Comment on "α decay in the complex-energy shell model"

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A recent paper by Id Betan and Nazarewicz re-opened the problem of the absolute width of the α-decay leading to a doubly magic nucleus. I point out a problematic aspect of this work, and reaffirm the correctness of the classical results.

The α-decay width ($\Gamma = 1.5 \times 10^{-15}$ MeV) of ^{212}Po was reproduced in a cluster-configuration shell model [1] as well as in a shell model with a stochastically optimized Gaussian [2] basis twenty years ago. It looked [3] as though the problem had been settled once and for all. In these models the widths are extracted from the tail of the wave function in the α-decay channel. The amount of clustering turned out to be comparable with unity ($\mathcal{S} \approx 0.3$), which shows that even the extreme cluster models are viable [4, 5]. In the meantime, the applications of the extreme cluster model to heavy nuclei and radioactive decay have been spectacularly extended (see, e.g., Ref. [6]).

In the paper I am commenting on [7], however, Γ is reproduced in a shell model that yields $\mathcal{S} = 0.011$, and that is obviously inconsistent with the cluster model and even with the classical microscopic calculations [1, 2, 8]. This calculation of Γ relies on the amount of clustering $\mathcal{S}$, thus the correctness of $\mathcal{S}$ is crucial. I will argue for the validity of the classical results.

The conventional α-formation amplitude $g(R)$ and the amplitude $G(R)$ of the amount of clustering $\mathcal{S}$ are defined as the radial factors of

$$g(\mathbf{R}) = \langle \mathcal{A}\{\Phi^{\rm D}\Phi^{\alpha}\delta(\mathbf{R}-\mathbf{R}_{\alpha\mathbf{D}})\}|\Phi^{\rm P}\rangle, \quad (1)$$

$$G(\mathbf{R}) = \mathcal{N}^{-1/2} g(\mathbf{R}), \quad (2)$$

where $\Phi^{\rm P}$, Φ^{α}, and $\Phi^{\rm D}$ are the intrinsic wave functions of the parent nucleus, the α-particle and the daughter, respectively, and $\mathcal{N}$ is the α–D norm operator. That is expressed as $\mathcal{N} = \sum_\nu |\varphi_\nu\rangle n_\nu \langle\varphi_\nu|$, where $\mathcal{N}\varphi_\nu = n_\nu\varphi_\nu$ is the eigenvalue equation of $\mathcal{N}$. The daughter being a heavy closed-shell core, the parent state is expressible as $\Phi^{\rm P} = \mathcal{A}\{\Phi^{\rm D}\Phi^{\rm val}\}$, with $\Phi^{\rm val}$ describing the valence nucleons.

The shapes of functions $g(R)$ and $G(R)$ are characteristic: $g(R)$ has 12 nodes [1], while $G(R)$ has an awkward shape with few nodes or none [1, 8], except for pure cluster models, in which $G(R)$ also has 12 nodes [1, 5, 9]. But the amplitudes in Ref. [7] are dissimilar: $g(R)$ has no nodes, while $G(R)$ has 11. Expounding a hint made in Ref. [1], I now show that the functions $g(R)$ and $G(R)$ are bound to behave like in Refs. [1, 8, 9].

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In the heavy-core approximation the daughter degrees of freedom can be eliminated, with the effect of $\mathcal{A}$ incorporated in an operator $\mathcal{P}$ that projects off the s.p. states occupied in $\Phi^{\rm D}$ [1]. The amplitude $g(\mathbf{R})$ becomes

$$g(\mathbf{R}) = \langle \Phi^\alpha \delta(\mathbf{R} - \mathbf{R}_{\alpha\mathbf{D}}) | \mathcal{P} | \Phi^{\rm val} \rangle, \tag{3}$$

and the eigenvalues of $\mathcal{N}$ can be expressed as

$$n_\nu = \langle \Phi^\alpha \varphi_\nu | \mathcal{P} | \Phi^\alpha \varphi_\nu \rangle \equiv (\langle \Phi^\alpha \varphi_\nu | \mathcal{P})(\mathcal{P} | \Phi^\alpha \varphi_\nu \rangle). \tag{4}$$

It is useful to introduce the normalized multiparticle state belonging to the eigenfunction φ_ν:

$$|\Xi_\nu\rangle = n_\nu^{-1/2} \mathcal{P} | \Phi^\alpha \varphi_\nu \rangle. \tag{5}$$

To have a feeling for the shapes of g and G, we should expand them in terms of the complete orthonormal set $\{\varphi_\nu\}$. In the shell model ($\Phi^{\rm val} = \Phi^{\rm sh}$)

$$\begin{aligned} g^{\rm sh}(\mathbf{R}) &= \sum_\nu \varphi_\nu(\mathbf{R}) \langle \Phi^\alpha \varphi_\nu | \mathcal{P} | \Phi^{\rm sh} \rangle \\ &= \sum_\nu n_\nu^{1/2} \langle \Xi_\nu | \Phi^{\rm sh} \rangle \varphi_\nu(\mathbf{R}). \end{aligned} \tag{6}$$

