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NORMALITY IN GROUP RINGS

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Dedicated to Professor P. M. Gudivok on the occasion of his 70th birthday

ABSTRACT. Let KG be the group ring of a group G over a commutative ring K with unity. The rings KG are described for which $xx^\sigma = x^\sigma x$ for all $x = \sum_{g\in G} \alpha_g g \in KG$, where $x \mapsto x^\sigma = \sum_{g\in G} \alpha_g f(g)\sigma(g)$ is an involution of KG; here $f : G \to U(K)$ is a homomorphism and σ is an antiautomorphism of order two of G.

Let R be a ring with unity. We denote by $U(R)$ the group of units of R. A (bijective) map $\diamond : R \to R$ is called an *involution* if for all $a, b \in R$ we have $(a + b)^\diamond = a^\diamond + b^\diamond$, $(ab)^\diamond = b^\diamond \cdot a^\diamond$ and $a^{\diamond^2} = a$. Let KG be the group ring of a group G over a commutative ring K with unity, let σ be an antiautomorphism of order two of G, and let $f : G \to U(K)$ be a homomorphism from G onto $U(K)$. For an element $x = \sum_{g\in G} \alpha_g g \in KG$, we define $x^\sigma = \sum_{g\in G} \alpha_g f(g)\sigma(g) \in KG$. Clearly, $x \mapsto x^\sigma$ is an involution of KG if and only if $g\sigma(g) \in \operatorname{Ker} f = \{h \in G \mid f(h) = 1\}$ for all $g \in G$.

The ring KG is said to be *σ-normal* if

$$xx^\sigma = x^\sigma x \tag{1}$$

for each $x \in KG$. The properties of the classical involution $x \mapsto x^*$ (where $* : g \mapsto g^{-1}$ for $g \in G$) and the properties of normal group rings (i.e., $xx^* = x^*x$ for each $x \in KG$) have been used actively for the investigation of the group of units $U(KG)$ of the group ring KG (see [1, 2]). Moreover, they also have important applications in topology (see [7, 8]). Our aim is to describe the structure of the σ-normal group ring KG for an arbitrary order 2 antiautomorphism σ of the group G. Note that descriptions of the classical normal group rings and the twisted group rings were obtained in [1, 3] and [4, 5], respectively.

The notation used throughout the paper is essentially standard. C_n denotes the cyclic group of order n; $\zeta(G)$ and $C_G(H)$ are the center of the group G and the centralizer of H in G, respectively; $(g, h) = g^{-1}h^{-1}gh = g^{-1}g^h$ $(g, h \in G)$; $\gamma_i(G)$ is the ith term of the lower central series of G, i.e., $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = \big(\gamma_i(G), G\big)$ for $i \geq 1$; $\Phi(G)$ denotes the Frattini subgroup of G. We say that $G = A \curlyvee B$ is a central product of its subgroups A and B if A and B commute elementwise and, taken together, they generate G, provided that $A \cap B$ is a subgroup of $\zeta(G)$.

A non-Abelian 2-generated nilpotent group $G = \langle a,b\rangle$ with an antiautomorphism σ of order 2 is called a σ-group if G' has order 2, $\sigma(a) = a(a, b)$, and $\sigma(b) = b(a, b)$.

Our main result reads as follows.

Theorem. *Let KG be the noncommutative group ring of a group G over a commutative ring K and $f : G \to U(K)$ a homomorphism. Assume that σ is an antiautomorphism of order two of G such that $x \mapsto x^\sigma$ is an involution of KG. Put $\mathfrak{R}(G) = \{g \in G \mid \sigma(g) = g\}$.*

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The group ring KG is σ-normal if and only if $f : G \to \{\pm 1\}$, G, K, and σ satisfy one of the following conditions:

(i) *G has an Abelian subgroup H of index 2 such that $G = \langle H, b\rangle$, $f(b) = -1$, $f(h) = 1$, $\sigma(b) = b$, and $\sigma(h) = b^{-1}hb = bhb^{-1}$ for all $h \in H$;*

