it} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \rho_{\mathfrak{a}}(n,t) W_{(n/|n|)(k/2),it}(4\pi|n|y)e(nx).$$

Finally, when $g(z)$ is a holomorphic cusp form of weight k, we write

$$g(z) = \sum_{n\geq 1} \rho_g(n)(4\pi n)^{k/2}e(nz), \tag{2.1}$$

keeping in mind that $y^{k/2}g(z)$ is a Maass form of weight k.

We will need the following general Voronoi-type summation formula ([HM04a, Proposition 2.1]).

Proposition 2.1. *Let g be a cusp form (holomorphic or Maass) of weight k, level D and nebentypus χ_g. Let $c \equiv 0 \pmod D$, and let a be an integer coprime to c. If $F \in C^{\infty}((0,\infty))$ is a Schwartz class function vanishing in a neighborhood of zero, then*

$$\sum_{n\geq 1}\sqrt{n}\rho_g(n)e\left(n\frac{a}{c}\right)F(n) = \frac{\chi_g(\bar{a})}{c}\sum_{\pm}\sum_{n\geq 1}\sqrt{n}\rho_g^{\pm}(n)e\left(\mp n\frac{\bar{a}}{c}\right)\mathscr{F}^{\pm}\left(\frac{n}{c^2}\right). \tag{2.2}$$

In this formula,

$$\rho_g^{+}(n) := \rho_g(n), \quad \rho_g^{-}(n) := \frac{\Gamma\left(\frac{1}{2}+it-\frac{k}{2}\right)}{\Gamma\left(\frac{1}{2}+it+\frac{k}{2}\right)}\rho_g(-n),$$

and

(2.3) $$\mathscr{F}^{\pm}(y) := \int_0^\infty F(x) J_g^{\pm}(4\pi\sqrt{xy})\, dx,$$

where $t = t_g$ is the spectral parameter of g in the Maass case, and

$$J_g^+(x) := 2\pi i^l J_{l-1}(x), \quad J_g^-(x) := 0,$$

if g is induced from a holomorphic form of weight l;

$$J_g^+(x) := \frac{-\pi}{\mathrm{ch}(\pi t)}\{Y_{2it}(x) + Y_{-2it}(x)\}, \quad J_g^-(x) := 4\,\mathrm{ch}(\pi t) K_{2it}(x),$$

if k is even, and g is not induced from a holomorphic form;

$$J_g^+(x) := \frac{\pi}{\mathrm{sh}(\pi t)}\{Y_{2it}(x) - Y_{-2it}(x)\}, \quad J_g^-(x) := -4i\,\mathrm{sh}(\pi t) K_{2it}(x),$$

if k is odd, and g is not induced from a holomorphic form.

2.2. Hecke operators. We recall that there is an action on the L^2-space of modular forms of level D and some given nebentypus by the commutative algebra $\mathbb{T}$ generated by the Hecke operators $\{T_n\}_{n\geq 1}$. We denote by $\mathbb{T}^{(D)}$ the subalgebra generated by $\{T_n\}_{(n,D)=1}$ and call a holomorphic or Maass cusp form a *Hecke–Maass* cusp form if it is an eigenform for $\mathbb{T}^{(D)}$. For a Hecke–Maass cusp form g we denote by χ_g its nebentypus and by $\lambda_g(n)$ its n-th Hecke eigenvalue. If Hypothesis H_θ is valid, one has

(2.4) $$|\lambda_g(n)| \leq \tau(n) n^\theta,$$

where $\tau(n)$ denote the divisor function. Moreover, for $(n, D) = 1$ the following relations hold:

(2.5) $$\sqrt{n}\rho_g(\pm n) = \rho_g(\pm 1)\lambda_g(n),$$

(2.6) $$\sqrt{m}\rho_g(m)\lambda_g(n) = \sum_{d|(m,n)} \chi_g(d)\rho_g\left(\frac{m}{d}\frac{n}{d}\right)\sqrt{\frac{mn}{d^2}},$$

(2.7) $$\sqrt{mn}\rho_g(mn) = \sum_{d|(m,n)} \chi_g(d)\mu(d)\rho_g\left(\frac{m}{d}\right)\sqrt{\frac{m}{d}}\lambda_g\left(\frac{n}{d}\right).$$

If g is a Hecke form and belongs to the *new* subspace (in the sense of Atkin–Lehner theory), then g an eigenform of all Hecke operators and the above relations hold with no restriction on n. In this case, we say that g is a *newform*.

2.3. Kuznetsov's formula and the large sieve. We make the following convention: if f and g are two Maass cusp forms of the same weight, same level D, and same nebentypus, then we normalize their Petersson inner product $\langle f, g\rangle$ as

$$(2.8)\qquad \langle f, g\rangle = \int_{\Gamma_0(D)\backslash\mathbb{H}} \bar{f}(z)g(z)\frac{dx\,dy}{y^2};$$

if f and g are holomorphic of weight k, $\langle f, g\rangle$ is given as above with an extra y^k factor. In particular, we say that g is L^2-normalized if $\langle g, g\rangle = 1$.

For a character χ to modulus D, we denote by $\mathscr{B}(D;\chi) = \{f_j\}_{j\geq 1}$ (resp. $\mathscr{B}_k^h(D;\chi)$) an orthonormal basis of the L^2-space of weight 0 Maass cusp forms (resp. of the space of holomorphic cusp forms of weight k) of level D and nebentypus χ. If χ is the trivial character, we simply omit it from the notation. We can always choose a basis formed of Hecke–Maass cusp forms, and we will pick a special basis later in section 3.1. Let us now recall Kuznetsov's trace formula (in the trivial nebentypus case, see [I87, Theorems 9.4, 9.5, 9.7]).

Theorem 2. *Let m, n, D be positive integers and $\varphi \in C_c^\infty((0,\infty))$. One has*

$$\frac{1}{4\sqrt{mn}}\sum_{c\equiv 0\ (D)}\frac{S(m,n;c)}{c}\varphi\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

$$= \sum_{k\equiv 0\ (2)}\Gamma(k)\tilde{\varphi}(k-1)\sum_{f\in\mathscr{B}_k^h(D)}\overline{\rho_f}(m)\rho_f(n) + \sum_{j\geq 1}\frac{\hat{\varphi}(t_j)}{\mathrm{ch}(\pi t_j)}\overline{\rho_j}(m)\rho_j(n)$$

$$+\frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^{+\infty}\frac{\hat{\varphi}(t)}{\mathrm{ch}(\pi t)}\overline{\rho_{\mathfrak{a}}}(m,t)\rho_{\mathfrak{a}}(n,t)\,dt,$$

and

$$\frac{1}{4\sqrt{mn}}\sum_{c\equiv 0\ (D)}\frac{S(m,-n;c)}{c}\varphi\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

$$= \sum_{j\geq 1}\frac{\check{\varphi}(t_j)}{\mathrm{ch}(\pi t_j)}\overline{\rho_j}(m)\rho_j(-n) + \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^{+\infty}\frac{\check{\varphi}(t)}{\mathrm{ch}(\pi t)}\overline{\rho_{\mathfrak{a}}}(m,t)\rho_{\mathfrak{a}}(-n,t)\,dt,$$

where the Bessel transforms are defined by

$$\tilde{\varphi}(k-1) := \int_0^\infty \varphi(x) i^k J_{k-1}(x)\frac{dx}{x};$$

$$(2.9)\qquad \hat{\varphi}(t) := \int_0^\infty \varphi(x)\frac{-\pi}{2\,\mathrm{ch}(\pi t)}\{Y_{2it}(x) + Y_{-2it}(x)\}\frac{dx}{x};$$

$$(2.10)\qquad \check{\varphi}(t) := \int_0^\infty \varphi(x) 2\,\mathrm{ch}(\pi t) K_{2it}(x)\frac{dx}{x}.$$

Remark 2.1. The kernels in (2.9) and (2.10) can be expressed alternatively as

$$\frac{-\pi}{2\,\mathrm{ch}(\pi t)}\{Y_{2it}(x)+Y_{-2it}(x)\}=\frac{\pi i}{2\,\mathrm{sh}(\pi t)}\{J_{2it}(x)-J_{-2it}(x)\};$$

$$2\,\mathrm{ch}(\pi t)K_{2it}(x)=\frac{\pi i}{2\,\mathrm{sh}(\pi t)}\{I_{2it}(x)-I_{-2it}(x)\}.$$

The next lemma is a variant of Lemma 7.1 of [DI82] and provides bounds for the various Bessel transforms of the test functions φ above.

Lemma 2.1. *Let $\varphi(x)$ be a smooth function, compactly supported in $(X,2X)$, satisfying*

$$\varphi^{(i)}(x)\ll_i (Z/X)^i$$

for some $Z\geq 1$ and for any integer $i\geq 0$, the implied constant depending only on i. Then, for $t\geq 0$ and for any real $k>1$, one has

$$\hat{\varphi}(it),\check{\varphi}(it)\ll\frac{1+(X/Z)^{-2t}}{1+X/Z}\qquad \text{for } 0\leq t<\frac{1}{4}; \tag{2.11}$$

$$\hat{\varphi}(t),\check{\varphi}(t),\tilde{\varphi}(t)\ll\frac{1+|\log(X/Z)|}{1+X/Z}\qquad \text{for } t\geq 0; \tag{2.12}$$

$$\hat{\varphi}(t),\check{\varphi}(t),\tilde{\varphi}(t)\ll\left(\frac{Z}{t}\right)\left(\frac{1}{t^{1/2}}+\frac{X}{t}\right)\qquad \text{for } t\geq 1; \tag{2.13}$$

$$\hat{\varphi}(t),\check{\varphi}(t),\tilde{\varphi}(t)\ll_k\left(\frac{Z}{t}\right)^k\left(\frac{1}{t^{1/2}}+\frac{X}{t}\right)\qquad \text{for } t\geq \max(2X,1). \tag{2.14}$$

Proof. The inequalities (2.11), (2.12), (2.13) can be proved exactly as (7.1), (7.2) and (7.3) in [DI82]. The last inequality (2.14) is an extension of (7.4) in [DI82], but we only claim it in the restricted range $t\geq\max(2X,1)$. On the one hand, we were unable to reconstruct the proof of (7.4) in [DI82] for the entire range $t\geq 1$; on the other hand, [DI82] only utilizes this inequality for $t\gg\max(X,Z)$ (cf. page 268 there, and note also that for $t\ll Z$ the bound (2.13) is stronger). For this reason we include a detailed proof of (2.14) in the case of $\check{\varphi}(t)$. For $\hat{\varphi}(t)$ and $\tilde{\varphi}(t)$ the proof is similar.

We may assume that $k=2j+1$ is a positive odd integer. The Bessel differential equation

$$x^2K''_{2it}(x)+xK'_{2it}(x)=(x^2-4t^2)K_{2it}(x)$$

gives an identity

$$(2.15)\quad \check{\varphi}(t) = (D_t\varphi)^\vee(t),$$

where

$$D_t\varphi(x) := x\left(\frac{x\varphi(x)}{x^2-4t^2}\right)'' + x\left(\frac{\varphi(x)}{x^2-4t^2}\right)'.$$

This transform $D_t\varphi$ is smooth and compactly supported in $(X, 2X)$, and it is straightforward to check that

$$\|(D_t\varphi)^{(i)}\|_\infty \ll_i (Z/t)^2(Z/X)^i \quad \text{for } t \geq \max(2X, 1).$$

By iterating (2.15) it follows that

$$\check{\varphi}(t) = (D_t^j\varphi)^\vee(t),$$

where $D_t^j\varphi$ is a smooth function, compactly supported in $(X, 2X)$, satisfying

$$\|(D_t^j\varphi)^{(i)}\|_\infty \ll_{j,i} (Z/t)^{2j}(Z/X)^i \quad \text{for } t \geq \max(2X, 1).$$

We bound $(D_t^j\varphi)^\vee(t)$ by (2.13) and obtain

$$\check{\varphi}(t) \ll_j \left(\frac{Z}{t}\right)^{2j+1}\left(\frac{1}{t^{1/2}} + \frac{X}{t}\right) \quad \text{for } t \geq \max(2X, 1). \qquad \square$$

Finally, we recall the large sieve inequalities from [DI82].