In the cluster model $\Phi^{\rm val} = \Phi^\alpha \phi^{\rm rel}$, with $\phi^{\rm rel}$ a relative-motion function. Hence

$$\begin{aligned} g^{\rm cl}(\mathbf{R}) &= \sum_\nu \varphi_\nu(\mathbf{R}) \langle \Phi^\alpha \varphi_\nu | \mathcal{P} | \Phi^\alpha \phi^{\rm rel} \rangle \\ &= \sum_{\nu\nu'} \varphi_\nu(\mathbf{R}) \langle \Phi^\alpha \varphi_\nu | \mathcal{P} | \Phi^\alpha \varphi_{\nu'} \rangle \langle \varphi_{n'} | \phi^{\rm rel} \rangle \\ &= \sum_\nu n_\nu \langle \varphi_\nu | \phi^{\rm rel} \rangle \varphi_\nu(\mathbf{R}). \end{aligned} \tag{7}$$

For $G(\mathbf{R})$ of Eq. (2), we thus have

$$G^{\rm sh}(\mathbf{R}) = \sum_\nu \langle \Xi_\nu | \Phi^{\rm sh} \rangle \varphi_\nu(\mathbf{R}), \tag{8}$$

$$G^{\rm cl}(\mathbf{R}) = \sum_\nu n_\nu^{1/2} \langle \varphi_\nu | \phi^{\rm rel} \rangle \varphi_\nu(\mathbf{R}). \tag{9}$$

Both $\langle \Xi_\nu | \Phi^{\rm sh} \rangle$ and $\langle \varphi_\nu | \phi^{\rm rel} \rangle$ are overlaps between normalized functions, and, for small ν values, both the bra and the ket functions are appreciable in the nuclear volume. The eigenvalues, if ordered conventionally, start with $n_0 \approx 10^{-9}$–10^{-7} [7, 1] and tend to 1 monotonously. Since φ_ν have $\nu = 0, 1, 2, \ldots$ nodes and $n_\nu < 0.1$ for $\nu \leq 11$, the functions $g^{\rm sh}$, $g^{\rm cl}$ and $G^{\rm cl}$ are approximately orthogonal to all φ_ν up to $\nu = 11$, hence they are bound to have at least 12 nodes. No such statement holds for $G^{\rm sh}$, whose expansion contains no n_ν. That agrees with the classical results and contradicts Ref. [7].

The situation becomes more clear-cut in a harmonic oscillator model. The valence protons in ^{212}Po would carry $5\hbar\omega$ each, while the neutrons $6\hbar\omega$ each, altogether $22\hbar\omega$. In evaluating $g(\mathbf{R}) = \langle \Phi^\alpha \delta(\mathbf{R} - \mathbf{R}_{\alpha\mathbf{D}}) | \mathcal{P} | \Phi^{\rm val} \rangle$ the oscillator functions in $\Phi^{\rm val}$ have to be transformed into a series of products of an intrinsic-motion function and a relative function. The overlap of the intrinsic motion with Φ^α is only non-zero if this factor carries $0\hbar\omega$, hence all $22\hbar\omega$ excitation must be carried by the relative-motion factor. The $g(\mathbf{R})$ being an $L = 0$ function, all $22\hbar\omega$ excitation must be concentrated in the radial

motion, i.e., the node number must be at least 11. (Since the asymptotic α-cluster is formed mostly by protons lying as high as the valence neutrons, node number 12 has turned out to be favored.) For realistic models these considerations only hold approximately.

It would be important to understand why the amplitudes in Ref. [7] look so unusual. One problem is obvious. The small eigenvalues ($n_\nu < 10^{-3}$) are discarded without good reason. In Ref. [7] it is stated that "a significant fraction of them accumulate at zero." This is incorrect as is well-known [10], for a two-cluster system there is an accumulation point at 1, but there is none at 0. This property represents the Pauli effects: a finite number of low-lying relative-motion states are strongly suppressed, while the suppression peters out as the node number increases. The finite number of small eigenvalues are those which belong to the "almost Pauli-forbidden" relative-motion states, which are strictly Pauli-forbidden in a single-$\hbar\omega$ harmonic-oscillator model. For the $L = 0$ motion of the ^{208}Pb+α system there are about 11 such states, which is borne out in all other works on α decay.

The authors omit the norm-operator eigenstates belonging to small eigenvalues because they consider them to be spurious. But that is not justifiable even though they might cause slight numerical inaccuracies [8]. The almost forbidden states are not forbidden; they give rise to well-defined Pauli-allowed multinucleon configurations [10], which must mix into the ground state. In calculating the amount of clustering $\mathcal{S} = \langle G|G\rangle$ their inclusion is crucial since the terms of the amplitude $G(\mathbf{R})$ of $\mathcal{S}$ are multiplied by $n_\nu^{-1/2}$ (implicit in Ξ_ν), which is large if n_ν is small. (It is quite another matter that in Ref. [7] the eigenvalues n_ν are very inaccurate, especially the small ones.) The enhancement by $n_\nu^{-1/2}$ is to compensate for the "oversuppression" of these components in the conventional formation amplitude $g(\mathbf{R})$ [cf. Eqs. (6), (7)].

The omission of the components with small n_ν only influences the inner region of the amplitudes and, especially, the amount of clustering. It is worth mentioning that the tail of $G(R)$ actually produces 36 times smaller value for the width than experiment, which is more or less what one can expect from such a model. (The result of using single-nucleon resonance states may not be expected to improve the α-width very much without a much more extensive basis.)

The idea, proposed in Ref. [7], of using $\Gamma = \mathcal{S}\Gamma^{\rm sp}$ to determine the width is sound since $\mathcal{S} = \langle G|G\rangle$, being an integrated quantity, is indeed less sensitive to the degree of completeness of the model state space than the tail of $g(R)$ or $G(R)$. This viable idea deserves confirmation with a more reliable calculation.

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