(ii) *$G = H \curlyvee \mathfrak{C}$ is a central product of a σ-group $H = \langle a, b\rangle$ and an Abelian group $\mathfrak{C}$ such that $G' = \langle c \mid c^2 = 1\rangle$ and $H \subset \mathrm{Ker}(f)$. Moreover, either $\sigma(d) = d$ for all $d \in \mathfrak{C}$, $\mathfrak{C} \subset \mathrm{Ker}(f)$, $\mathfrak{R}(G) = \zeta(G)$, and*

$$G/\mathfrak{R}(G) = \langle a\zeta(G), b\zeta(G)\rangle \cong C_2 \times C_2,$$

or $\mathfrak{R}(G)$ is of index 2 in $\zeta(G)$ and

$$G/\mathfrak{R}(G) = \langle g\mathfrak{R}(G), h\mathfrak{R}(G), d\mathfrak{R}(G)\rangle \cong C_2 \times C_2 \times C_2,$$

where $d \in \mathfrak{C}$, $\sigma(d) = dc$, and $f(d) = -1$;

(iii) *$\mathrm{char}(K) = 2$, $G = S \curlyvee \mathfrak{C}$ is a central product of $S = \curlyvee_{i=1}^{n} H_i$ and an Abelian group $\mathfrak{C}$ such that $H_i = \langle a_i, b_i\rangle$ is a σ-group and $G = \mathrm{Ker}(f)$. Moreover, $G' = \langle c \mid c^2 = 1\rangle$, $n \geq 2$, where n is not necessarily a finite number, $\sigma(a_i) = a_i c$, $\sigma(b_i) = b_i c$ for all $i = 1, 2, \ldots,$ and $\exp(G/\mathfrak{R}(G)) = 2$.*

Furthermore, if n is finite, then either $\sigma(d) = d$ for all $d \in \mathfrak{C}$ and

$$G/\mathfrak{R}(G) = \bigtimes_{i=1}^{n} \langle a_i\zeta(G), b_i\zeta(G)\rangle \cong \bigtimes_{i=1}^{2n} C_2,$$

or $\mathfrak{R}(G)$ is of index 2 in $\zeta(G)$ and

$$G/\mathfrak{R}(G) = \bigtimes_{i=1}^{n} \langle a_i\mathfrak{R}(G), b_i\mathfrak{R}(G)\rangle \times \langle d\mathfrak{R}(G)\rangle \cong \bigtimes_{i=1}^{2n+1} C_2,$$

where $d \in \mathfrak{C}$ and $\sigma(d) = dc$.

Note that, in parts (i) and (ii) of the theorem, the group $\mathfrak{C}$ may be equal to 1.

To make the statements less cumbersome, in what follows we shall often talk of σ-normal group rings KG without specifying the homomorphism $f : G \to U(K)$ and the antiautomorphism σ of order two of G. In order to prove the main theorem, we need some preliminary lemmas.

Lemma 1. *Let $U(R)$ be the group of units of the ring R, and let $x \mapsto x^\diamond$ be an involution of R. Suppose that $xx^\diamond = x^\diamond x$ for all $x \in R$. If $a \in U(R)$, then $a^\diamond = ta$, $at = ta$, and $t^\diamond = t^{-1}$, where $t \in U(R)$.*

Proof. Clearly $a^\diamond = at$ for some $t \in U(R)$, and $a^\diamond a = aa^\diamond$ implies that $ata = a^2t$ and $at = ta$. Now $a = a^{\diamond^2} = (at)^\diamond = t^\diamond at = t^\diamond ta$, whence $t^\diamond = t^{-1}$. □

Lemma 2. *Let K be a commutative ring, let $H = \langle a, b\rangle$ be a non-Abelian 2-generated subgroup of a group G, and let $f : G \to U(K)$ be a homomorphism. If the group ring KG is σ-normal, then $f : H \to \{\pm 1\}$ and one of the following conditions is fulfilled*:

(i) *$f(a) = 1$, $f(b) = -1$, $\sigma(a) = (a, b)a$, $\sigma(b) = b$, $(b^2, a) = 1$, $(ab)^2 = (ba)^2$, $((a, b), a) = 1$, and $((a, b), b) = (a, b)^{-2}$;*