Theorem 3. *Let D be a positive integer, $M, K, T \geq 1$, and let $(a_m)_{m\sim M}$ be a sequence of complex numbers supported on $[M, 8M]$. Then, for any $\varepsilon > 0$,*

$$(2.16)\quad \sum_{\substack{k\equiv 0\ (2)\\ k\leq K}} \Gamma(k) \sum_{f\in\mathscr{B}_k^h(D)} \left|\sum_m a_m\sqrt{m}\rho_f(m)\right|^2 \ll_\varepsilon M^\varepsilon\left(K^2 + \frac{M}{D}\right)\sum_m |a_m|^2;$$

$$(2.17)\quad \sum_{|t_j|\leq T} \frac{1}{\mathrm{ch}(\pi t_j)} \left|\sum_m a_m\sqrt{m}\rho_j(m)\right|^2 \ll_\varepsilon M^\varepsilon\left(T^2 + \frac{M}{D}\right)\sum_m |a_m|^2;$$

$$(2.18)\quad \sum_{\mathfrak{a}} \int_{-T}^{T} \frac{1}{\mathrm{ch}(\pi t)} \left|\sum_m a_m\sqrt{m}\rho_{\mathfrak{a}}(m;t)\right|^2 dt \ll_\varepsilon M^\varepsilon\left(T^2 + \frac{M}{D}\right)\sum_m |a_m|^2.$$

3 A large sieve inequality

Theorem 4. *Let D, q, r be positive integers. For $M, N, C, Z \geq 1$, let $g(m, n; c)$ be a smooth function compactly supported on*

$$[M/2, 2M] \times [N/2, 2N] \times [C/2, 2C]$$

satisfying

$$\frac{\partial^{i+j+k}}{\partial m^i \partial n^j \partial c^k} g(m,n;c) \ll_{i,j,k} \frac{Z^{i+j+k}}{M^i N^j C^k},$$

and let $(a_m)_{m \in [M/2,2M]}$, $(b_n)_{n \in [N/2,2N]}$ *be two sequences of complex numbers satisfying*

$$\text{(3.1)} \qquad \begin{aligned} &a_m \neq 0 \Rightarrow q|m, \quad (q, m/q) = 1, \\ &b_n \neq 0 \Rightarrow r|n, \quad (r, n/r) = 1. \end{aligned}$$

Then for any $\varepsilon > 0$ *one has, under Hypothesis* H_θ,

$$\begin{aligned} \Sigma^{\pm}(M,N;C) &:= \sum_{c \equiv 0\ (D)} \sum_m \sum_n a_m b_n \frac{S(m, \pm n; c)}{c} g(m,n;c) \\ &\ll_\varepsilon (qrDZMNC)^\varepsilon (qr)^\theta \left(1 + \frac{C}{\sqrt{MN}}\right)^{2\theta} \\ &\quad \times Z^2 \left(Z^{3/2} + Z\frac{\sqrt{MN}}{C} + Z^{2\theta}\frac{(q,D)M}{qD}\right)^{1/2} \\ &\quad \times \left(Z^{3/2} + Z\frac{\sqrt{MN}}{C} + Z^{2\theta}\frac{(r,D)N}{rD}\right)^{1/2} \|a\|_2 \|b\|_2. \end{aligned}$$

Proof. This is a variant of Theorem 9 of [DI82]. We only treat the bound for $\Sigma^+(M,N;C)$, the bound for $\Sigma^-(M,N;C)$ being similar. We first proceed as in [DI82], and put the test function in a shape appropriate for the use of Kuznetsov's trace formula: We define G by

$$g\left(x_1, x_2, \frac{4\pi\sqrt{x_1 x_2}}{x}\right) = \iint_{\mathbb{R}^2} G(\xi_1, \xi_2; x) e(\xi_1 x_1 + \xi_2 x_2)\, d\xi_1\, d\xi_2,$$

so that by Fourier inversion one has

$$\text{(3.2)} \qquad G(\xi_1, \xi_2; x) = \int_{M/2}^{2M} \int_{N/2}^{2N} g\left(x_1, x_2, \frac{4\pi\sqrt{x_1 x_2}}{x}\right) e(-\xi_1 x_1 - \xi_2 x_2)\, dx_1\, dx_2,$$

and

$$\Sigma^+(M,N;C) = \iint_{\mathbb{R}^2} \sum_{c\equiv 0\ (D)} \sum_m \sum_n a_m e(\xi_1 m) b_n e(\xi_2 n) \frac{S(m,n;c)}{c} G\left(\xi_1,\xi_2;\frac{4\pi\sqrt{mn}}{c}\right) d\xi_1\, d\xi_2. \tag{3.3}$$

Note that $G(\xi_1,\xi_2;x)$ as a function of the x variable is supported in the interval $[X,16X]$ with

$$X := \frac{\pi\sqrt{MN}}{C}. \tag{3.4}$$

Let p_1, p_2, k be 3 positive integers. We integrate (3.2) by parts p_1 times with respect to x_1 and p_2 times with respect to x_2, and differentiate it k times with respect to x getting

$$\frac{\partial^k}{\partial x^k} G(\xi_1,\xi_2;x) \ll_{p_1,p_2,k} \left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2}\left(\frac{Z}{X}\right)^{k} MN. \tag{3.5}$$

We postpone the integration over ξ_1, ξ_2 and the choice of p_1, p_2 to the end of section 3. Having these parameters fixed for the moment, we simplify the notation and set

$$\varphi(x) := G(\xi_1,\xi_2;x),$$

and by slight abuse of notation we denote by a_m and b_n the complex numbers $a_m e(\xi_1 m)$ and $b_n e(\xi_2 n)$, respectively. This, of course, does not change the values of $\|a\|_2$, $\|b\|_2$ or the support of these sequences. We apply Theorem 2, so that the integrand in (3.3) is the sum of three terms:

$$T^{\text{Holo}} + T^{\text{Maass}} + T^{\text{Eisen}}, \tag{3.6}$$

where

$$T^{\text{Holo}} := 4\sum_{k\equiv 0\ (2)} \tilde{\varphi}(k-1)\Gamma(k) \sum_{f\in\mathcal{B}_k^h(D)} \left(\sum_{m\sim M} a_m\sqrt{m}\overline{\rho_f}(m)\right)\left(\sum_{n\sim N} b_n\sqrt{n}\rho_f(n)\right),$$

$$T^{\text{Maass}} := 4\sum_{j\geq 1} \frac{\hat{\varphi}(t_j)}{\text{ch}(\pi t_j)} \left(\sum_{m\sim M} a_m\sqrt{m}\overline{\rho_j}(m)\right)\left(\sum_{n\sim N} b_n\sqrt{n}\rho_j(n)\right),$$

$$T^{\text{Eisen}} := \frac{1}{\pi}\sum_{\mathfrak{a}} \int_{-\infty}^{+\infty} \frac{\hat{\varphi}(t)}{\text{ch}(\pi t)} \left(\sum_{m\sim M} a_m\sqrt{m}\overline{\rho_{\mathfrak{a}}}(m,t)\right)\left(\sum_{n\sim N} b_n\sqrt{n}\rho_{\mathfrak{a}}(n,t)\right) dt.$$

3.1. Contribution of the cuspidal spectrum. In this section we bound the contribution from the holomorphic and the Maass spectrum. The proof of the bound for T^{Holo} is similar to that of T^{Maass}, but sharper, so we only display the proof for T^{Maass}. By Cauchy–Schwarz we have

(3.7)
$$|T^{\mathrm{Maass}}|^2 \ll \left(\sum_{j\geq 1}\frac{|\hat{\varphi}(t_j)|}{\mathrm{ch}(\pi t_j)}\left|\sum_{m\sim M} a_m\sqrt{m}\rho_j(m)\right|^2\right)\times\left(\sum_{j\geq 1}\frac{|\hat{\varphi}(t_j)|}{\mathrm{ch}(\pi t_j)}\left|\sum_{n\sim N} b_n\sqrt{n}\rho_j(n)\right|^2\right),$$

so that it is sufficient to bound each factor separately. Our aim is to establish multiplicative properties of the coefficients a_m, b_n in order to exploit the condition (3.1). Since the Hecke operators T_n with $(n,D)=1$ are normal and commute with the Laplacian, one may choose an orthonormal basis $\{f_j\}$ of $\mathscr{B}(D)$ made of Hecke–Maass eigenforms. More precisely, it follows from Atkin–Lehner theory that an orthonormal basis of Hecke–Maass eigenforms can be obtained as follows. Let f be any Maass newform of level D_f dividing D, then the complex vector space generated by $\left\{f_d(z) := f(dz), d\,|\,\frac{D}{D_f}\right\}$ is a $\tau(D/D_f)$-dimensional subspace of the space of cusp forms of level D formed of Hecke eigenforms of the T_n, $(n,D)=1$, with eigenvalues being the same as those of f, that is, $\lambda_f(n)$. By Gram–Schmidt there is an L^2-orthonormal basis $\left\{f_{(d)}(z), d\,|\,\frac{D}{D_f}\right\}$ of this subspace of the form

$$f_{(d)}(z) = \sum_{d'|D/D_f}\alpha_d(d')f(d'z),\quad \alpha_d(d')\in\mathbf{C},$$

(the $\alpha_d(d')$ depend also on f but we suppress it from the notations). Now we form an orthonormal basis of Hecke–Maass cusp forms of level D by the union of the $\left\{f_{(d)}(z), d\,|\,\frac{D}{D_f}\right\}$ for f ranging over the L^2-normalized cuspidal newforms of level dividing D. Let us fix one of these basis elements $f_{(d)}$ for a moment. We have for any $m\geq 1$,

$$\sqrt{m}\rho_{f_{(d)}}(m) = \sum_{d'|D}\alpha_d(d')\sqrt{m}\rho_{f_{d'}}(m) = \sum_{d'|(D,m)}\sqrt{d'}\alpha_d(d')\sqrt{m/d'}\rho_f(m/d'),$$

where, in order to simplify notation, we made the convention that $\alpha_d(d')=0$ if d or d' does not divide D/D_f. We suppose now that $m=qm'$ with $(q,m')=1$. Setting $q'=(q,D)$, we obtain by (2.7) that

$$\begin{aligned}\sqrt{m}\rho_{f_{(d)}}(m) &= \sum_{\substack{d_1|(D,m')\\ d_2|q'}}\sqrt{d_1d_2}\alpha_d(d_1d_2)\sqrt{\frac{qm'}{d_2d_1}}\rho_f\left(\frac{qm'}{d_2d_1}\right)\\ &= \sum_{\substack{d_1|(D,m')\\ d_2|q'}}\sqrt{d_1d_2}\alpha_d(d_1d_2)\lambda_f\left(\frac{q}{d_2}\right)\sqrt{\frac{m'}{d_1}}\rho_f\left(\frac{m'}{d_1}\right),\end{aligned}$$

since f is a Hecke eigenform for all Hecke operators. For the same reason, we have the identity

$$\lambda_f\left(\frac{q}{d_2}\right) = \lambda_f\left(\frac{q}{q'}\frac{q'}{d_2}\right) = \sum_{d_3 \mid (q/q', q'/d_2)} \chi_f(d_3)\mu(d_3)\lambda_f\left(\frac{q}{q'd_3}\right)\lambda_f\left(\frac{q'}{d_2 d_3}\right).$$

Since $(q', m') = 1$, we conclude that

$$\begin{aligned}
&\sqrt{m}\rho_{f_{(d)}}(m)\\
&\quad = \sum_{d_3 \mid (q/q', q')} \chi_f(d_3)\mu(d_3)\lambda_f\left(\frac{q}{q'd_3}\right) \sum_{\substack{d_1 \mid (D, m')\\ d_2 \mid q'/d_3}} \sqrt{d_1 d_2}\alpha_d(d_1 d_2)\sqrt{\frac{q'm'}{d_3 d_2 d_1}}\rho_f\left(\frac{q'm'}{d_3 d_2 d_1}\right)\\
&\quad = \sum_{d_3 \mid (q/q', q')} \chi_f(d_3)\mu(d_3)\lambda_f\left(\frac{q}{q'd_3}\right) \sum_{d' \mid (D, q'm'/d_3)} \sqrt{d'}\alpha_d(d')\sqrt{\frac{q'm'}{d_3 d'}}\rho_f\left(\frac{q'm'}{d_3 d'}\right)\\
&\quad = \sum_{d_3 \mid (q/q', q')} \chi_f(d_3)\mu(d_3)\lambda_f\left(\frac{q}{q'd_3}\right)\sqrt{\frac{q'm'}{d_3}}\rho_{f_{(d)}}\left(\frac{q'm'}{d_3}\right).
\end{aligned}$$

Hence it follows from (3.1) and (2.4) that for f_j an element of Hecke eigenbasis described above (and by writing d for $\frac{q'}{d_3}$),

$$\left|\sum_m a_m\sqrt{m}\rho_{f_j}(m)\right|^2 \le \tau(q)^2 q^{2\theta} \sum_{d \mid (q, D)} \left|\sum_{(m', q) = 1} a_{qm'}\sqrt{dm'}\rho_{f_j}(dm')\right|^2,$$

and therefore

(3.8)
$$\sum_{j \ge 1} \frac{|\hat{\varphi}(t_j)|}{\mathrm{ch}(\pi t_j)} \left|\sum_m a_m\sqrt{m}\rho_j(m)\right|^2 \le \tau(q)^2 q^{2\theta} \sum_{d \mid (q, D)} \sum_{j \ge 1} \frac{|\hat{\varphi}(t_j)|}{\mathrm{ch}(\pi t_j)} \left|\sum_{m' \sim M/q} a_{qm'}\sqrt{dm'}\rho_j(dm')\right|^2.$$

To estimate the j-sum we set

$$T_0 := \max(16X, (ZX)^{1/2}, Z^{2/3}),$$

where X is given by (3.4), and we split the sum as

$$\sum_{j \ge 1} \frac{|\hat{\varphi}(t_j)|}{\mathrm{ch}(\pi t_j)} \left|\sum_{m' \sim M/q} a_{qm'}\sqrt{dm'}\rho_j(dm')\right|^2 = \sum_{|t_j| \le 1} \cdots + \sum_{1 < |t_j| \le T_0} \cdots + \sum_T \sum_{T < |t_j| \le 2T} \cdots,$$

where T runs through the numbers of the form $2^\nu T_0$, $\nu \in \mathbb{N}_0$.

