# DETERMINANT PRESERVING MAPS: AN INFINITE DIMENSIONAL VERSION OF A THEOREM OF FROBENIUS

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ABSTRACT. In this paper we investigate the structure of maps on classes of Hilbert space operators leaving the determinant of linear combinations invariant. Our main result is an infinite dimensional version of the famous theorem of Frobenius about determinant preserving linear maps on matrix algebras. In that theorem of ours, we use the notion of (Fredholm) determinant of bounded Hilbert space operators which differ from the identity by an element of the trace class. The other result of the paper describes the structure of those transformations on sets of positive semidefinite matrices which preserve the determinant of linear combinations with fixed coefficients.

The determinant of square matrices (or linear operators on a finite dimensional vector space) is one of the most basic notions in matrix theory which has several applications also in other areas of mathematics. In light of its fundamental role, it is not surprising that maps on sets of matrices preserving related quantities have been extensively studied in the field of preserver problems. Indeed, the statement which is generally regarded as the first result in that branch of mathematics also concerns such transformations. It is the famous theorem of Frobenius from 1897 which reads as follows. In this paper, for a positive integer $n$ and a field $F$, the space of all $n \times n$ matrices with entries in $F$ is denoted by $M_n(F)$ and $^t$ stands for the transpose.

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**Theorem** (Frobenius [5])**.** *If $\phi\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear map satisfying*

$$\det \phi(A) = \det A \quad (A \in M_n(\mathbb{C})),$$

*then there exist matrices $M, N \in M_n(\mathbb{C})$ such that $\det MN = 1$ and $\phi$ can be written in the form*

$$\phi(A) = MAN \quad (A \in M_n(\mathbb{C}))$$

*or in the form*

$$\phi(A) = MA^tN \quad (A \in M_n(\mathbb{C})).$$

Later on this theorem has been generalized and several similar results have been obtained. In the paper [3], Eaton has described the structure of certain linear maps $\phi$ on the space $S_n(\mathbb{R})$ of all symmetric matrices in $M_n(\mathbb{R})$ for which $\det \phi(A) = c \det A$ $(A \in S_n(\mathbb{R}))$ holds with a real number $c \neq 0$. In the previous decade, non-linear transformations leaving the determinant of linear combinations invariant have been investigated. We remark that this property is more general than the linearity and the determinant preserving property together. In the articles [1, 2, 14], the authors have studied maps $\phi$ of subsets of $M_n(\mathbb{C})$ satisfying

$$\det(\alpha\phi(A) + \phi(B)) = \det(\alpha A + B)$$

for all numbers $\alpha \in \mathbb{C}$ and each pair of elements $A, B$ in the domain of $\phi$. This domain is $M_n(\mathbb{C})$ in [2, 14], $S_n(\mathbb{R})$ in [1] and the set of all upper triangular matrices in $M_n(\mathbb{C})$ in [14]. In the second part of the paper, we will investigate those maps $\phi$ on collections of definite and semidefinite matrices which satisfy the last displayed equality for a fixed number $\alpha > 0$. In many cases, it has turned out that transformations preserving the determinant of linear combinations can be written as the composition of the transposition and a two-sided multiplication by matrices whose product has determinant 1. Observe that maps of such form have the latter invariance property.

The concept of determinant has been extended to the infinite dimensional setting, i.e., to certain bounded linear operators acting on an infinite dimensional Hilbert space. Before we can discuss the corresponding notion in more details, some basic notation used in the paper should be fixed. Let $H$ be a complex Hilbert space and $B(H)$ be the algebra of bounded linear operators on $H$. We denote by $C_1(H)$ the ideal of trace-class operators, which by definition consists of all elements of $B(H)$ whose absolute value has a finite trace and $I$ stands for the identity map of $H$. The affine subspace $I + C_1(H) \subset B(H)$ will be denoted by $T(H)$. We remark that $T(H)$ is a multiplicative semigroup which is closed under taking inverses of its bijective elements and that $T(H) = B(H)$ in the case $\dim H < \infty$. There are several equivalent definitions for the (Fredholm) determinant $\det T$ of an operator $T \in T(H)$. For example, on [13, p. 33] it is mentioned that one of them can be given as follows. Let $(\lambda_n)_{n\in N}$ be a sequence of the nonzero eigenvalues of the compact operator $T - I$ counted according to their algebraic multiplicities. Then the product $\prod_{n\in N}(1+\lambda_n)$ exists, and $\det T$ is defined by the formula