(ii) *$f(a) = -1$, $f(b) = 1$, $\sigma(a) = a$, $\sigma(b) = (a, b)b$, $(b, a^2) = 1$, $(ab)^2 = (ba)^2$, $((a, b), b) = 1$, and $((a, b), a) = (a, b)^{-2}$;*

(iii) *$f(a) = f(b) = -1$, $\sigma(a) = a$, $\sigma(b) = b$, $(a^2, b) = (a, b^2) = 1$, $(ab)^2 = (ba)^2$, and $((a, b), ab) = 1$;*

(iv) *$f(a) = f(b) = 1$, $\sigma(a) = (a, b)a$, $\sigma(b) = (a, b)b$, and $\langle a, b\rangle$ is nilpotent of class 2 and such that $\gamma_2(\langle a, b\rangle)$ is of order 2.*

Proof. Let KG be a σ-normal ring. For any noncommutative $a, b \in G$ we can put $\sigma(a) = at$ and $\sigma(b) = bs$, where $s, t \in G$. By Lemma 1, $at = ta$, $bs = sb$, $\sigma(t) = t^{-1}$, and $\sigma(s) = s^{-1}$. Set $x = a + b \in KG$. Clearly, $x^\sigma = f(a)\sigma(a) + f(b)\sigma(b)$, and by (1) we have

$$f(b)a\sigma(b) + f(a)b\sigma(a) = f(a)\sigma(a)b + f(b)\sigma(b)a. \tag{2}$$

If $a\sigma(b) = b\sigma(a) = \sigma(a)b = \sigma(b)a$, then we get $s = t$ and $ab = ba$, a contradiction. Observe that if three of the elements $\{a\sigma(b), b\sigma(a), \sigma(a)b, \sigma(b)a\}$ coincide, then $s = t$ and $ab = ba$, a contradiction. We consider the following cases.

1. $a\sigma(b) = b\sigma(a)$. By (2), it follows that

$$f(a) + f(b) = 0, \quad a\sigma(b) = b\sigma(a), \quad \sigma(a)b = \sigma(b)a. \tag{3}$$

2. $a\sigma(b) = \sigma(a)b$. This yields $asb = atb$, so that $s = t \in \zeta(H)$, and (2) ensures

$$f(a) = f(b), \quad \sigma(a) = at, \quad \sigma(b) = bt, \quad t \in \zeta(H). \tag{4}$$

3. $a\sigma(b) = \sigma(b)a$. Since $b\sigma(b) = \sigma(b)b$, we get $\sigma(b) \in \zeta(H)$, a contradiction.

Now put $x = a(1 + b)$. Then $x^\sigma = (1 + f(b)\sigma(b))f(a)\sigma(a)$ and, by (1),

$$f(ab)a\sigma(ab) + f(a)ab\sigma(a) = f(a)\sigma(a)ab + f(ab)\sigma(ab)a. \tag{5}$$

We shall treat the following cases separately.

1. $a\sigma(ab) = ab\sigma(a)$. Formula (5) implies that

$$f(b) = -1, \quad \sigma(b) = b, \quad (\sigma(a)a) \cdot b = b \cdot (\sigma(a)a). \tag{6}$$

2. $a\sigma(ab) = \sigma(a)ab$ and $ab\sigma(a) = \sigma(ab)a$. By (1) we have $ab = \sigma(b)a$ and $\sigma(b) = aba^{-1}$. Since $a\sigma(ab) = \sigma(a)ab$, we get $aba^{-1}at = atb$ and $(b, t) = 1$. Recall that $\sigma(b) = bs = sb$. So, by (5),

$$f(b) = 1, \quad \sigma(b) = aba^{-1}, \quad t \in \zeta(H), \quad s = (a^{-1}, b^{-1}) = (b, a^{-1}). \tag{7}$$

3. $a\sigma(b)\sigma(a) = \sigma(b)\sigma(a)a$. Then $a\sigma(b) = \sigma(b)a$ and $\sigma(b) \in \zeta(H)$, a contradiction.