By the large sieve inequality (2.17) combined with (3.5), (2.11) and (2.12), we obtain

$$\sum_{|t_j|\le 1}\cdots \ll_{\varepsilon,p_1,p_2} ((1+X)Z)^{2\varepsilon}\left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MN\left(1+\frac{Z}{X}\right)^{2\theta}\left(1+\frac{dM}{qD}\right)\|a\|_2^2.$$

Similarly, a combination of (2.17), (3.5) and (2.12) shows that

$$\begin{aligned}\sum_{1<|t_j|\le T_0}\cdots &\ll_{\varepsilon,p_1,p_2} ((1+X)Z)^{2\varepsilon}\left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MN\left(\frac{Z}{Z+X}\right)\left(T_0^2+\frac{dM}{qD}\right)\|a\|_2^2\\ &\ll_{\varepsilon,p_1,p_2} ((1+X)Z)^{2\varepsilon}\left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MN\left(Z^{4/3}+ZX+\frac{dM}{qD}\right)\|a\|_2^2.\end{aligned}$$

For each $T=2^\nu T_0$ such that $T\le Z^{1+\varepsilon}$ we can combine (2.17), (3.5) and (2.13) to see that

$$\begin{aligned}\sum_{T<|t_j|\le 2T}\cdots &\ll_{\varepsilon,p_1,p_2} Z^{\varepsilon}\left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MN\left(\frac{Z}{T}\right)\left(\frac{1}{T^{1/2}}+\frac{X}{T}\right)\left(T^2+\frac{dM}{qD}\right)\|a\|_2^2\\ &\ll_{\varepsilon,p_1,p_2} Z^{2\varepsilon}\left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MN\left(Z^{3/2}+ZX+\frac{dM}{qD}\right)\|a\|_2^2.\end{aligned}$$

The contribution of each $T=2^\nu T_0$ such that $T>Z^{1+\varepsilon}$ is negligible as follows from (2.17), (3.5) and (2.14) with $k=10/\varepsilon$. In fact, we have $Z<T^{1-\varepsilon/2}$, therefore

$$\begin{aligned}\sum_{T<|t_j|\le 2T}\cdots &\ll_{\varepsilon,p_1,p_2} T^{\varepsilon}\left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MN\left(\frac{Z}{T}\right)^{10/\varepsilon}\left(\frac{1}{T^{1/2}}+\frac{X}{T}\right)\left(T^2+\frac{dM}{qD}\right)\|a\|_2^2\\ &\ll_{\varepsilon,p_1,p_2} \left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MNT^{-2}\left(1+\frac{dM}{qD}\right)\|a\|_2^2.\end{aligned}$$

By summing over all $T\ge T_0$ and using also (3.8) we infer that

$$\begin{aligned}&\sum_{j\ge 1}\frac{|\hat{\varphi}(t_j)|}{\mathrm{ch}(\pi t_j)}\left|\sum_m a_m\sqrt{m}\rho_j(m)\right|^2\\ &\quad\ll_{\varepsilon,p_1,p_2} ((1+X)Zq)^{3\varepsilon}\left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MN\\ &\qquad\times q^{2\theta}\left(1+\frac{1}{X}\right)^{2\theta}\left(Z^{3/2}+ZX+Z^{2\theta}\frac{(q,D)M}{qD}\right)\|a\|_2^2.\end{aligned}$$

We have a similar bound for the second factor in (3.7). Therefore we obtain, for any $\varepsilon > 0$,

$$(3.9)\qquad T^{\mathrm{Maass}} \ll_{\varepsilon,p_1,p_2} ((1+X)Zqr)^{\varepsilon}\left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MN$$

$$\times (qr)^{\theta}\left(1+\frac{1}{X}\right)^{2\theta}\left(Z^{3/2}+ZX+Z^{2\theta}\frac{(q,D)M}{qD}\right)^{1/2}$$

$$\times\left(Z^{3/2}+ZX+Z^{2\theta}\frac{(r,D)N}{rD}\right)^{1/2}\|a\|_2\|b\|_2.$$

The contribution of holomorphic forms is treated similarly using (2.16); however, since the Ramanujan–Petersson conjecture holds true for holomorphic forms, one obtains the stronger bound

$$(3.10)\qquad T^{\mathrm{Holo}} \ll_{\varepsilon,p_1,p_2} ((1+X)Zqr)^{\varepsilon}\left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MN$$

$$\times\left(Z^{3/2}+ZX+\frac{(q,D)M}{qD}\right)^{1/2}\left(Z^{3/2}+ZX+\frac{(r,D)N}{rD}\right)^{1/2}\|a\|_2\|b\|_2.$$

3.2. Contribution of the Eisenstein spectrum. We now evaluate T^{Eisen} in (3.6). By Cauchy–Schwarz one has

$$|T^{\mathrm{Eisen}}|^2 \ll \left(\int_{\mathbb{R}}\frac{|\hat{\varphi}(t)|}{\mathrm{ch}(\pi t)}\sum_{\mathfrak{a}}\left|\sum_m a_m\sqrt{m}\rho_{\mathfrak{a}}(m;t)\right|^2 dt\right)\times\left(\int_{\mathbb{R}}\frac{|\hat{\varphi}(t)|}{\mathrm{ch}(\pi t)}\sum_{\mathfrak{a}}\left|\sum_n b_n\sqrt{n}\rho_{\mathfrak{a}}(n)\right|^2 dt\right),$$

and it is sufficient to bound each factor separately. We wish to imitate the argument given above, but a slight difficulty occurs as the Eisenstein series $E_{\mathfrak{a}}(z,s)$ are not Hecke eigenforms in general. The problem of diagonalizing Hecke operators on the space of Eisenstein series was dealt with by Rankin [Ra90, Ra93, Ra92, Ra94]. However, we proceed (as in [M04a, HM04a]) by computing directly the Fourier coefficients of the Eisenstein series. Recall (see [DI82, Lemma 2.3]) that the cusps $\{\mathfrak{a}\}$ of $\Gamma_0(D)$ are uniquely represented by the rationals

$$\left\{\frac{u}{w}\colon w|D, u\in\mathscr{U}_w\right\},$$

where, for each $w|D$, $\mathscr{U}_w$ is a set of integers coprime with w representing each reduced residue class modulo $\tilde{w} := (w, D/w)$ exactly once. In the half-plane $\Im t < 0$ we have for $m \neq 0$ (see [DI82, (1.17) and p. 247]),

$$\sqrt{|m|}\,\rho_{\mathfrak{a}}(m,t) = \frac{\pi^{1/2+it}|m|^{it}}{\Gamma\left(\frac{1}{2}+it\right)}\left(\frac{(w,D/w)}{wD}\right)^{1/2+it}\sum_{(\gamma,D/w)=1}\frac{1}{\gamma^{1+2it}}\sum_{\substack{\delta\ (\mathrm{mod}\ \gamma w),\,(\delta,\gamma w)=1\\ \delta\gamma\equiv u\ (\mathrm{mod}\ \tilde{w})}} e\left(-m\frac{\delta}{\gamma w}\right)$$

with analytic continuation to $\Im t = 0$. The congruence condition on δ can be analyzed by means of multiplicative characters modulo $\tilde{w}$:

$$\sum_{(\gamma,D/w)=1}\frac{1}{\gamma^{1+2it}}\sum_{\substack{\delta\ (\mathrm{mod}\ \gamma w),\,(\delta,\gamma w)=1\\ \delta\gamma\equiv u\ (\mathrm{mod}\ \tilde{w})}} e\left(-m\frac{\delta}{\gamma w}\right)$$

$$= \frac{1}{\varphi(\tilde{w})}\sum_{\psi\ \mathrm{mod}\ \tilde{w}}\bar{\psi}(-u)\sum_{(\gamma,D/w)=1}\frac{\psi(\gamma)}{\gamma^{1+2it}}G_{\psi}(m;\gamma w),$$

where

$$G_{\psi}(m,q) := \sum_{b\ (\mathrm{mod}\ q)}^{*}\psi(b)e\left(\frac{bm}{q}\right)$$

is the Gauss sum. Note that we may replace ψ by its underlying primitive character, since we only sum over b coprime with q. For brevity we write

$$S_{\psi}(m;w) := \sum_{(\gamma,D/w)=1}\frac{\psi(\gamma)}{\gamma^{1+2it}}G_{\psi}(m;\gamma w),$$

so that

$$\sum_{\mathfrak{a}}\left|\sum_{m}a_m\sqrt{m}\rho_{\mathfrak{a}}(m,t)\right|^2$$

$$= \frac{\pi}{\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2}\sum_{w|D}\frac{\tilde{w}}{wD\varphi^2(\tilde{w})}\sum_{u\in\mathcal{U}_w}\left|\sum_{\psi\ \mathrm{mod}\ \tilde{w}}\bar{\psi}(-u)\sum_{m}a_m m^{it}S_{\psi}(m;w)\right|^2.$$

By Parseval it follows that

$$\sum_{\mathfrak{a}}\left|\sum_{m}a_m\sqrt{m}\rho_{\mathfrak{a}}(m,t)\right|^2 = \frac{\pi}{\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2}\sum_{w|D}\frac{\tilde{w}}{wD\varphi(\tilde{w})}\sum_{\psi\ \mathrm{mod}\ \tilde{w}}\left|\sum_{m}a_m m^{it}S_{\psi}(m;w)\right|^2.$$

In the following it will be useful to perform the summation over the primitive characters underlying the ψ's. For each character ψ mod $\tilde{w}$, we record by w^* its conductor and by ψ^* its underlying primitive character, so that

$$(3.11)\quad \sum_{\mathfrak{a}}\left|\sum_m a_m\sqrt{m}\rho_{\mathfrak{a}}(m,t)\right|^2$$

$$= \frac{\pi}{\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2}\sum_{w^{*2}\mid D}\ \sideset{}{^*}\sum_{\psi^*\ \mathrm{mod}\ w^*}\ \sum_{w^*\mid w\mid D/w^*}\frac{\tilde{w}}{wD\varphi(\tilde{w})}\left|\sum_m a_m m^{it}S_{\psi^*}(m;w)\right|^2,$$

where the ψ^*-sum is over primitive characters only. To compute $S_{\psi^*}(m;w)$, we decompose w as

$$w = w^*w'w'',\quad w'|(w^*)^\infty,\quad (w'',w^*)=1.$$

Our aim is now to establish (3.13) below, i.e. for $m = qm'$ with $(q,m')=1$ we want to express $S_{\psi^*}(qm',w)$ in terms of $S_{\psi^*}(dm',*)$ with $d\,|\,(q,D)$. Let us first note that according to our decomposition of w the Gauss sum factors as

$$\begin{aligned}G_{\psi^*}(m;\gamma w) &= \psi^*(\gamma w'')G_{\psi^*}(m;w^*w')r(m;\gamma w'')\\ &= \delta_{w'|m}w'\psi^*(\gamma w'')G_{\psi^*}(m/w';w^*)r(m;\gamma w''),\end{aligned}$$

where $r(m;q) := G_1(m;q)$ is the Ramanujan sum and $\delta_{w'|m}=1$ if $w'|m$ and else it vanishes. With this notation,

$$\begin{aligned}S_{\psi^*}(m;w) &= \frac{\delta_{w'|m}w'\overline{\psi^*}(m/w')\psi^*(w'')G_{\psi^*}(1;w^*)}{L^{(D)}(\psi^{*2},1+2it)}\\ &\quad\times\Bigg(\sum_{\substack{\gamma|D^\infty\\(\gamma,D/w)=1}}\frac{\psi^{*2}(\gamma)}{\gamma^{1+2it}}r(m;\gamma w'')\Bigg)\Bigg(\sum_{\substack{a|m\\(a,D)=1}}\frac{\psi^{*2}(a)}{a^{2it}}\Bigg)\\ &= \frac{\delta_{w'|m}w'\overline{\psi^*}(m/w')\psi^*(w'')G_{\psi^*}(1;w^*)}{L^{(D)}(\psi^{*2},1+2it)}R_{\psi^*}(m;w'')\eta_{\psi^*}(m),\end{aligned}$$

say, where the superscript (D) indicates that the local factors at the primes dividing D have been removed. We consider the γ-sum

$$R_{\psi^*}(m;w'') = \sum_{\substack{\gamma|D^\infty\\(\gamma,D/w)=1}}\frac{\psi^{*2}(\gamma)}{\gamma^{1+2it}}r(m;\gamma w'').$$

Since $(\gamma,w^*)=(\gamma,D/w)=1$, it follows that in fact $\gamma|(w'')^\infty$ (justifying our notation), and one has

$$R_{\psi^*}(m;w'') = \prod_{\substack{p^\alpha \| w'' \\ p^\alpha \| D}} \left(\sum_{\beta \geq 0} \frac{\psi^{*2}(p^\beta)}{p^{\beta(1+2it)}} r(p^{v_p(m)}; p^{\alpha+\beta}) \right).$$

We suppose now that m is of the form $m = qm'$ with $(q,m') = 1$, and we factor w', w'' (recall that they are coprime) as

$$w' = w'_q w'^{(q)},\ w'' = w''_q w''^{(q)}, \quad \text{where } w'_q, w''_q | q^\infty \text{ and } (w'^{(q)} w''^{(q)}, q) = 1.$$

Moreover, since $(q,m') = 1$ and $w' \mid qm'$, it follows that $w'_q = (w',q)$. With these notations we find that

$$S_{\psi^*}(qm';w) = \delta_{w'_q|q} w'_q \overline{\psi^*}\left(\frac{q}{w'_q}\right) \psi^*(w''_q) R_{\psi^*}(q;w''_q) \eta_{\psi^*}(q)$$

$$\times \delta_{w'^{(q)}|m'} w'^{(q)} \overline{\psi^*}\left(\frac{m'}{w'^{(q)}}\right) \psi^*(w''^{(q)}) \frac{G_{\psi^*}(1;w^*)}{L^{(D)}(\psi^{*2}, 1+2it)} R_{\psi^*}(m';w''^{(q)}) \eta_{\psi^*}(m').$$

Setting

$$v_q := \left(\frac{w''}{(w'',2)}, q\right) \quad \text{and} \quad \tilde{v}_q := \prod_{\substack{p^\alpha \| w''_q \\ \alpha \leq v_p(q)+1}} p^\alpha,$$

an explicit calculation shows that $R_{\psi^*}(v_q;\tilde{v}_q)$ is nonzero and in fact

$$\left| \frac{R_{\psi^*}(q;w''_q)}{R_{\psi^*}(v_q;\tilde{v}_q)} \right| \leq \prod_{p|(q,D)} (v_p(q)+1) \frac{1+\frac{1}{p}}{\kappa(p)} \leq 3\tau(D)\tau(q), \tag{3.12}$$

where

$$\kappa(p) := \begin{cases} \frac{1}{2}, & p = 2; \\ 1 - \frac{2}{p}, & p > 2. \end{cases}$$

Note also that $w'_q v_q \mid (q,D)$ and $\tilde{v}_q | w''_q$. In particular, $(w'_q v_q, m') = 1$ and $\eta_{\psi^*}(w'_q v_q) = 1$, therefore by the above formula

$$S_{\psi^*}(qm';w) = \delta_{w'_q|q} \overline{\psi^*}\left(\frac{q}{w'_q v_q}\right) \psi^*\left(\frac{w''_q}{\tilde{v}_q}\right) \frac{R_{\psi^*}(q;w''_q)}{R_{\psi^*}(v_q;\tilde{v}_q)} \eta_{\psi^*}(q) S_{\psi^*}(w'_q v_q m'; w^* w' \tilde{v}_q w''^{(q)}). \tag{3.13}$$