$$\det T = \prod_{n\in N}(1 + \lambda_n).$$

One can observe that in the case $\dim H < \infty$, the quantity $\det T$ is the usual determinant of $T$. We remark that the map det has a central role in the Fredholm theory of trace class operators, i.e., elements of $C_1(H)$. For details of this theory and for more information about infinite dimensional determinants, one can consult, e.g. the books [10, 13] and the paper [12]. In [12], a lot of their properties are presented and several references to publications on their equivalent definitions can be found.

The main aim of this paper is to establish an infinite dimensional version of Theorem. To the best of our knowledge, maps preserving infinite dimensional determinants have never been investigated before. We are going to do this in the next part of the paper. First, we make the following observations. We infer also also from [13, Theorem 5.4] that $T(H)$ and the determinant are invariant under the transposition and under similarity transformations. Moreover, according to [13, Theorem 3.5.(a),(b)], det is multiplicative and its zeroes are exactly the non-invertible elements of $T(H)$. Therefore, on the one hand, $\det I = 1$ and for any invertible operator $A \in T(H)$ the equality $\det A^{-1} = (\det A)^{-1}$ holds. On the other hand, the multiplication by an element $B \in T(H)$ with $\det B = 1$ preserves the determinant. Now we see that for each pair of invertible operators $M, N \in B(H)$ satisfying $MN \in T(H)$ and $\det MN = 1$, the selfmaps

$$A \mapsto MAN,\ A \mapsto MA^tN \quad (A \in T(H))$$

of $T(H)$ are surjective, affine and they leave det invariant. We emphasize that in this paper, a transformation $\phi\colon T(H) \to T(H)$ is called affine if

$$\phi(\lambda A + (1-\lambda)B) = \lambda\phi(A) + (1-\lambda)\phi(B) \quad (\lambda \in \mathbb{C};\ A, B \in T(H)).$$

After these observations, we formulate our main result stating that if $\dim H = \infty$, then the surjective determinant preserving maps of $T(H)$ which satisfy the last displayed equality have one of the above forms.

**Theorem 1.** *If* $\dim H = \infty$ *and* $\phi\colon T(H) \to T(H)$ *is a surjective affine map with the property that*

$$\det \phi(A) = \det A \quad (A \in T(H)),$$

*then there exist invertible operators* $M, N \in B(H)$ *such that* $MN \in T(H)$, $\det MN = 1$ *and* $\phi$ *is of the form*

$$\phi(A) = MAN \quad (A \in T(H))$$

*or of the form*

$$\phi(A) = MA^tN \quad (A \in T(H)).$$

*Proof.* By the preserver property of $\phi$ and the discussion before Theorem 1, we obtain that $\phi(I)$ has an inverse and the transformation $A \mapsto \phi(A)\phi(I)^{-1}$ $(A \in T(H))$ is a selfmap of $T(H)$ which leaves det invariant. Moreover, it is surjective affine and unital, i.e., it sends $I$ to $I$. Now it is straightforward to see that the map $\psi\colon C_1(H) \to C_1(H)$ defined by $\psi(A) = \phi(A+I)\phi(I)^{-1} - I$ is onto, affine and has the properties $\psi(0) = 0$ and

$$\det(\psi(A) + I) = \det(A + I) \quad (A \in C_1(H)).$$

Thus $\psi$ is a surjective linear transformation and then it follows that

$$\det(\mu\psi(A) + I) = \det(\mu A + I) \quad (A \in C_1(H),\ \mu \in \mathbb{C}).$$