Assume that (3) and (6) are true. Then $f(b) = -1$, $f(a) = 1$, $\sigma(b) = b$, $\sigma(a) = b^{-1}ab = bab^{-1}$, whence $(b^2, a) = 1$. Since $\sigma(a) = a(a^{-1}b^{-1}ab) = (bab^{-1}a^{-1})a$, we get $a^{-1}b^{-1}ab = bab^{-1}a^{-1}$ and $(ab)^2 = (ba)^2$. Obviously,

$$b^{-1}(a, b)b = b^{-1}(bab^{-1}a^{-1})b = ab^{-1}a^{-1}b = aba^{-1}b^{-1} = (a, b)^{-1},$$

so that $((a, b), b) = (a, b)^{-2}$, and statement (i) of our lemma follows.

If (3) and (7) are fulfilled, then $f(b) = 1$, $f(a) = -1$, $\sigma(a) = a$, $\sigma(b) = a^{-1}ba$, and $(a^2, b) = 1$. Since $\sigma(b) = b(b^{-1}a^{-1}ba) = (aba^{-1}b^{-1})b$, we obtain $s = b^{-1}a^{-1}ba = aba^{-1}b^{-1}$ and $(ab)^2 = (ba)^2$. Therefore, $\sigma(a) = a$, $\sigma(b) = a^{-1}ba$, $(a^2, b) = 1$, and we arrive at statement (ii).

Assume (4) and (6). Then $f(a) = f(b) = -1$, $\sigma(a) = a$, $\sigma(b) = b$, and $(a^2, b) = 1$. Moreover, $f(ab) = 1$ and $\sigma(ab) = ba = a^{-1}(ab)a$. We put $x = b(1 + a)$. Clearly, $x^\sigma = (a - 1)b$, and (1) implies $b^2a + b^2a^2 = ab^2a + ab^2$, whence $(b^2, a) = 1$. Thus, statement (iii) of our lemma is fulfilled.

Finally, if (4) and (7) are true, then $f(a) = f(b) = 1$ and $(b, a^{-1}) \in \zeta(H)$. Using the identity $(\alpha\beta, \gamma) = (\alpha, \gamma)(\alpha, \gamma, \beta)(\beta, \gamma)$, where $\alpha, \beta, \gamma \in G$, we see that $1 = (a^{-1}a, b) = (a^{-1}, b)(a, b)$, whence $s = (b, a^{-1}) = (a, b) \in \zeta(H)$ and $\sigma(a) = (a, b)a$, $\sigma(b) = (a, b)b$. Since $a = \sigma^2(a)$ and $(a, b) \in \zeta(H)$, we have $(a, b)^2 = 1$, which yields statement (iv). The proof is complete. ☐

Lemma 3. *Let KG be a σ-normal group ring of a non-Abelian group G. Then $H = \langle w \in G \mid \sigma(w) \neq w \rangle$ is a normal subgroup in G. If H is Abelian, then G satisfies statement* (i) *of the theorem.*

Proof. Set $W = \{w \in G \mid \sigma(w) \neq w\}$. Let $g \notin W$ be such that $g^2 \notin \zeta(G)$. Then $(g^2, h) \neq 1$ and $(g, h) \neq 1$, respectively, for some $h \in G$.

We consider the following cases.

1. $\mathrm{char}(K) \neq 2$. Since $\sigma(g^2) = g^2$, we can use Lemma 2 for the group $\langle g^2, h\rangle$ to show that $-1 = f(g^2) = (\pm 1)^2 = 1$, a contradiction.

2. $\mathrm{char}(K) = 2$. Using Lemma 2 for $\langle g, h\rangle$, we get $(g^2, h) = 1$, again a contradiction.

Thus, $g^2 \in \zeta(G)$ for any $g \notin W$. Now, if $w \in W$, $g \in G \setminus W$, and $g^{-1}wg \notin W$, then $\sigma(g^{-1}wg) = g^{-1}wg$ and

$$g^{-1}wg = \sigma(g^{-1}wg) = g\sigma(w)g^{-2}g = g^{-1}\sigma(w)g$$

so that $\sigma(w) = w$, a contradiction. Therefore, $g^{-1}wg \in W$ and the subgroup $H = \langle W\rangle$ is normal in G.

Suppose that $H = \langle W\rangle$ is Abelian. If $a \in W$ and $c \in C_G(W) \setminus H$, then $ca \notin H$. Therefore, $ca = \sigma(ca) = \sigma(a)c$, whence $\sigma(a) = a$, a contradiction. This shows that $C_G(W) = H$ and for each $b \notin H$ there exists $w \in W$ such that $(b, w) \neq 1$.