Combining (3.12) with (3.13) and noting that $|\eta_{\psi^*}(q)| \leq \tau(q)$ we obtain by a trivial estimation

$$\left| \sum_m a_m m^{it} S_\psi(m;w) \right|^2 \leq 9\tau(D)^2 \tau(q)^4 \sum_{d|(q,D)} \left| \sum_{(m',q)=1} a_{qm'} (m')^{it} S_\psi(dm'; w^* w' \tilde{v}_q w''^{(q)}) \right|^2.$$

Given w^* such that $w^{*2}|D$ and given w such that $w^*|w|\frac{D}{w^*}$, the number $w_1 := w^*w'\tilde{v}_q w''^{(q)}$ also satisfies $w^*|w_1|w|\frac{D}{w^*}$. Moreover, if we set $\tilde{w}_1 := (w_1, D/w_1)$, one has

$$\frac{\tilde{w}}{\varphi(\tilde{w})} \leq \tau(D)\frac{\tilde{w}_1}{\varphi(\tilde{w}_1)},$$

so we deduce from this discussion and from (3.11) that

$$\begin{aligned} \sum_{\mathfrak{a}}\left|\sum_m a_m\sqrt{m}\rho_{\mathfrak{a}}(m,t)\right|^2 &\leq 9\tau(D)^3\tau(q)^4 \sum_{d|(q,D)} \frac{\pi}{\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2} \\ &\quad \times \sum_{w^{*2}|D}\ \sum_{\psi^* \bmod w^*}\ \sum_{w^*|w_1|D} \frac{\tilde{w}_1}{w_1 D\varphi(\tilde{w}_1)}\left|\sum_{(m',q)=1} a_{qm'}(m')^{it} S_\psi(dm'; w_1)\right|^2 \\ &= 9\tau(D)^3\tau(q)^4 \sum_{d|(q,D)}\sum_{\mathfrak{a}}\left|\sum_{(m',q)=1} a_{qm'}\sqrt{dm'}\rho_{\mathfrak{a}}(dm',t)\right|^2. \end{aligned}$$

We are now in a similar situation as in (3.8). Applying the large sieve inequality (2.18) we obtain analogously the bound

$$\begin{aligned} T^{\text{Eisen}} &\ll_{\varepsilon,p_1,p_2} ((1+X)DZqr)^{\varepsilon}\left(\frac{Z}{|\xi_1|M}\right)^{p_1}\left(\frac{Z}{|\xi_2|N}\right)^{p_2} MN \\ &\quad \times \left(Z^{3/2}+ZX+\frac{(q,D)M}{qD}\right)^{1/2}\left(Z^{3/2}+ZX+\frac{(r,D)N}{rD}\right)^{1/2}\|a\|_2\|b\|_2 \end{aligned} \tag{3.14}$$

for any $\varepsilon > 0$.

Collecting (3.4), (3.9), (3.10), (3.14), and integrating over the ξ_1, ξ_2 variables (with $p_1 = 0$ if $|\xi_1| \leq Z/M$, and $p_1 > 1$ if $|\xi_1| > Z/M$ and similarly for (ξ_2, p_2)) we conclude the proof of Theorem 4. □

4 A shifted convolution problem

Let q, l_1, l_2 be positive integers, and g be a cuspidal newform of level D and nebentypus χ_g and Hecke eigenvalues $\lambda_g(n)$, $n \geq 1$. Let $F(x, y)$ be a smooth function supported on $[X/2, 2X] \times [Y/2, 2Y]$ which satisfies

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j}F(x,y) \ll \frac{Z^{i+j}}{X^iY^j} \tag{4.1}$$

for some $X, Y, Z \geq 1$ and for all $i, j \geq 0$, the implied constant depending only on i, j.

In this section we estimate, for positive integers l_1, l_2, q, the following average of shifted convolution sums:

$$\mathscr{D}(g, l_1, l_2, q) := \sum_{h \neq 0} \phi(qh) \sum_{l_1 m - l_2 n = qh} \overline{\lambda_g}(m) \lambda_g(n) F(l_1 m, l_2 n),$$

where by symmetry we may assume that $Y \geq X$ and $\phi(qh)$ is a redundancy factor (borrowed from [DFI94a] to ease the forthcoming computations) arising from a smooth even function ϕ such that $\phi_{|[-2Y, 2Y]} \equiv 1$, $\operatorname{supp} \phi \subset [-4Y, 4Y]$ and $\phi^{(i)}(x) \ll_i Y^{-i}$. Our analysis follows closely Sections 4.1–5.2 of [HM04a], but we also make use of the essential ingredients of [Bl04b], namely the square mean bound for shifted convolution sums taken from [J96] and the spectral large sieve of [DI82] in our improved form of Theorem 4. The proof of Proposition 2.4 of [HM04a] yields the following uniform estimate for exponential sums associated with g:

$$S_g(\alpha, x) := \sum_{n \leq x} \lambda_g(n) e(n\alpha) \ll_\varepsilon (D\mu_g x)^\varepsilon \omega_g D^{1/2} \mu_g^{1/2} x^{1/2}, \tag{4.2}$$

where

$$\omega_g := \frac{\|g\|_\infty}{\|g\|_2}.$$

In the same proof we also demonstrated by an elementary argument that

$$\omega_g \ll_\varepsilon (D\mu_g)^\varepsilon D^{1/2} \mu_g^{3/2}. \tag{4.3}$$

With this bound at hand we derive the following square mean bound for shifted convolution sums, a variant of Lemma 3 in [J96] (cf. Lemma 3.2 in [Bl04b]): for $l_1, l_2 \in \mathbb{Z}$ and $x, y \geq 1$ we have by Rankin–Selberg theory

$$\begin{aligned} \sum_{h' \in \mathbb{Z}} \Bigg| \sum_{\substack{m \leq x, n \leq y \\ l_1 m \pm l_2 n = h'}} \overline{\lambda_g}(m) \lambda_g(n) \Bigg|^2 &= \int_0^1 |S_g(-l_1\alpha, x) S_g(\pm l_2 \alpha, y)|^2 \, d\alpha \\ &\ll_\varepsilon (D\mu_g x)^\varepsilon \omega_g^2 D\mu_g x \int_0^1 |S_g(\pm l_2 \alpha, y)|^2 \, d\alpha \\ &= (D\mu_g x)^\varepsilon \omega_g^2 D\mu_g x \sum_{n \leq y} |\lambda_g(n)|^2 \\ &\ll_\varepsilon (D\mu_g xy)^{2\varepsilon} \omega_g^2 D\mu_g xy \end{aligned} \tag{4.4}$$

the implied constant depending on ε alone. Our goal is

Theorem 5. *Assume Hypothesis H_θ. Set*

$$T := qD\mu_g l_1 l_2 XYZ,$$

and assume (by symmetry) that $Y \geq X$. Then

$$\mathscr{D}(g, l_1, l_2, q) \ll_\varepsilon T^\varepsilon \omega_g D \mu_g^{63/4+3\theta} Z^{25/4+\theta} q^{\theta-1/2} Y \left(\left(\frac{Y}{X} \right)^{1/2} + \frac{(q, Dl_1 l_2)}{qDl_1 l_2} Y \right)^{1/2}.$$

The remaining part of this section is devoted to the proof of this theorem.

4.1. Setting up the circle method. We detect the summation condition $l_1 m - l_2 n - qh = 0$ by means of additive characters:

$$\mathscr{D}(g, l_1, l_2, q) = \int_{\mathbb{R}} G(\alpha) 1_{[0,1]}(\alpha)\, d\alpha$$

with

$$G(\alpha) := H(\alpha)K(\alpha) := \sum_{h \neq 0} \phi(qh) e(-\alpha q h) \times \sum_{m,n \geq 1} \overline{\lambda_g}(m) \lambda_g(n) F(l_1 m, l_2 n) e(\alpha(l_1 m - l_2 n)).$$

As in [H03a, HM04a, Bl04b], we apply Jutila's method of overlapping intervals [J92, J96] to approximate the characteristic function of the unit interval $I(\alpha) = 1_{[0,1]}(\alpha)$ by sums of characteristic functions of intervals centered at well chosen rationals. Let $C \geq Y$ be a large parameter to be chosen later, and let w be a smooth function supported on $[C/2, 3C]$ with values in $[0,1]$ equal to 1 on $[C, 2C]$ such that $w^{(i)}(x) \ll_i C^{-i}$. We also set

$$\delta := C^{-1}, \quad D' := Dl_1 l_2, \quad L := \sum_{c \equiv 0\ (D')} w(c)\varphi(c), \quad P := TC = qD\mu_g l_1 l_2 XYZC.$$

Note that, assuming $C \geq D'$, L satisfies the inequality

$$L \gg_\varepsilon \frac{C^{2-\varepsilon}}{Dl_1 l_2} \tag{4.5}$$

for any $\varepsilon > 0$. The approximation to $I(\alpha)$ is provided by

$$\tilde{I}(\alpha) := \frac{1}{2\delta L} \sum_{c \equiv 0\ (D')} w(c) \sum_{\substack{a\ (\mathrm{mod}\ c) \\ (a,c)=1}} 1_{[a/c-\delta,\, a/c+\delta]}(\alpha)$$

(which is supported in $[-1, 2]$), and by the main theorem in [J92] one has

$$(4.6)\qquad \int_{[-1,2]} |I(\alpha) - \tilde{I}(\alpha)|^2\, d\alpha \ll_\varepsilon \frac{C^{2+\varepsilon}}{\delta L^2} \ll_\varepsilon C^{3\varepsilon} \frac{(Dl_1 l_2)^2}{C}.$$

Next, we introduce the corresponding approximation of $\mathscr{D}(g, l_1, l_2, q)$:

$$\tilde{\mathscr{D}}(g, l_1, l_2, q) := \int_{[-1,2]} G(\alpha)\tilde{I}(\alpha)\, d\alpha.$$

Then it follows from (4.6) that

$$|\mathscr{D}(g, l_1, l_2, q) - \tilde{\mathscr{D}}(g, l_1, l_2, q)| \le \|I - \tilde{I}\|_2 \|G\|_2 \ll_\varepsilon C^{2\varepsilon} \frac{Dl_1 l_2}{C^{1/2}} \|G\|_2.$$

By Parseval,

$$\|G\|_2 \le \|H\|_2 \|K\|_\infty \ll \left(\frac{Y}{q}\right)^{1/2} \|K\|_\infty,$$

while an integration by parts shows that

$$K(\alpha) = l_1 l_2 \int_0^\infty \int_0^\infty F^{(1,1)}(l_1 x, l_2 y) \overline{S_g(-l_1\alpha, x)} S_g(-l_2\alpha, y)\, dx\, dy,$$

so that by (4.1) and (4.2),

$$\|K\|_\infty \ll_\varepsilon T^{2\varepsilon} \omega_g^2 D \mu_g Z^2 \left(\frac{XY}{l_1 l_2}\right)^{1/2}.$$

Collecting the above estimates, we find that

$$(4.7)\qquad \mathscr{D} - \tilde{\mathscr{D}} \ll_\varepsilon P^{2\varepsilon} \omega_g^2 D^2 \mu_g Z^2 \left(\frac{l_1 l_2 X Y^2}{qC}\right)^{1/2}.$$

4.2. $\tilde{\mathscr{D}}$ as a sum of Kloosterman sums. We have

$$\tilde{\mathscr{D}} = \frac{1}{L} \sum_{c \equiv 0\ (D')} w(c) \sum_{\substack{a\ (\mathrm{mod}\ c)\\ (a,c)=1}} \mathfrak{I}_{\delta, a/c},$$

where

$$\mathfrak{I}_{\delta, a/c} := \sum_h e\left(\frac{-aqh}{c}\right) \sum_{m,n} \overline{\lambda_g}(m) \lambda_g(n) e\left(\frac{al_1 m}{c}\right) e\left(\frac{-al_2 n}{c}\right) E(m, n, h)$$

and

$$E(x,y,z) := F(l_1x, l_2y)\phi(qz)\frac{1}{2\delta}\int_{-\delta}^{\delta} e(\alpha(l_1x - l_2y - qz))\,d\alpha.$$

By applying Proposition 2.1 to the variables m, n and by summing over a, c, we get (observe that the factor $\bar{\chi}_g(\bar{a})$ from the m-sum is cancelled by $\chi_g(\bar{a})$ coming from the n-sum) together with (2.5)

$$\tilde{\mathscr{D}} = \sum_{\pm,\pm} \varepsilon_g^{\pm}\varepsilon_g^{\pm}\tilde{\mathscr{D}}^{\pm,\pm},$$

where

$$\tilde{\mathscr{D}}^{\pm,\pm} := \frac{1}{L}\sum_{m,n}\overline{\lambda_g}(m)\lambda_g(n)\sum_{c\equiv 0\ (D')}\sum_{h\neq 0}\frac{S(qh, \mp l_1m \pm l_2n; c)}{c}\mathscr{E}^{\pm,\pm}(m,n,h;c), \tag{4.8}$$

and

$$\mathscr{E}^{\pm,\pm}(m,n,h;c) := \frac{l_1l_2w(c)}{c}\int_0^\infty\int_0^\infty E(x,y,h)J_g^{\pm}\left(\frac{4\pi l_1\sqrt{mx}}{c}\right)J_g^{\pm}\left(\frac{4\pi l_2\sqrt{ny}}{c}\right)dx\,dy$$

and where $\varepsilon_g^+ = 1$ and $\varepsilon_g^- = \pm 1$ is the sign of g if it is not induced from a holomorphic form, i.e., ε_g^- satisfies $\rho_g(-n) = \varepsilon_g^- \frac{\Gamma\left(\frac{1}{2}+it+\frac{k}{2}\right)}{\Gamma\left(\frac{1}{2}+it-\frac{k}{2}\right)}\rho_g(n)$ for all $n \geq 1$.