According to [13, Theorem 3.5.(b)], the latter equality yields that for all operators $A \in C_1(H)$ and numbers $\mu \in \mathbb{C}$ the equivalence

$$0 \in \mu\sigma(\psi(A)) + 1 \iff 0 \in \mu\sigma(A) + 1$$

holds (in this paper $\sigma$ denotes the spectrum). It implies that

$$\forall A \in C_1(H),\ \mu \in \mathbb{C} \setminus \{0\} :\ -\mu^{-1} \in \sigma(\psi(A)) \iff -\mu^{-1} \in \sigma(A).$$

By the previous observations and the properness of the ideal $C_1(H)$ (here we use the condition $\dim H = \infty$), we infer that $0 \in \sigma(A)$ and that $\psi$ preserves the spectrum, i.e. $\sigma(\psi(A)) = \sigma(A)$ $(A \in C_1(H))$.

To proceed, we need some arguments from the paper [7] in which spectrum preserving linear maps between the spaces $B(X)$ and $B(Y)$ of all bounded linear operators on the Banach space $X$, resp. $Y$ were studied. Lemma 2 there tells us that such a map is injective and Theorem 1 provides a characterization of rank-1 elements in $B(X)$ in terms of their spectra. One can check that the proofs of those statements can be used to show that any spectrum preserving linear transformation of $C_1(H)$ is one-to-one and that the following equivalence holds. We denote the set of the rank-1, resp. finite rank operators in $B(H)$ by

$F_1(H)$, resp. $F(H)$. Given a nonzero operator $A \in C_1(H)$ we have $A \in F_1(H)$ if and only if

$$\sigma(R+A) \cap \sigma(R+cA) \subset \sigma(R) \quad (\forall R \in C_1(H),\ c \in \mathbb{C} \setminus \{1\}).$$

Now, referring to the previous paragraph, we obtain that $\psi$ is injective and then that it preserves the rank-1 operators in both directions, i.e. an arbitrary operator $A \in C_1(H)$ is in $F_1(H)$ if and only if $\psi(A) \in F_1(H)$. Since the elements of $F(H)$ can be decomposed into sums of operators in $F_1(H)$, it follows that the linear map $\psi$ leaves $F(H)$ invariant. It is trivial that the inverse of the bijection $\psi$ has the same preserver properties as $\psi$, and this yields that the transformation $\psi|_{F(H)} \colon F(H) \to F(H)$ is onto, additive and preserves the elements of $F_1(H)$ in both directions. Applying [9, Theorem 3.3] to it, we deduce that there are bijective semilinear operators $S_1, S_2, T_1, T_2 \colon H \to H$ such that one of the following possibilities holds.

(i) $\psi(x \otimes y) = S_1 x \otimes T_1 y$ $(x, y \in H)$
(ii) $\psi(x \otimes y) = S_2 y \otimes T_2 x$ $(x, y \in H)$

As for the symbol $\otimes$, we denote by $x \otimes y$ the element of $B(H)$ defined by $x \otimes y(u) = \langle u, y\rangle x$ $(u, x, y \in H)$. Operators of this form are called elementary tensors. Now the linearity of $\psi$ implies that $S_1, T_1$ are linear and $S_2, T_2$ are conjugate-linear. Moreover, in case (ii) we have

$$\psi(x \otimes y) = \tilde{S}_2 \overline{y} \otimes \tilde{T}_2 \overline{x} \quad (x, y \in H),$$