We claim that if $b_1, b_2 \in G \setminus H$, then $b_1b_2 \in H$. The following cases will be treated separately:

1. $\mathrm{char}(K) \neq 2$ and $b_1b_2 \in G\setminus H$. For each b_i we choose $w_i \in W$ such that $(b_i, w_i) \neq 1$. By (i) or (ii) of Lemma 2, in $\langle w_i, b_i\rangle$ we have $f(b_i) = -1$, so that $f(b_1b_2) = 1$ and there exists $w \in W$ for which $(b_1b_2, w) \neq 1$. Since $\sigma(b_1b_2) = b_1b_2$, by (i) or (ii) of Lemma 2 we get $f(b_1b_2) = -1$, a contradiction.

2. $\mathrm{char}(K) = 2$ and $b_1b_2 \in G \setminus H$. Obviously, $b_1b_2 = \sigma(b_1b_2) = b_2b_1$, whence $(b_1, b_2) = 1$. Now, there is $w \in W$ with $(w, b_1) \neq 1$, and by Lemma 2 we get $\sigma(w) = b_1^{-1}wb_1 = b_1wb_1^{-1}$. Furthermore, $b_1b_2w \in G \setminus H$ and

$$b_1b_2w = \sigma(b_1b_2w) = \sigma(w)b_1b_2 = b_1wb_2,$$

implying $(b_2, w) = 1$. Now $(b_1, b_2w) = (b_1, w) \neq 1$ and $b_2w \in G \setminus H$; applying Lemma 2 in $\langle b_1, b_2w\rangle$, we obtain $b_2w = \sigma(b_2w) = \sigma(w)b_2$ and $\sigma(w) = w$, a contradiction.

We have proved that $b_1b_2 \in H$ for every $b_1, b_2 \in G \setminus H$. Hence, $G = \langle H, b \mid b \notin H, b^2 \in H\rangle$, $f(b) = -1$, and $f(h) = 1$ for all $h \in H$.

Finally, let $w \in W$ be such that $(b, w) = 1$. Since $b \notin H = C_G(W)$, there exists $w_1 \in W$ with $w_1 \neq b^{-1}w_1b = \sigma(w_1)$. Clearly, we have $(ww_1, b) \neq 1$; using Lemma 2 for $\langle ww_1, b\rangle$, we obtain

$$\sigma(w)\sigma(w_1) = \sigma(w_1w) = b^{-1}w_1wb = \sigma(w_1)w,$$

whence $\sigma(w) = w$, a contradiction. Thus, $b^{-1}hb = \sigma(h)$ for all $h \in H$. □

Lemma 4. *Let KG be a σ-normal group ring, let $W = \{w \in G \mid \sigma(w) \neq w\}$, and let $a, b \in W$ be such that $(a, b) \neq 1$. Put $\mathfrak{R} = \{g \in G \mid \sigma(g) = g\}$ and $\mathfrak{C} = C_G(\langle a, b\rangle)$. Then $\langle a, b\rangle$ is a σ-group, $\Phi(\langle a, b\rangle) = \zeta(\langle a, b\rangle) = \{g \in \langle a, b\rangle \mid \sigma(g) = g\}$, and*

$$\sigma(g) = \begin{cases} g & \text{if } g \in \zeta(\langle a, b\rangle), \\ g(a, b) & \text{if } g \notin \zeta(\langle a, b\rangle). \end{cases}$$

Moreover, $G = \langle a, b\rangle \curlyvee \mathfrak{C}$, and either $\sigma(c) = (a, b)c$, or $\sigma(c) = c$, where $c \in \mathfrak{C}$. Also, the following is true:

(i) *if $\mathfrak{C}$ is Abelian, then G satisfies statement* (ii) *of the theorem*;

(ii) *if $\mathfrak{C}$ is not Abelian, then* $\mathrm{char}(K) = 2$.