4.3. Estimates for $\mathscr{E}^{\pm,\pm}$ and its derivatives. Notice that the definition of E and the various assumptions made so far imply that

$$E(x,y,z) = 0 \quad \text{unless } x \sim X/l_1,\ y \sim Y/l_2,\ |qz| \leq 4Y. \tag{4.9}$$

Moreover,

$$E^{(i,j,k)}(x,y,z) \ll_{i,j,k} \frac{Z^{i+j}l_1^i l_2^j q^k}{X^i Y^{j+k}}, \tag{4.10}$$

so that for any fixed h

$$\|E^{(i,j,k)}(*,*,h)\|_1 \ll_{i,j,k} \frac{Z^{i+j}l_1^{i-1}l_2^{j-1}q^k XY}{X^i Y^{j+k}}, \tag{4.11}$$

and therefore

$$\|E^{(i,j,k)}\|_1 \ll_{i,j,k} \frac{Z^{i+j}l_1^{i-1}l_2^{j-1}q^{k-1}XY^2}{X^i Y^{j+k}}.$$

Next, we evaluate $\mathscr{E}^{\pm,\pm}(m,n,h;c)$ and its partial derivatives. Depending on the case, $\mathscr{E}^{\pm,\pm}(m,n,h;c)$ can be written as a linear combination (with constant coefficients) of integrals of the form

$$\frac{l_1 l_2 w(c)}{c}\int_0^\infty\int_0^\infty E(x,y,h)J_{1,\nu_1}\left(\frac{4\pi l_1\sqrt{mx}}{c}\right)J_{2,\nu_2}\left(\frac{4\pi l_2\sqrt{ny}}{c}\right)dx\,dy, \tag{4.12}$$

where

$$\{J_{1,\nu}(x),J_{2,\nu}(x)\}\subset\left\{\frac{Y_\nu(x)}{\mathrm{ch}(\pi t)},\mathrm{ch}(\pi t)K_\nu(x)\right\}$$

with $\nu\in\{\pm 2it_g\}$ if g is a Maass form of weight 0 and spectral parameter t_g; or

$$\{J_{1,\nu}(x),J_{2,\nu}(x)\}\subset\left\{\frac{Y_\nu(x)}{\mathrm{sh}(\pi t)},\mathrm{sh}(\pi t)K_\nu(x)\right\}$$

with $\nu\in\{\pm 2it_g\}$ if g is a Maass form of weight 1; or

$$J_{1,\nu}(x)=J_{2,\nu}(x)=J_{k_g-1}(x),$$

if g is a holomorphic form of weight k_g.

In order to estimate (4.12) efficiently, we integrate by parts i (resp. j) times with respect to x (resp. y), where i (resp. j) will be determined later in terms of m (resp. n) and ε. Using (6.1), we see that $\mathscr{E}^{\pm,\pm}(m,n,h;c)$ can be written as a linear combination (with constant coefficients) of expressions of the form

$$\frac{l_1 l_2 w(c)}{c}\left(\frac{l_1\sqrt{m}}{c}\right)^{-2i}\left(\frac{l_2\sqrt{n}}{c}\right)^{-2j}\int_0^\infty\int_0^\infty\frac{\partial^{i+j}}{\partial x^i\partial y^j}\{E(x,y,h)W_1^{-\nu_1}W_2^{-\nu_2}\}$$
$$\times W_1^{\nu_1+i}W_2^{\nu_2+j}J_{1,\nu_1+i}(W_1)J_{2,\nu_2+j}(W_2)\,dx\,dy,$$

where $\{\nu_1,\nu_2\}\subset\{\pm 2it_g\}$ (or $\nu_1,\nu_2=k_g-1$) and

$$W_1:=\frac{4\pi l_1\sqrt{mx}}{c}\sim\frac{\sqrt{ml_1X}}{C},\quad W_2:=\frac{4\pi l_2\sqrt{ny}}{c}\sim\frac{\sqrt{nl_2Y}}{C},$$

in view of (4.9). Using (4.11) and Proposition 6.2 in the slightly weaker form

$$J_{1,\nu_1+i}(W_1)\ll_{i,\varepsilon}\mu_g^{i+\varepsilon}(1+W_1^{-1})^{i+2|\Im t_g|+\varepsilon}(1+W_1)^{-1/2},$$

$$J_{2,\nu_2+j}(W_2)\ll_{j,\varepsilon}\mu_g^{j+\varepsilon}(1+W_2^{-1})^{j+2|\Im t_g|+\varepsilon}(1+W_2)^{-1/2},$$

we can deduce for any $i, j \geq 0$ that

$$\mathscr{E}^{\pm,\pm}(m,n,h;c) \ll_{i,j,\varepsilon} P^{\varepsilon}(\mu_g^2 Z)^{i+j}\left\{\frac{C^2}{l_1 mX} + \left(\frac{C^2}{l_1 mX}\right)^{1/2}\right\}^{i} \times \left\{\frac{C^2}{l_2 nY} + \left(\frac{C^2}{l_2 nY}\right)^{1/2}\right\}^{j} \Xi(m,n),$$

where

$$\Xi(m,n) := \frac{XY}{C}\left\{\left(1+\frac{C^2}{l_1 mX}\right)\left(1+\frac{C^2}{l_2 nY}\right)\right\}^{|\Im t_g|}\left\{\left(1+\frac{l_1 mX}{C^2}\right)\left(1+\frac{l_2 nY}{C^2}\right)\right\}^{-1/4}. \tag{4.13}$$

This shows, upon choosing i and j appropriately, that $\mathscr{E}^{\pm,\pm}(m,n,h;c)$ is very small unless

$$q|h| \leq 4Y, \quad c \sim C, \quad m \ll_{\varepsilon} P^{\varepsilon}\frac{\mu_g^4 Z^2 C^2}{l_1 X}, \quad n \ll_{\varepsilon} P^{\varepsilon}\frac{\mu_g^4 Z^2 C^2}{l_2 Y}, \tag{4.14}$$

and in this range we retain the bound (by taking $i = j = 0$)

$$\mathscr{E}^{\pm,\pm}(m,n,h;c) \ll_{\varepsilon} P^{\varepsilon}\Xi(m,n). \tag{4.15}$$

The partial derivatives

$$m^i n^j h^k c^l \frac{\partial^{i+j+k+l}}{\partial m^i \partial n^j \partial h^k \partial c^l}\mathscr{E}^{\pm,\pm}(m,n,h;c)$$

can be estimated similarly. We shall restrict our attention to the range (4.14). The same argument as above yields that outside this range the partial derivatives are very small. By (6.1) applied to the m and n variables and (6.2) applied to the c variable, the above partial derivative is a linear combination of integrals of the form

$$R_l(t_g)c^{a_3}\frac{\partial^{a_3}}{\partial c^{a_3}}\left(\frac{w(c)}{c}\right)\int_0^{\infty}\int_0^{\infty} h^k \frac{\partial^k}{\partial h^k}E(x,y,h)W_1^{a_1}W_2^{a_2}J_{1,\nu_1-a_1}(W_1)J_{2,\nu_2-a_2}(W_2)\,dx\,dy,$$

where R_l is a polynomial of degree $\leq l$ and the nonnegative integers a_1, a_2, a_3 satisfy

$$a_1 + a_2 + a_3 \leq i + j + l.$$

Therefore we obtain using (4.14)

$$(4.16)\quad m^i n^j h^k c^l (\mathscr{E}^{\pm,\pm})^{(i,j,k,l)}(m,n,h;c)$$

$$\ll_{i,j,k,l,\varepsilon} P^{\varepsilon}\left(\frac{q|h|}{Y}\right)^k \mu_g^l \left(1+\frac{\sqrt{l_1 m X}}{C}+\frac{\sqrt{l_2 n Y}}{C}\right)^{i+j+l} \Xi(m,n)$$

$$\ll_{i,j,k,l,\varepsilon} P^{\varepsilon}\mu_g^l (P^{\varepsilon}\mu_g^2 Z)^{i+j+l}\Xi(m,n).$$

4.4. Bounding $\tilde{\mathscr{D}}^{\pm,\pm}$ via the large sieve. We only treat $\tilde{\mathscr{D}}^{-,-}$, the other terms being similar. To simplify notation, we rename $\tilde{\mathscr{D}}^{-,-}$ as $\tilde{\mathscr{D}}$ and $\mathscr{E}^{-,-}$ as $\mathscr{E}$. We collect the terms in the definition (4.8) according to

$$h' := l_1 m - l_2 n.$$

Thus we have the natural splitting

$$\tilde{\mathscr{D}} = \tilde{\mathscr{D}}^0 + \tilde{\mathscr{D}}^+ + \tilde{\mathscr{D}}^-,$$

where

$$\tilde{\mathscr{D}}^0 := \frac{1}{L}\sum_{l_1 m = l_2 n} \overline{\lambda_g}(m)\lambda_g(n) \sum_{c\equiv 0\ (D')}\sum_{h\neq 0}\frac{r(qh;c)}{c}\mathscr{E}(m,n,h;c)$$

with

$$r(qh;c) = S(qh,0;c) = \sum_{c'|(qh,c)} \mu(c/c')c'$$

the Ramanujan sum, and

$$(4.17)\quad \tilde{\mathscr{D}}^{\pm} := \frac{1}{L}\sum_{c\equiv 0\ (D')}\sum_{h\neq 0}\sum_{\pm h'>0}\frac{S(qh,h';c)}{c}\sum_{l_1 m - l_2 n = h'}\overline{\lambda_g}(m)\lambda_g(n)\mathscr{E}(m,n,h;c).$$

4.4.1. Bounding $\tilde{\mathscr{D}}^{\mathbf{0}}$. We set $l_1' := l_1/(l_1,l_2)$, $l_2' := l_2/(l_1,l_2)$; then

$$\tilde{\mathscr{D}}^0 = \frac{1}{L}\sum_{m\geq 1}\overline{\lambda_g}(l_2' m)\lambda_g(l_1' m)\sum_{c\equiv 0\ (D')}\frac{1}{c}\sum_{h\neq 0} r(qh;c)\mathscr{E}(l_2' m, l_1' m, h; c).$$

The c-sum equals

$$\sum_{c''}\frac{\mu(c'')}{c''}\sum_{(D'/(c'',D'))|c'}\sum_{h\neq 0}\mathscr{E}\left(l_2' m, l_1' m, \frac{c'}{(c',q)}h; c'c''\right),$$

therefore by (4.14) and (4.15) it is bounded by

$$\ll_\varepsilon P^\varepsilon \frac{Y}{q} \Xi(l_2'm, l_1'm) \sum_{c''} \sum_{\substack{(D'/(c'',D'))\mid c' \\ c'c'' \sim C}} \frac{\mu(c'')(c',q)}{c'c''} \ll P^\varepsilon \frac{(q, Dl_1l_2)}{Dl_1l_2} \frac{Y}{q} \Xi(l_2'm, l_1'm).$$

In summing over the m variable we may restrict ourselves to the range

$$[l_1, l_2]m \ll_\varepsilon P^\varepsilon \mu_g^4 Z^2 (C^2/Y),$$

as the remaining contribution is negligible. If $Y/X \ll_\varepsilon P^\varepsilon \mu_g^4 Z^2$, then we split the m-sum into three parts,

$$\sum_{[l_1,l_2]m < C^2/Y} \cdots + \sum_{C^2/Y \le [l_1,l_2]m < C^2/X} \cdots + \sum_{C^2/X \le [l_1,l_2]m \ll_\varepsilon P^\varepsilon \mu_g^4 Z^2 (C^2/Y)} \cdots,$$

and combine (4.5) and (4.13) with basic properties of the Hecke eigenvalues $\lambda_g(n)$ to infer that

$$\tilde{\mathscr{D}}^0 \ll_\varepsilon P^{2\varepsilon} \frac{(q, Dl_1l_2)}{q[l_1,l_2]^{1-\theta}} \frac{XY^2}{C} (X^{-\theta} Y^{\theta-1} + X^{-3/4} Y^{-1/4} + \mu_g^2 Z X^{-1/4} Y^{-3/4}).$$

If $Y/X \gg_\varepsilon P^\varepsilon \mu_g^4 Z^2$, then we split the m-sum into two parts,

$$\sum_{[l_1,l_2]m < C^2/Y} \cdots + \sum_{C^2/Y \le [l_1,l_2]m \ll_\varepsilon P^\varepsilon \mu_g^4 Z^2 (C^2/Y)} \cdots,$$

and infer similarly that

$$\tilde{\mathscr{D}}^0 \ll_\varepsilon P^{2\varepsilon} \frac{(q, Dl_1l_2)}{q[l_1,l_2]^{1-\theta}} \frac{XY^2}{C} (X^{-\theta} Y^{\theta-1} + \mu_g^{3-4\theta} Z^{3/2-2\theta} X^{-\theta} Y^{\theta-1}).$$

In both cases we conclude that

$$\tilde{\mathscr{D}}^0 \ll_\varepsilon P^{3\varepsilon} \mu_g^2 Z \frac{X^{3/4} Y^{5/4}}{C}. \tag{4.18}$$