where $\tilde{S}_2$, resp. $\tilde{T}_2$ is the operator obtained from $S_2$, resp. $T_2$ by composing it from the right with the conjugation $^-$. Then $\tilde{S}_2, \tilde{T}_2$ are linear and we are going to show that the bijections $S_1, T_1, \tilde{S}_2, \tilde{T}_2$ are bounded and hence invertible in $B(H)$, furthermore $T_1 = (S_1^{-1})^*$, $\tilde{T}_2 = (\tilde{S}_2^{-1})^*$. To do this, we need the simple fact that $\sigma(x \otimes y) = \{\langle x, y\rangle, 0\}$ $(x, y \in H)$. Referring to the spectrum preserving property of $\psi$, the latter relation implies that in case (i), resp. (ii) the equality $\langle S_1 x, T_1 y\rangle = \langle x, y\rangle$, resp. $\langle \tilde{S}_2 \overline{y}, \tilde{T}_2 \overline{x}\rangle = \langle x, y\rangle$ $(x, y \in H)$ holds. Since $\langle x, y\rangle = \langle \overline{y}, \overline{x}\rangle$, in case (ii) it follows that $\langle \tilde{S}_2 u, v\rangle = \langle u, \tilde{T}_2^{-1} v\rangle$, while in case (i) we get that

$$\langle S_1 u, v\rangle = \langle u, T_1^{-1} v\rangle \quad (u, v, x, y \in H).$$

It is easy to check that if two linear Hilbert space operators satisfy the defining equality of the adjoint, then they are bounded and the adjoints of each other. The previous observations imply that $S_1, T_1, \tilde{S}_2, \tilde{T}_2 \in B(H)$ and $T_1 = (S_1^{-1})^*$, $\tilde{T}_2 = (\tilde{S}_2^{-1})^*$. Using the equality displayed last but one and the one in (i), it follows that

$$\psi(x \otimes y) = S_1 (x \otimes y) S_1^{-1} \quad (x, y \in H)$$

or

$$\psi(x \otimes y) = \tilde{S}_2(\overline{y} \otimes \overline{x})\tilde{S}_2^{-1} = \tilde{S}_2(x \otimes y)^t \tilde{S}_2^{-1} \quad (x, y \in H).$$

Since the elements of $F_1(H)$ are elementary tensors, we arrive at the following conclusion. There is an invertible operator $S \in B(H)$ such that $\psi(A) = SAS^{-1}$ $(A \in F_1(H))$ or $\psi(A) = SA^tS^{-1}$ $(A \in F_1(H))$, which, by the linearity of $\psi$, yields that $\psi(A) = SAS^{-1}$ $(A \in F(H))$ or $\psi(A) = SA^tS^{-1}$ $(A \in F(H))$.

In what follows, we will prove that $\psi$ is of one of the latter forms. To this end, we introduce a map $\tilde{\psi} \colon C_1(H) \to C_1(H)$ in the following way. If $\psi(A) = SAS^{-1}$ $(\forall A \in F(H))$, then we define $\tilde{\psi}(A) = S^{-1}\psi(A)S$; otherwise $\tilde{\psi}$ is given by the formula

$$\tilde{\psi}(A) = (S^{-1}\psi(A)S)^t \quad (A \in C_1(H)).$$

Now on the one hand, by the properties of $\psi$, for all elements $A \in C_1(H)$, $T \in F(H)$ one has $\sigma(\psi(A) + \psi(T)) = \sigma(A + T)$. On the other hand, the spectrum is clearly invariant under similarity transformations and under the transposition. The latter properties and the last equality imply that

$$\sigma(\tilde{\psi}(A) + T) = \sigma(A + T) \quad (A \in C_1(H),\ T \in F(H)). \tag{1}$$

Next we recall two facts. The first one is that any compact operator on $H$ is the limit of a sequence of elements in $F(H)$. The second one can be derived from a theorem of Newburgh stating that if $\sigma(a)$ is totally disconnected for an element $a$ in a complex unital Banach algebra $\mathcal{A}$, then $\sigma$ is continuous at $a$. This yields that if $\sigma(a)$ is countable, then we have the latter conclusion. We infer the continuity of $\sigma$ at any point in $B(H)$ which is compact. These facts together with the equality (1) give us that it is valid for any compact operator $T \in B(H)$, so in particular for each member $T$ of $C_1(H)$. Now let $A, T \in C_1(H)$; $\lambda \in \mathbb{C}$ be arbitrary elements. Using the last part of the proof, we get that