Proof. Let $a, b \in W$ satisfy $(a, b) \neq 1$. By Lemma 2, $f(a) = f(b) = 1$, $\langle a, b\rangle$ is nilpotent of class 2 and such that $|\gamma_2(\langle a, b\rangle)| = 2$, and $\sigma(a) = b^{-1}ab$, $\sigma(b) = a^{-1}ba$. Thus $\langle a, b\rangle$ is a σ-group. Any element $g \in \langle a, b\rangle$ can be written as $g = a^ib^j(a, b)^k$, where $i, j, k \in \mathbb{N}$.

Since $\sigma(g) = g$, we conclude that i and j are even. Now by [6, Theorems 10.4.1 and 10.4.3] we obtain

$$\Phi(\langle a, b\rangle) = \zeta(\langle a, b\rangle) = \{g \in \langle a, b\rangle \mid \sigma(g) = g\}.$$

Suppose $c \in W$ and $(a, c) \neq 1$. Again by Lemma 2, $\langle a, c\rangle$ is nilpotent of class 2 and $\sigma(a) = c^{-1}ac = b^{-1}ab$, so that $(a, b) = (a, c)$. Now, let $c, d \in W$ be such that $(c, d) \neq 1$ and $\langle c, d\rangle \in \mathfrak{C}$. Obviously, $(ac, b) = (a, b) \neq 1$ and $(ac, d) = (c, d) \neq 1$. By Lemma 2, $\sigma(ac) = b^{-1}(ac)b = d^{-1}(ac)d$ and $(a, b) = (c, d)$, which shows that H' has order two and is central in G.

Let $g \in G \setminus \mathfrak{C} \cdot \langle a, b\rangle$. If $(a, g) \neq 1$, then, using Lemma 2 for $\langle a, g\rangle$, we get $\sigma(a) = g^{-1}ag = b^{-1}ab$ and $(a, g) = (a, b)$. Similarly, if $(b, g) \neq 1$, then $(b, g) = (a, b)$.

The following cases are possible:

1. $(g, a) = 1$ and $(g, b) \neq 1$. Then we have $(ga, b) = (ga, a) = 1$, which implies that $g = (ga) \cdot a^{-1} \in \mathfrak{C} \cdot \langle a, b\rangle$.

2. $(g, a) \neq 1$ and $(g, b) = 1$. Then we have $(gb, a) = (gb, b) = 1$, which implies that $g = (gb) \cdot b^{-1} \in \mathfrak{C} \cdot \langle a, b\rangle$.

3. $(g, a) \neq 1$ and $(g, b) \neq 1$. Then we have $(gab, b) = (gab, a) = 1$, which implies that $g = (gab) \cdot (ab)^{-1} \in \mathfrak{C} \cdot \langle a, b\rangle$.

Since each of these cases leads to a contradiction, we have $G = \mathfrak{C} \mathbin{\mathsf{Y}} \langle a, b\rangle$.

Let $d \in \mathfrak{C} \setminus H$. Since $\sigma(ad) = ad$, we get $ad = \sigma(ad) = \sigma(a)d$, whence $\sigma(a) = a$, a contradiction. Since $G = \mathfrak{C} \cdot \langle a, b\rangle$, it follows that $G = H = \langle W\rangle$. If $d \in \zeta(G) \cap W$, then $\sigma(ad) = ad$ and $(ad, b) = (a, b) \neq 1$; using Lemma 2 for $\langle ad, b\rangle$, we obtain

$$-1 = f(ad) = f(a)f(d) = f(d).$$

Now, we let $\zeta(G) \cap W = \varnothing$ and put $x = ac + b$, where $c \in \mathfrak{C}$. Then there exists $d \in G$ such that $(c, d) \neq 1$, and Lemma 2 implies that $f(g) = 1$ for all $g \in G$. Thus, $x^\sigma = (a\sigma(c) + b)(a, b)$, and by (1) we have $(\sigma(c) - c)(1 - (a, b)) = 0$. It follows that either $\sigma(c) = c$, or $\mathrm{char}(K) = 2$ and $\sigma(c) = (a, b)c$. Therefore, if $\mathfrak{C}$ is Abelian, we obtain statement (ii) of the theorem.