4.4.2. Bounding $\tilde{\mathscr{D}}^\pm$. Following [Bl04b], we decompose the inner sum in (4.17) as

$$\int_1^\infty \sum_{\substack{n \le y \\ l_1m - l_2n = h'}} \overline{\lambda_g}(m) \lambda_g(n) \frac{\partial}{\partial y} \mathscr{E}(x, y, h; c)\, dy, \tag{4.19}$$

where the variables x and y are connected by the equation

$$l_1 x - l_2 y = h' \tag{4.20}$$

(note that $h', y > 0$ implies $x > 0$). Then $\tilde{\mathscr{D}}^+$ decomposes accordingly as

$$\tilde{\mathscr{D}}^+ = \frac{1}{L}\int_1^\infty \tilde{\mathscr{D}}^+(y)\,dy, \tag{4.21}$$

where

$$\tilde{\mathscr{D}}^+(y) := \sum_{c\equiv 0\ (D')}\sum_{h\neq 0}\sum_{h'>0} b_{y,h'}\frac{S(qh,h';c)}{c}g_y(qh,h';c) \tag{4.22}$$

and

$$b_{y,h'} := \sum_{\substack{n\le y\\ l_1m-l_2n=h'}} \overline{\lambda_g}(m)\lambda_g(n), \quad g_y(h,h';c) := \frac{\partial}{\partial y}\mathscr{E}\left(\frac{l_2y+h'}{l_1}, y, \frac{h}{q}; c\right). \tag{4.23}$$

In particular, $g_y(h,h';c)$ and all its partial derivatives are very small unless

$$|h| \le 4Y, \quad c \sim C, \quad x \ll_\varepsilon P^\varepsilon \frac{\mu_g^4 Z^2 C^2}{l_1 X}, \quad y \ll_\varepsilon P^\varepsilon \frac{\mu_g^4 Z^2 C^2}{l_2 Y}, \tag{4.24}$$

and in this range they obey by (4.16) and (4.20) the bound

$$\begin{aligned} &h^i h'^j c^k g_y^{(i,j,k)}(h,h';c)\\ &\quad = \left(\frac{h}{q}\right)^i\left(\frac{h'}{l_1}\right)^j c^k\left\{\frac{l_2}{l_1}\mathscr{E}^{(1+j,0,i,k)}\left(x,y,\frac{h}{q};c\right) + \mathscr{E}^{(j,1,i,k)}\left(x,y,\frac{h}{q};c\right)\right\}\\ &\quad \ll_{i,j,k,\varepsilon} \mu_g^{2+2j+3k}(P^\varepsilon Z)^{1+j+k}y^{-1}\Xi(x,y). \end{aligned}$$

Using the definition (4.13) it can be checked that in the range (4.24) we have the uniform bound

$$(xy)^{1/2}\Xi(x,y) \ll_\varepsilon P^\varepsilon \mu_g^2 ZC\left(\frac{XY}{l_1l_2}\right)^{1/2} =: W, \tag{4.25}$$

so that in fact

$$h^i h'^j c^k g_y^{(i,j,k)}(h,h';c) \ll_{i,j,k,\varepsilon} \mu_g^{2+2j+3k}(P^\varepsilon Z)^{1+j+k} W x^{-1/2} y^{-3/2}.$$

This also shows that we can decompose $g_y(h,h';c)$ dyadically in the h and h' variables such that

$$g_y(h,h';c) = \sum_{H,H' \geq 1} g_{y,H,H'}(h,h';c),$$

where H and H' run through the powers of 2 independently, and $g_{y,H,H'}(h,h';c)$ as a function of h (resp. h') is supported on $H/2 \leq |h| \leq 2H$ (resp. $H'/2 \leq h' \leq 2H'$) and satisfies

$$(4.26) \quad g_{y,H,H'}^{(i,j,k)}(h,h';c) \ll_{i,j,k,\varepsilon} \frac{P^\varepsilon \mu_g^2 Z W}{\tilde{x}^{1/2} y^{3/2}} \cdot \frac{\tilde{Z}^{j+k}}{H^i H'^j C^k},$$

where

$$(4.27) \quad \tilde{Z} := P^\varepsilon \mu_g^3 Z,$$

and $\tilde{x}$ is connected to y by the equation

$$l_1 \tilde{x} - l_2 y = H'.$$

The same argument also shows that all these partial derivatives are very small unless

$$(4.28) \quad H \leq 8Y, \quad H' \ll_\varepsilon P^\varepsilon \frac{\mu_g^4 Z^2 C^2}{X}, \quad c \sim C, \quad y \ll_\varepsilon P^\varepsilon \frac{\mu_g^4 Z^2 C^2}{l_2 Y}.$$

Accordingly, $\tilde{\mathscr{D}}^+(y)$ of (4.22) decomposes into a double sum over the H and the H'. We further decompose all the pieces according to the q-part of the h variable and we find (after replacing qh by h) that

$$(4.29) \quad \tilde{\mathscr{D}}^+(y) = \sum_{H,H' \geq 1} \sum_{q' \mid q^\infty} \tilde{\mathscr{D}}^+_{H,H',q'}(y),$$

where

$$\tilde{\mathscr{D}}^+_{H,H',q'}(y) := \sum_{c \equiv 0\ (D')} \sum_{\substack{qq' \mid h \\ (h/qq',qq')=1}} \sum_{h'>0} b_{y,h'} \frac{S(h,h';c)}{c} g_{y,H,H'}(h,h';c).$$

We are ready to apply the large sieve for the sums $D^+_{H,H',q'}(y)$. By Theorem 4 and (4.26),

$$\tilde{\mathscr{D}}^+_{H,H',q'}(y) \ll_\varepsilon \frac{P^\varepsilon \mu_g^2 Z W}{\tilde{x}^{1/2} y^{3/2}} (q'P)^{2\varepsilon} (qq')^\theta \left(1 + \frac{C}{\sqrt{HH'}}\right)^{2\theta}$$
$$\times \tilde{Z}^2 \left(\tilde{Z}^{3/2} + \tilde{Z}\frac{\sqrt{HH'}}{C} + \tilde{Z}^{2\theta} \frac{(qq', D')H}{qq'D'}\right)^{1/2}$$
$$\times \left(\tilde{Z}^{3/2} + \tilde{Z}\frac{\sqrt{HH'}}{C} + \tilde{Z}^{2\theta} \frac{H'}{D'}\right)^{1/2} \left(\frac{H}{qq'}\right)^{1/2} \|b_y\|_2.$$

Here, by (4.23) and (4.4),

$$\|b_y\|_2 \ll_\varepsilon P^\varepsilon \omega_g D^{1/2} \mu_g^{1/2} (\tilde{x}y)^{1/2}.$$

If we choose $C \geq \tilde{Z}^2 D' Y$, then by our general assumption $Y \geq X$ it follows that in the range (4.28)

$$\left(1 + \frac{C}{\sqrt{HH'}}\right)^{2\theta} \left(\tilde{Z}^{3/2} + \tilde{Z}\frac{\sqrt{HH'}}{C} + \tilde{Z}^{2\theta}\frac{H'}{D'}\right)^{1/2} \left(\frac{H}{qq'}\right)^{1/2} \ll_\varepsilon P^\varepsilon \mu_g^2 Z \tilde{Z}^\theta C \left(\frac{Y}{qq'D'X}\right)^{1/2}.$$

These additional estimates yield (for the relevant range (4.28))

$$\tilde{\mathscr{D}}^+_{H,H',q'}(y) \ll_\varepsilon y^{-1} (q'P)^{5\varepsilon} \omega_g D^{1/2} \mu_g^{11/2} Z^{5/2} \tilde{Z}^{11/4+\theta} C W (qq')^\theta$$
$$\times \left(\frac{Y}{qq'D'X}\right)^{1/2} \left(\left(\frac{Y}{X}\right)^{1/2} + \frac{(qq', D')}{qq'D'} Y\right)^{1/2},$$

and by the definition of W and $\tilde{Z}$ (see (4.25) and (4.27)) the right hand side is

$$\ll_\varepsilon y^{-1} \frac{C^2}{l_1 l_2} (q'P)^{11\varepsilon} \omega_g \mu_g^{63/4+3\theta} Z^{25/4+\theta} (qq')^{\theta-1/2} Y \left(\left(\frac{Y}{X}\right)^{1/2} + \frac{(q, D')}{qD'} Y\right)^{1/2}.$$

Finally, by (4.29), (4.28), (4.21) and (4.5) we conclude that

$$\tilde{\mathscr{D}}^+ \ll_\varepsilon P^{12\varepsilon} \omega_g D \mu_g^{63/4+3\theta} Z^{25/4+\theta} q^{\theta-1/2} Y \left(\left(\frac{Y}{X}\right)^{1/2} + \frac{(q, Dl_1 l_2)}{qDl_1 l_2} Y\right)^{1/2}. \tag{4.30}$$

The treatment of $\tilde{\mathscr{D}}^-$ is very similar to that of $\tilde{\mathscr{D}}^+$ except that instead of (4.19) we decompose the inner sum in (4.17) as

$$\int_1^\infty \sum_{\substack{m \leq x \\ l_1 m - l_2 n = h'}} \overline{\lambda_g}(m) \lambda_g(n) \frac{\partial}{\partial x} \mathscr{E}(x, y, h; c)\, dx.$$

4.5. Concluding Theorem 5. Theorem 5 follows immediately from (4.7), (4.18) and (4.30) upon choosing C to be a large power of T, say

$$C := (qD\mu_g l_1 l_2 XYZ)^{100}.$$

5 Application to subconvexity bounds for twisted L-functions

Let g be a cuspidal newform (i.e., either a holomorphic or a Maass cusp form) as in (1.1), let $\Re s = \frac{1}{2}$, and let χ be a primitive character of conductor q.

5.1. Approximate functional equation. Using the functional equation of the L-function attached to the cuspidal automorphic representation $\pi_g \otimes \chi = \bigotimes_v (\pi_{g,v} \otimes \chi_v)$ and a standard technique involving Mellin transforms, we can express the special value $L(g \otimes \chi, s) = L(\pi_g \otimes \chi, s)$ as a sum of two Dirichlet series of essentially $\sqrt{C}$ terms, where

$$C := C(\pi_g \otimes \chi, s) \leq |s|^2 \mu_g^2 D q^2$$

is the analytic conductor defined by [IS00]. For example, Theorem 1 in [H02] shows that

$$L(g \otimes \chi, s) = \sum_{n \geq 1} \frac{\lambda_{\pi_g \otimes \chi}(n)}{n^s} W\left(\frac{n}{C^{1/2}}\right) + \kappa \sum_{n \geq 1} \frac{\overline{\lambda_{\pi_g \otimes \chi}}(n)}{n^{1-s}} \overline{W}\left(\frac{n}{C^{1/2}}\right),$$

where κ is of modulus one, $\lambda_{\pi_g \otimes \chi}(n)$ are the coefficients of the Dirichlet series $L(g \otimes \chi, s)$, and W is a smooth bounded function of $(0, \infty)$ (depending on $\pi_g \otimes \chi$ and s) satisfying the uniform estimates

$$x^i W^{(i)}(x) \ll_{i,A} (1 + x)^{-A}$$

for any $x > 0$, any integer $i \geq 0$ and any $A \geq 0$ with implied constants depending only on i and A. The coefficient $\lambda_{\pi_g \otimes \chi}(n)$ may be complicated for n not coprime with qD, however for $\Re s > 1$ one has

$$L(g \otimes \chi, s) = \prod_{p \mid qD} \frac{1 - \lambda_g(p)\chi(p)p^{-s} + \chi(p^2)p^{-2s}}{L^{-1}(\pi_{g,p} \otimes \chi_p, s)} \sum_{n \geq 1} \frac{\lambda_g(n)\chi(n)}{n^s},$$

so that by (2.4)

$$\sum_{n \geq 1} \frac{\lambda_{\pi_g \otimes \chi}(n)}{n^s} W\left(\frac{n}{C^{1/2}}\right) \ll_\varepsilon \sum_{d \mid (qD)^\infty} \frac{1}{d^{1/2-\theta-\varepsilon}} \left| \sum_{n \geq 1} \frac{\lambda_g(n)\chi(n)}{n^s} W\left(\frac{dn}{C^{1/2}}\right) \right|$$

for any $\varepsilon > 0$, the implied constant depending only on ε.

By the rapid decay properties of W, the contribution of the d, n such that $dn \geq C^{1/2+\varepsilon}$ is negligible for any given $\varepsilon > 0$. For the rest of the sum we apply a smooth dyadic decomposition, so that we are left with estimating $O(\log C)$ sums of the form

$$N^{-1/2}\Sigma(g \otimes \chi, N) = N^{-1/2} \sum_{n \geq 1} \lambda_g(n)\chi(n) W_{N,s}(n),$$

where $1 \leq N \leq C^{1/2+\varepsilon}$ and $W_{N,s}$ is some smooth function supported in $[N, 2N]$ satisfying

$$x^i W_{N,s}^{(i)}(x) \ll_i |s|^i,$$

the implied constant depending only on i.