Finally, assume that $\mathrm{char}(K) \neq 2$. Suppose there exist $c, d \in \mathfrak{C}$ such that $(c, d) \neq 1$. If $\sigma(c) = c$, then $f(c) = 1$ by what has already been proved, but, by Lemma 2 in $\langle c, d\rangle$, we have $f(c) = -1$, a contradiction. Therefore, $c \in W$ and similarly $d \in W$. We put $x = ac + d$. Clearly, $x^\sigma = ac + d(a, b)$ and $(a, b) = 1$ by (1), a contradiction. Thus, if $\mathfrak{C}$ is not Abelian, then $\mathrm{char}(K) = 2$, and the proof is complete. □

Now we are in a position to prove our main theorem.

Proof of the "if" part of the theorem. Set $W = \{w \in G \mid \sigma(w) \neq w\}$ and $H = \langle W\rangle$. If H is Abelian, then, by Lemma 3, statement (i) of the theorem is valid for G.

Suppose that H is non-Abelian and that $a, b \in W$ satisfy $(a, b) \neq 1$. By Lemma 4, $G = \langle a, b\rangle \mathbin{\mathsf{Y}} \mathfrak{C} = \langle W\rangle$, where $\mathfrak{C} = C_G(\langle a, b\rangle)$. If $\mathfrak{C}$ is Abelian, then statement (ii) of our theorem is valid for G by Lemma 4.

Let $c, d \in C_G(\langle a, b\rangle)$ be such that $(c, d) \neq 1$ (i.e., $\mathfrak{C}$ is non-Abelian). By Lemma 4, we have $\mathrm{char}(K) = 2$. If $c, d \in W$, then, by Lemma 4,

$$G = C_G(\langle a, b\rangle) \cdot \langle a, b\rangle = C_G(\langle c, d\rangle) \cdot \langle c, d\rangle.$$

Obviously, $C_G(\langle a, b\rangle) \cap \langle a, b\rangle \subseteq \zeta(G)$. Therefore, G contains the subgroup $H_2 = \langle a, b\rangle \mathbin{\mathsf{Y}} \langle c, d\rangle$, which cannot be a direct product because G' has order 2.

Since $G' \subseteq \mathfrak{R}(G)$, we see that $G/\mathfrak{R}(G)$ is an elementary Abelian 2-group. Let $\tau : G \to G/\mathfrak{R}(G) = \bigtimes_{i\geq 1}\langle a_i \mid a_i^2 = 1\rangle$ be such that $\tau^{-1}(a_1) = a$ and $\tau^{-1}(a_2) = b$. We put $\overline{a}_i = \tau^{-1}(a_i)$ for all $i \geq 3$ and $\mathfrak{B} = \{a_i \mid i \geq 3\}$.

Suppose that for some $s \geq 3$ we have $(\overline{a}_s, \overline{a}_i) = 1$ for all $i \geq 3$. Such an element is unique, because if $\overline{a}_t \neq \overline{a}_s$ commutes with all $\overline{a}_s$, then $\sigma(\overline{a}_t\overline{a}_s) = \overline{a}_s\overline{a}_t$, whence $a_s a_t = 1$, a contradiction. Put $\mathfrak{B} = \mathfrak{B} \setminus a_s$, $b_0 = a_s$, $b_1 = a_1$, and $b_2 = a_2$. Note that if such an element a_s does not exist, then we put $b_0 = 1$.

Choose $a_i \in \mathfrak{B}$. There is $a_j \in \mathfrak{B}$ such that $(\overline{a}_i, \overline{a}_j) \neq 1$, and we consider the following cases.

1. $\overline{a}_i, \overline{a}_j \in W$. Put $b_3 = a_i$, $b_4 = a_j$ and $\mathfrak{B} = \mathfrak{B} \setminus \{a_i, a_j\}$.

2. $\overline{a}_i \in W$ and $\overline{a}_j \notin W$. Clearly, $\langle \overline{a}_1, \overline{a}_2\rangle \curlyvee \langle \overline{a}_i, \overline{a}_j\rangle \cong \langle \overline{a}_1\overline{a}_i, \overline{a}_2\rangle \curlyvee \langle \overline{a}_i, \overline{a}_2\overline{a}_j\rangle$ and $\overline{a}_1\overline{a}_i, \overline{a}_2, \overline{a}_i, \overline{a}_2\overline{a}_j \in W$. Put $b_1 = \tau(\overline{a}_1\overline{a}_i)$, $b_2 = a_2$, $b_3 = a_i$, $b_4 = \tau(\overline{a}_2\overline{a}_j)$, and $\mathfrak{B} = \mathfrak{B} \cup \{a_1, a_2\} \setminus \{b_1, b_2, b_3, b_4\}$.