5.2. Amplification. We shall estimate the sums $\Sigma(g \otimes \chi, N)$ by the amplification method of [DFI93]. That is, we choose some L so that $\log L \asymp \log q$ and estimate the amplified second moment

$$S(g, N) := \sum_{\chi' \ (\mathrm{mod}\, q)} \left| \sum_{L \leq l \leq 2L} \bar{\chi}(l)\chi'(l) \right|^2 |\Sigma(g \otimes \chi', N)|^2,$$

where χ' runs over the characters modulo q and

$$\Sigma(g \otimes \chi', N) := \sum_{n \geq 1} \lambda_g(n)\chi'(n) W_{N,s}(n).$$

By orthogonality of characters one has (see [M04b])

$$S(g, N) \leq \varphi(q) \sum_{L \leq l_1, l_2 \leq 2L} \bar{\chi}(l_1)\chi(l_2) \sum_h \sum_{l_1 m - l_2 n = hq} \overline{\lambda_g}(m)\lambda_g(n) F(l_1 m, l_2 n),$$

where

$$F(x, y) := \overline{W_{N,s}(x/l_1)}\, W_{N,s}(y/l_2).$$

The total contribution of $h = 0$ can be estimated by Cauchy–Schwarz and the Rankin–Selberg bound:

$$\sum_{\substack{L \leq l_1, l_2 \leq 2L \\ l_1 m = l_2 n}} \overline{\lambda_g}(m)\lambda_g(n) F(l_1 m, l_2 n) \ll_\varepsilon T^\varepsilon NL, \quad T := |s|\mu_g Dq.$$

For each pair l_1, l_2 coprime with q, the contribution of $h \neq 0$ can be estimated directly by Theorem 5 with the parameters (when $l_1 \leq l_2$) $X = l_1 N,\ Y = l_2 N,\ Z = |s|$:

$$\sum_{h\neq 0}\sum_{l_1m-l_2n=hq}\overline{\lambda_g}(m)\lambda_g(n)F(l_1m,l_2n)\ll_\varepsilon T^\varepsilon Uq^{\theta-1/2}NL\left(1+\frac{N}{qL}\right)^{1/2},$$

where we have put for convenience

$$(5.1)\qquad U := |s|^{25/4+\theta}\omega_g\mu_g^{63/4+3\theta}D.$$

We also note that

$$(5.2)\qquad N\leq T^\varepsilon Vq,\quad V := |s|\mu_g D^{1/2},$$

whence the obvious lower bound

$$N^{-1/2}\Sigma(g\otimes\chi,N)\ll_\varepsilon q^\varepsilon\left(\frac{S(g,N)}{NL^2}\right)^{1/2}$$

together with the above estimates imply that

$$(5.3)\qquad L(g\otimes\chi,s)\ll_\varepsilon T^{4\varepsilon}\left\{\frac{q}{L}+Uq^{1/2+\theta}L\left(1+\frac{V}{L}\right)^{1/2}\right\}^{1/2}.$$

The expression on the right hand side suggests that we choose L of the form

$$(5.4)\qquad L := q^{(1-2\theta)/4}/R,$$

where $R\gg 1$ depends only on s, μ_g and D. If R is not too large then we can guarantee with the help of the convexity bound

$$L(g\otimes\chi,s)\ll_\varepsilon T^\varepsilon V^{1/2}q^{1/2}$$

that $L\geq q^\delta$ with some $\delta>0$. More precisely, the bound (5.3) follows from the convexity bound unless

$$Uq^{1/2+\theta}L\left(1+\frac{V}{L}\right)^{1/2}\ll Vq,$$

which by (5.4) is equivalent to

$$\max\left\{\frac{U}{VR^2},\left(\frac{U^2}{VR^4}\right)^{1/3}\right\}\ll L.$$

Fixing any $\delta>0$, this inequality can be rewritten as

$$\max\left\{\frac{U}{VR^2},\left(\frac{U^2}{VR^4}\right)^{1/3}\right\}^{1-\delta}\left(\frac{q^{(1-2\theta)/4}}{R}\right)^{\delta}\ll L,$$

which implies that

$$R^{\delta/(1-\delta)}\ll\max\left\{\frac{U}{VR^2},\left(\frac{U^2}{VR^4}\right)^{1/3}\right\}\Rightarrow q^{\delta(1-2\theta)/4}\ll L.$$

This justifies the choice

$$R:=\left(\frac{U^2}{V}\right)^{(1-\delta)/(4-\delta)},\tag{5.5}$$

since then (5.3) holds true either because L is admissible or as a consequence of the convexity bound.

5.3. Concluding Theorem 1. We choose L according to (5.4)–(5.5), where δ is a very small positive number depending on ε and U, V are defined in (5.1)–(5.2). Then (5.3) implies that

$$L(g\otimes\chi,s)\ll_{\varepsilon}T^{4\varepsilon}\{UV^{1/2}q^{1/2+\theta}L\}^{1/2}\ll_{\varepsilon}T^{5\varepsilon}U^{1/4}V^{3/8}q^{(3+2\theta)/8}.$$

In view of (4.3), this is the bound of Theorem 1.

Appendix 1: Bounds for Bessel functions

In this appendix we recall some facts about Bessel functions. Proofs of Propositions 6.1 and 6.2 can be found in the Appendix of [HM04a].

For $s\in\mathbb{C}$, the Bessel functions satisfy the recurrence relations

$$(x^sJ_s(x))'=x^sJ_{s-1}(x),\quad (x^sY_s(x))'=x^sY_{s-1}(x),\quad (x^sK_s(x))'=-x^sK_{s-1}(x).$$

In particular, if $r>0$ and H_s denotes either J_s, Y_s or K_s, then

$$\frac{d}{dx}((r\sqrt{x})^{s+1}H_{s+1}(r\sqrt{x}))=\pm(r^2/2)(r\sqrt{x})^sH_s(r\sqrt{x}),\tag{6.1}$$

and for any $j\geq 0$,

$$x^j\frac{d^j}{d^jx}H_s\left(\frac{r}{x}\right)=Q_j(s)H_s\left(\frac{r}{x}\right)+Q_{j-1}(s)\left(\frac{r}{x}\right)^1H_{s-1}\left(\frac{r}{x}\right)+\cdots+Q_0(s)\left(\frac{r}{x}\right)^jH_{s-j}\left(\frac{r}{x}\right),\tag{6.2}$$

where each Q_i is a polynomial of degree i whose coefficients depend only on i and j.

Proposition 6.1. *For any integer $k \geq 1$, the following uniform estimate holds:*

$$J_{k-1}(x) \ll \begin{cases} \frac{x^{k-1}}{2^{k-1}\Gamma\left(k-\frac{1}{2}\right)}, & 0 < x \leq 1; \\ kx^{-1/2}, & 1 < x. \end{cases}$$

The implied constant is absolute.

Proposition 6.2. *For any $\sigma > 0$ and any $\varepsilon > 0$, the following uniform estimates hold in the strip $|\Re s| \leq \sigma$:*

$$e^{-\pi|\Im s|/2}\, Y_s(x) \ll \begin{cases} (1+|\Im s|)^{\sigma+\varepsilon} x^{-\sigma-\varepsilon}, & 0 < x \leq 1+|\Im s|; \\ (1+|\Im s|)^{-\varepsilon} x^{\varepsilon}, & 1+|\Im s| < x \leq 1+|s|^2; \\ x^{-1/2}, & 1+|s|^2 < x. \end{cases}$$

$$e^{\pi|\Im s|/2} K_s(x) \ll \begin{cases} (1+|\Im s|)^{\sigma+\varepsilon} x^{-\sigma-\varepsilon}, & 0 < x \leq 1+\pi|\Im s|/2; \\ e^{-x+\pi|\Im s|/2} x^{-1/2}, & 1+\pi|\Im s|/2 < x. \end{cases}$$

The implied constants depend only on σ and ε.

Appendix 2: Improved bound for the Fourier coefficients of holomorphic half-integral weight cusp forms

By Zhengyu Mao

In this appendix we apply the estimate obtained in Theorem 1 along with the work of [BM05] to get an improved upper bound for the Fourier coefficients of holomorphic half-integral weight cusp forms. For positive integers k and M and an even Dirichlet character χ modulo $4M$ let $S_{k+1/2}(4M,\chi)$ denote the space of holomorphic forms of weight $k+\frac{1}{2}$, level M, and nebentypus χ. The functions $f(z)$ in this space satisfy (cf. [Wa81])

$$f\left(\frac{az+b}{cz+d}\right) = j(\sigma,z)^{2k+1}\chi(d)f(z), \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4M), \tag{7.1}$$

where

$$j(\sigma,z) := \theta\left(\frac{az+b}{cz+d}\right)\Big/\theta(z), \quad \theta(z) := \sum_{n=-\infty}^{\infty} e(n^2 z).$$

Let $S'_{k+1/2}(4M,\chi)$ denote the orthogonal complement in $S_{k+1/2}(4M,\chi)$ of the space of theta series in one variable. Note that $S'_{k+1/2}(4M,\chi)$ is the entire space $S_{k+1/2}(4M,\chi)$

for $k \geq 2$, while for $k = 1$ it equals the subspace $V(4M;\chi)$ defined in [U93]. In the following we prove:

Theorem 6. *Let B, C, θ be the constants as in Theorem 1. If*

$$f(z) = \sum_{n=1}^{\infty} \rho_f(n)(4\pi n)^{k/2+1/4} e(nz)$$

is an L^2-normalized cusp form in $S'_{k+1/2}(4M,\chi)$ (cf. (2.1), (2.8)), then for $\varepsilon > 0$ and for any $n \geq 1$ whose square part is coprime with $2M$ we have

$$\sqrt{n}\rho_f(n) \ll_\epsilon (kMn)^\epsilon \left(\Gamma\left(k+\frac{1}{2}\right)\right)^{-1/2} k^{(B+1)/2} M^{C+1} n^{1/4-(1/16)(1-2\theta)}. \tag{7.2}$$

Remark 7.1. A similar bound holds for Maass forms. For example, when

$$f(x+iy) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \rho_f(n) W_{(n/|n|)(1/4),\,it}(4\pi|n|y)e(nx)$$

is an L^2-normalized Maass cusp form of weight $\frac{1}{2}$, level $4M$, and Laplacian eigenvalue $\frac{1}{4}+t^2$ ($t \in \mathbb{R}$), one has, for any $n \geq 1$ whose square part is coprime with $2M$,

$$\sqrt{n}\rho_f(n) \ll_\epsilon ((1+|t|)M|n|)^\epsilon (\mathrm{ch}(\pi t))^{1/2} (1+|t|)^{B'} M^{C'} |n|^{1/4-(1/16)(1-2\theta)}$$

for some positive constants B' and C'. We leave the details to a future work.

Remark 7.2. The first breakthrough in obtaining nontrivial bounds for the Fourier coefficients of half-integral weight cusp forms was achieved by Iwaniec [I87] and reads with slight refinements as (cf. Lemma 2 in [DSp90])

$$\sqrt{n}\rho_f(n) \ll_\epsilon (kMn)^\epsilon \left(\Gamma\left(k+\frac{1}{2}\right)\right)^{-1/2} k^{9/2} n^{1/4-1/28}$$

with the above normalization and under the above assumptions, *providing that n is square-free*. Estimates that are valid for all n have been obtained in [Bl04a], Theorem 1 and Lemma 4.4, which are particularly useful for applications with ternary quadratic forms. Combining (7.2) with Lemma 4.2 in [Bl04a], we obtain (cf. [Bl04a, p. 5] for the notation)

$$r(\mathrm{spn}\, f, n) - r(f,n) \ll_\varepsilon N^{41/16} n^{1/2-(1/16)(1-2\theta)+\epsilon} \tag{7.3}$$

for integers n whose square part is coprime with N.

7.1. Cusp forms and cuspidal automorphic representations. Let $\mathbb{A}_{\mathbb{Q}}$ be the ring of adeles of the rational field $\mathbb{Q}$. Define the additive character ψ of $\mathbb{A}_{\mathbb{Q}}$ such that $\psi_\infty(x) = e^{2\pi i x}$ over $\mathbb{Q}_\infty = \mathbb{R}$ and for $x \in \mathbb{Q}_p$, $\psi_p(x) = e^{-2\pi i \hat{x}}$ where $p^l \hat{x} \in \mathbb{Z}$ for some integer l and $x - \hat{x} \in \mathbb{Z}_p$, the ring of integers of $\mathbb{Q}_p$. The additive measures on $\mathbb{Q}_v$ are defined to be self-dual with respect to ψ_v for $v = p$ or ∞.

Let $f(z) \in S'_{k+1/2}(4M, \chi)$, then associated to $f(z)$ is a vector $\tilde{\varphi} = t(f)$ in the space of cuspidal automorphic representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_{\mathbb{Q}})$, the two-fold cover of $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$. Here $\tilde{\varphi} = t(f)$ is a function on $\widetilde{\mathrm{SL}}_2(\mathbb{A}_{\mathbb{Q}})$ which is continuous and left-invariant under $\mathrm{SL}_2(\mathbb{Q})$ and satisfies

$$\left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, 1, 1, \dots\right) = y^{k/2+1/4} e^{i(k+1/2)\theta} f(x + yi), \tag{7.4}$$

where $y > 0$, $x \in \mathbb{R}$, and $-\pi < \theta \le \pi$. By the strong approximation theorem for SL_2, we see that $\tilde{\varphi} = t(f)$ is unique.

It is clear that $\tilde{\varphi}$ decomposes into a sum of $\tilde{\varphi}_i = \bigotimes_v \tilde{\varphi}_{i,v}$, where each $\tilde{\varphi}_i$ is a vector in some irreducible cuspidal representation $\tilde{\pi}_i$ of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_{\mathbb{Q}})$. We will first establish the bound (7.2) in the case when $\tilde{\varphi} = t(f)$ itself is a vector $\tilde{\varphi} = \bigotimes_v \tilde{\varphi}_v$ in an irreducible cuspidal representation $\tilde{\pi}$ of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_{\mathbb{Q}})$.

In [Wa91] a map S_ψ is defined from the set of irreducible cuspidal representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_{\mathbb{Q}})$ to the set of irreducible automorphic representations of $\mathrm{PGL}_2(\mathbb{A}_{\mathbb{Q}})$. With our assumption of orthogonality to one-variable theta series, we see that $\pi := S_\psi(\tilde{\pi})$ is a cuspidal representation. Let $\varphi = \bigotimes_v \varphi_v$ be the unique (up to scalar multiple) new vector in the space of π (cf. [Ca73, Sc02]). Define

$$g(z) := \varphi(\sigma)(cz + d)^{-2k}, \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \quad z := \frac{ai + b}{ci + d},$$

where $\mathrm{SL}_2(\mathbb{R})$ is identified with its image in $\mathrm{PGL}_2(\mathbb{A}_{\mathbb{Q}})$ under the embedding $\sigma \mapsto (\sigma, 1, 1, \dots)$.

Then $g(z)$ is a newform of weight $2k$, some level N, and trivial nebentypus. We can be more precise on the size of N.

Lemma 7.1. *We have $N \mid (2M)^\infty$ and $N \le (4M)^2$.*

Proof. If p does not divide $2M$, then $\tilde{\pi}_p$ is unramified, thus so is π_p, which implies that N is not divisible by p. Let $c(\pi)$ denote the conductor of π (cf. [Sc02]), and let v_p denote the p-adic valuation on $\mathbb{Q}$. Clearly, the conductor of the character χ over $\mathbb{Q}_p^*$ is at most $v_p(4M)$. We see from Section 1.3.2 of [Wa81] that the newform corresponding to the representation $\pi \otimes \chi$ has level at most $2M$, that is, $c(\pi_p \otimes \chi) \le v_p(2M)$. Then, from the table of conductors of local representations (see [Sc02]), $c(\pi_p) \le 2v_p(4M)$, which is equivalent to $N \le (4M)^2$. $\square$

We adopt the notation (2.1) to the present situation, that is,

$$g(z) = \sum_{n \geq 1} \rho_g(n)(4\pi n)^k e(nz).$$

7.2. A theorem from [BM05]. We recall Theorem 4.3 in [BM05]. The statement we need is:

Theorem 7. *Let $\pi = S_\psi(\tilde{\pi})$. Let $S = \{\infty\} \cup \{p : p \mid 2M\}$ be a finite set of places. For all square-free integers $D \geq 1$, either $\tilde{W}^D(\tilde{\varphi}) = 0$ or*

$$\frac{|\tilde{W}^D(\tilde{\varphi})|^2}{\|\tilde{\varphi}\|^2} = \frac{|W(\varphi)|^2 L\left(\pi \otimes \chi_D, \frac{1}{2}\right)}{\|\varphi\|^2} \prod_{v \in S} E_v(\varphi_v, \tilde{\varphi}_v, \psi_v, D), \tag{7.5}$$

where

$$E_v(\varphi_v, \tilde{\varphi}_v, \psi_v, D) := \frac{e(\varphi_v, \psi_v)}{e(\tilde{\varphi}_v, \psi_v^D) L_v\left(\pi_v \otimes \chi_D, \frac{1}{2}\right) |D|_v}. \tag{7.6}$$ □

The notations are as in [BM05] which we will explain along the way. χ_D is the quadratic character $\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$ associated with the quadratic extension $\mathbb{Q}(\sqrt{D})$. $\tilde{W}^D$ and W are Whittaker functionals and are related to the Fourier coefficients:

$$\tilde{W}^D(\tilde{\varphi}) := \int \tilde{\varphi}\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \psi(-Dx)\, dx = e^{-2\pi D}(4\pi D)^{k/2+1/4} \rho_f(D), \tag{7.7}$$

$$W(\varphi) := \int \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \psi(-x)\, dx = e^{-2\pi}(4\pi)^k \rho_g(1). \tag{7.8}$$

The Petersson norms are related by (cf. (2.8))

$$\frac{\|\varphi\|^2}{\|\tilde{\varphi}\|^2} = \frac{\langle g, g\rangle/\mathrm{vol}(\Gamma_0(N)\backslash\mathbb{H})}{\langle f, f\rangle/\mathrm{vol}(\Gamma_0(4M)\backslash\mathbb{H})} = \langle g, g\rangle \frac{\mathrm{vol}(\Gamma_0(4M)\backslash\mathbb{H})}{\mathrm{vol}(\Gamma_0(N)\backslash\mathbb{H})}. \tag{7.9}$$

When $v = \infty$, $\tilde{\varphi}_v$ and φ_v are the lowest weight vectors in $\tilde{\pi}_\infty$ and π_∞, respectively. The constant $E_v(\varphi_v, \tilde{\varphi}_v, \psi_v, D)$ is computed in [BM05]; it equals

$$E_\infty(\varphi_\infty, \tilde{\varphi}_\infty, \psi_\infty, D) = \frac{1}{2} e^{4\pi(1-D)} D^{k-1/2} \pi^{-k} \Gamma(k) L\left(\pi_\infty \otimes \chi_D, \frac{1}{2}\right)^{-1}. \tag{7.10}$$

Note that

$$L\left(\pi \otimes \chi_D, \frac{1}{2}\right) = L_\infty\left(\pi_\infty \otimes \chi_D, \frac{1}{2}\right) L\left(g \otimes \chi_D, \frac{1}{2}\right) l_2(g, D),$$

where

$$l_2(g, D) := \frac{L_2\left(g \otimes \chi_D, \frac{1}{2}\right)}{L_2\left(\pi_2 \otimes \chi_D, \frac{1}{2}\right)}$$

is the quotient of the 2-factors of $L(g \otimes \chi_D, s)$ and $L(\pi \otimes \chi_D, s)$. Thus from Theorem 7 we have

$$D|\rho_f(D)|^2 \ll \frac{|\rho_g(1)|^2 4^k \Gamma(k)}{\langle g, g\rangle/\mathrm{vol}(\Gamma_0(N)\backslash \mathbb{H})} M^{-1} L\left(g \otimes \chi_D, \frac{1}{2}\right) l_2(g, D) \prod_{p|2M} E_p(\varphi_p, \tilde{\varphi}_p, \psi_p, D). \tag{7.11}$$

The method of [HL94] implies that (cf. [HM04a, Section 2.6], [DFI94b, p. 219])

$$\frac{|\rho_g(1)|^2 \Gamma(2k)}{\langle g, g\rangle/\mathrm{vol}(\Gamma_0(N)\backslash \mathbb{H})} \ll_\epsilon (kN)^\epsilon.$$

We note that by the duplication formula for the Gamma function we have

$$\frac{4^k \Gamma(k)}{\Gamma(2k)} = \frac{2\sqrt{\pi}}{\Gamma\left(k + \frac{1}{2}\right)},$$

so that the above bound is equivalent to

$$\frac{|\rho_g(1)|^2 4^k \Gamma(k)}{\langle g, g\rangle/\mathrm{vol}(\Gamma_0(N)\backslash \mathbb{H})} \ll_\epsilon (kN)^\epsilon \left(\Gamma\left(k + \frac{1}{2}\right)\right)^{-1}. \tag{7.12}$$

As $L_2\left(g \otimes \chi_D, \frac{1}{2}\right) \leq \frac{2}{(\sqrt{2}-1)^2}$ and $L_2\left(\pi_2 \otimes \chi_D, \frac{1}{2}\right) \geq \frac{2}{(\sqrt{2}+1)^2}$, we see that

$$l_2(g, D) \ll 1. \tag{7.13}$$

Therefore in order to obtain an upper bound for $|\rho_f(D)|$, we only need to find upper bounds for $E_p(\varphi_p, \tilde{\varphi}_p, \psi_p, D)$.

7.3. Estimating $E_p(\varphi_p, \tilde{\varphi}_p, \psi_p, D)$. In the definition (7.6) of $E_v(\varphi_v, \tilde{\varphi}_v, \psi_v, D)$, we have

$$e(\varphi_v, \psi_v) := \frac{\|\varphi_v\|^2}{|L_v(\varphi_v)|^2},$$

$$e(\tilde{\varphi}_v, \psi_v^D) := \frac{\|\tilde{\varphi}_v\|^2}{|\tilde{L}_v^D(\tilde{\varphi}_v)|^2}.$$

Here L_v and $\tilde{L}_v^D$ are the local (D-th) Whittaker functionals, $\|\varphi_v\|^2$ is the local Hermitian form defined as

$$\|\varphi_v\|^2 := \int \left| L_v\left(\pi_v\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\varphi_v\right)\right|^2 \frac{da}{|a|_v},$$

and $\|\tilde{\varphi}_v\|^2$ is the local Hermitian form defined as

$$\|\tilde{\varphi}_v\|^2 := \frac{|2|_v}{2}\sum_{\delta} \int \left| \tilde{L}_v^{D\delta}\left(\tilde{\pi}_v\left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\right)\tilde{\varphi}_v\right)\right|^2 \frac{da}{|a|_v}.$$

The δ is summed over the representatives of square classes of $\mathbb{Q}_v^*$, and $\tilde{L}_v^{D\delta}$ are local Whittaker functionals fixed to be compatible with $\tilde{L}^D$.

When $v = p$ with $p \mid 2M$, $\tilde{\varphi}_p$ is a vector with

(7.14) $\quad \tilde{\pi}_p(k)\tilde{\varphi}_p = \chi_p(k)\tilde{\varphi}_p, \quad k \in K_p(M),$

where $K_p(M)$ is the subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, d \in \mathbb{Z}_p^*$, $b \in \mathbb{Z}_p$ and $c \in 4M\mathbb{Z}_p^*$; $\chi_p(k)$ is some unitary character of the group $K_p(M)$. We note that the double cover $\widetilde{\mathrm{SL}}_2$ splits over $K_v(M)$, thus we can consider $K_v(M)$ as a subgroup of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$.

Lemma 7.2. *When $\tilde{\varphi}_p$ satisfies equation* (7.14), *$e(\tilde{\varphi}_p, \psi_p^D)^{-1} \leq \frac{2}{|2|_p}(1 - p^{-1})^{-1}$ for any $D \in \mathbb{Q}_p^*$.*

Proof. Clearly,

$$\|\tilde{\varphi}_p\|^2 \geq \frac{|2|_p}{2}\int_{|a|_p = 1} \left| \tilde{L}_p^D\left(\tilde{\pi}_p\left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\right)\tilde{\varphi}_p\right)\right|^2 da.$$

When $|a|_p = 1$, $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \in K_p(M)$ and we get, for some unitary character χ',

$$\tilde{\pi}_p\left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\right)\tilde{\varphi}_p = \chi'(a)\tilde{\varphi}_p.$$

As the Whittaker functional $\tilde{L}^D$ is a linear form, we see that the integrand is identically $|\tilde{L}_p^D(\tilde{\varphi}_p)|^2$. The integral thus gives $(1 - p^{-1})|\tilde{L}_p^D(\tilde{\varphi}_p)|^2$. From the definition of $e(\tilde{\varphi}_p, \psi_p^D)$ we get the lemma. $\square$

The vector φ_p is a new vector in π_p. Let $F_p(x)$ be its image in the Kirillov model of π_p. Then $L_p(\varphi_p) = F_p(1)$ which we will assume to be 1, and

$$\|\varphi_p\|^2 = \int |F_p(x)|^2 \frac{dx}{|x|}.$$

Looking through the table of $F_p(x)$ in [Sc02], we see either π_p is an unramified representation or $|F_p(x)| \le |x|^{1/2} \operatorname{char}_{\mathbb{Z}_p}(x)$. In the latter case we get $\|\varphi_p\|^2 \le 1$. In the former case we computed $\|\varphi_p\|^2$ in [BM05] and obtained

$$\|\varphi_p\|^2 = (1+p^{-1})|1-p^{-2s-1}|^{-2}.$$

Here s is a purely imaginary parameter (by the Ramanujan conjecture established by Deligne) associated to π_p. Thus we have

Lemma 7.3. $e(\varphi_p, \psi_p) \le (1+p^{-1})(1-p^{-1})^{-2}$. □

Clearly, $|D|_p \ge p^{-1}$. Looking through the table of $L\left(\pi_p, \frac{1}{2}\right)$ in [Go70] we get

$$L\left(\pi_p \otimes \chi_D, \frac{1}{2}\right) \ge (1+p^{1/2})^{-2}.$$

Combining this with (7.6) and Lemmata 7.2–7.3 we get

Lemma 7.4. $E_p(\varphi_p, \tilde{\varphi}_p, \psi_p, D) \le \frac{2}{|2|_p}(1+p^{-1})(p+p^{1/2})^2(1-p^{-1})^{-3}$. □

7.4. Concluding Theorem 6. The above Lemma gives

$$\prod_{p|2M} E_p(\varphi_p, \tilde{\varphi}_p, \psi_p, D) \ll_\varepsilon M^{2+\varepsilon}.$$

Combining this estimate with (7.11), (7.12), (7.13), and Lemma 7.1 we obtain

$$D|\rho_f(D)|^2 \ll_\epsilon (kM)^\epsilon \left(\Gamma\left(k+\frac{1}{2}\right)\right)^{-1} ML\left(g \otimes \chi_D, \frac{1}{2}\right).$$

By Theorem 1 and Lemma 7.1,

$$L\left(g \otimes \chi_D, \frac{1}{2}\right) \ll_\epsilon (kMD)^\epsilon k^B M^{2C} D^{1/2-(1/8)(1-2\theta)}.$$

Thus we get, for square-free D,

$$\sqrt{D}\rho_f(D) \ll_\epsilon (kMD)^\epsilon \left(\Gamma\left(k+\frac{1}{2}\right)\right)^{-1/2} k^{B/2} M^{C+1/2} D^{1/4-(1/16)(1-2\theta)}. \tag{7.15}$$

From the theory of the Shimura correspondence [Sh73, Theorem 1.9], we see that for $n = Dt^2$ with D square-free and $(t, 2M) = 1$, we have

$$\sqrt{n}\rho_f(n) = \sqrt{D}\rho_f(D) \sum_{rs=t} \frac{\mu(r)\chi(r)}{\sqrt{r}} \left(\frac{-1}{r}\right)^k \left(\frac{D}{r}\right) \lambda(s),$$

where $\lambda(s)$ is the s-th Hecke eigenvalue of a fixed newform of weight $2k$. By Deligne's bound, $|\lambda(s)| \leq \tau(s) \leq \tau(n)$ which shows that (7.15) remains valid if D is replaced by any positive integer n whose square part is coprime with $2M$.

Now an arbitrary $f(z)$ with $\langle f, f \rangle = 1$ is a linear combination $\sum_i b_i f_i(z)$, where

$$f_i(z) = \sum \rho_{f_i}(n)(4\pi n)^{k/2+1/4} e(nz),$$

$\langle f_i, f_j \rangle = \delta_{i,j}$ (thus $\sum_i |b_i|^2 = 1$), and the Fourier coefficients $\rho_{f_i}(n)$ satisfy the bound of (7.15). By Cauchy–Schwarz,

$$|\rho_f(n)|^2 \leq \sum_i |\rho_{f_i}(n)|^2,$$

and by Theorem 4.2.1 of [Ra77], the dimension of $S'_{k+1/2}(4M, \chi)$ is at most

$$\frac{k + \frac{1}{2}}{12} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4M)] \ll_\varepsilon kM^{1+\varepsilon},$$

therefore we conclude the bound (7.2).

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