3. $\overline{a}_i, \overline{a}_j \notin W$. Obviously, we have $\overline{a}_i\overline{a}_j \in W$, so that this case reduces to the preceding one.

Furthermore, if $C_G(\langle \overline{b}_1, \overline{b}_2\rangle \curlyvee \langle \overline{b}_3, \overline{b}_4\rangle)$ contains a noncommuting pair of elements, then this pair can be chosen in W. By continuing this process, we can conclude that G contains a subgroup $\mathfrak{M} = A_1 \curlyvee A_2 \curlyvee \cdots$ that is a central product, where each $A_i = \langle g_i, h_i\rangle$ is a σ-group and $C_G(\mathfrak{M})$ is Abelian. Applying Lemma 4, we arrive at statement (iii) of the theorem, and the proof is complete. □

Proof of the "only if" part of the theorem. (i) We can write any $x \in KG$ as $x = x_1 + x_2 b$, where $x_i \in KH$. Clearly, $x^\sigma = x_1^\sigma + f(b)\sigma(b)x_2^\sigma = x_1^\sigma - x_2 b$ and

$$xx^\sigma = x_1x_1^\sigma - x_2x_2^\sigma b^2 = x_1^\sigma x_1 - x_2^\sigma x_2 b^2 = x^\sigma x,$$

so that KG is a σ-normal ring.

(ii) Any $x \in KH$ can be written as $x = x_0 + x_1 g + x_2 h + x_3 gh$, where $x_i \in K\langle g^2, h^2, c\rangle$ and $c = (g, h)$. Clearly, $x^\sigma = x_0 + (x_1 g + x_2 h + x_3 gh)c$ and $xx^\sigma = x^\sigma x$, so that KH is σ-normal. Suppose that $\sigma(d) = dc$, with $c = (a, b)$. Any $x \in KG$ can be written as $x = (w_0 + u_1) + (w_2 + u_3)d$, where $u_1 = \alpha_1 a + \alpha_2 b + \alpha_3 ab$, $u_3 = \beta_1 a + \beta_2 b + \beta_3 ab$, and $\alpha_i, \beta_i, w_0, w_2 \in K\mathfrak{R}$. Then $x^\sigma = (w_0 + u_1 c) - (w_2 + u_3 c)dc$ and $xx^\sigma - x^\sigma x = (u_3u_1 - u_1u_3)(1 + c)d$. Since $ab - ba = ba(c - 1)$ and $c^2 = 1$, it follows that $xx^\sigma - x^\sigma x = 0$. Thus, KG is σ-normal. In the case where $\sigma(d) = d$, the proof is similar.

(iii) Put $G_n = A_1 \curlyvee \cdots \curlyvee A_n$, where $A_i = \langle a_i, b_i \mid c = (a_i, b_i)\rangle$ is a σ-subgroup. We use induction on n. Any $x \in KG_n$ can be written as $x = x_0 + x_1 a_n + x_2 b_n + x_3 a_n b_n$, where $x_i \in K\langle G_{n-1}, a_n^2, b_n^2\rangle$. Obviously, $x^\sigma = x_0^\sigma + (x_1^\sigma a_n + x_2^\sigma b_n + x_3^\sigma a_n b_n)c$. Since KG_{n-1} is σ-normal, we get $x_i x_i^\sigma = x_i^\sigma x_i$ and $x_i^\sigma(1 + c) = x_i(1 + c)$. The formula

$$(x_i + x_j)(x_i + x_j)^\sigma = (x_i + x_j)^\sigma(x_i + x_j)$$

shows that

$$x_i x_j^\sigma + x_j x_i^\sigma = x_i^\sigma x_j + x_j^\sigma x_i.$$

Proceeding as in the preceding case, we conclude that

$$xx^\sigma = x^\sigma x,$$

and the proof is complete. □

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