$$\sigma(\tilde{\psi}(A) + (T + \lambda I)) = \sigma(A + (T + \lambda I)),$$

implying

$$\rho(\tilde{\psi}(A) + (T + \lambda I)) = \rho(A + (T + \lambda I)),$$

where $\rho$ is the spectral radius. This means that while $A$ is kept fixed and $R$ is running through the unital standard operator algebra of all members of $B(H)$ which are sums of trace class and scalar operators, the relation $\rho(\tilde{\psi}(A) + R) = \rho(A + R)$ holds. Applying [4, Theorem 4.2], we obtain that $\tilde{\psi}(A) = A$. Since this holds for arbitrary elements $A \in C_1(H)$ we infer that $\psi(A) = SAS^{-1}$ $(A \in C_1(H))$ or $\psi(A) = SA^tS^{-1}$ $(A \in C_1(H))$. Transforming back to $\phi$, we conclude that $\phi(A) = MAN$ $(A \in T(H))$ or $\phi(A) = MA^tN$ $(A \in T(H))$ is valid for the invertible operators $M = S$, $N = S^{-1}\phi(I)$. One has $MN \in T(H)$, $\det MN = 1$ and now the proof of Theorem 1 is complete. $\square$

Before formulating the second result in the paper, we introduce some notation that will be used in the rest of it. Let $n$ be a positive integer and $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We denote by $PD_n(\mathbb{K})$ the set of all positive definite matrices in $M_n(\mathbb{K})$. Moreover, let $\mathcal{M}(\mathbb{K}) \subset M_n(\mathbb{K})$ be either $PD_n(\mathbb{K})$ or the cone of all positive semidefinite matrices. We remark that both of the previous sets are strongly related to the notion of determinant. Now our second result follows in which, for a fixed real number $\alpha > 0$, we describe the structure of those maps $\phi\colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(\mathbb{K})$ which satisfy

$$\det(\alpha\phi(A) + \phi(B)) = \det(\alpha A + B) \tag{2}$$

for any elements $A, B \in \mathcal{M}(\mathbb{K})$.

**Theorem 2.** *If $\alpha > 0$ is a real number and $\phi\colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(\mathbb{K})$ is a transformation with the property that the relation* (2) *holds for all elements $A, B \in \mathcal{M}(\mathbb{K})$, then there exists a matrix $M \in M_n(\mathbb{K})$ such that $|\det M| = 1$ and $\phi$ is of the form*

$$\phi(A) = MAM^* \quad (A \in \mathcal{M}(\mathbb{K})) \tag{3}$$

*or of the form*

$$\phi(A) = MA^tM^* \quad (A \in \mathcal{M}(\mathbb{K})). \tag{4}$$

*Proof.* First, observe that by inserting equal elements $A, B \in \mathcal{M}(\mathbb{K})$ in the relation (2), we see that $\phi$ preserves det and therefore $\phi(PD_n(\mathbb{K})) \subset PD_n(\mathbb{K})$. This yields that $\phi(I) \in PD_n(\mathbb{K})$ and $\det\phi(I) = 1$. By the latter inclusion, in the case $\mathbb{K} = \mathbb{R}$, resp. $\mathbb{K} = \mathbb{C}$ there is an orthogonal, resp. unitary matrix $Q \in M_n(\mathbb{K})$ and a diagonal one $D \in M_n(\mathbb{R})$ such that the diagonal of $D$ consists of positive numbers and $\phi(I) = QDQ^*$. Now set $A_0 = Q\tilde{D}^{-1}Q^*$, where $\tilde{D}$ is the matrix given by $\tilde{D}_{ij} = \sqrt{D_{ij}}$ $(i, j = 1, \dots, n)$. Then $A_0$ is a well-defined element of $PD_n(\mathbb{K})$ such that $\det A_0 = (\det D)^{-1/2} = 1$. Let $\psi\colon \mathcal{M}(\mathbb{K}) \to M_n(\mathbb{K})$ be the map given by $\psi(A) = A_0\phi(A)A_0$ $(A \in \mathcal{M}(\mathbb{K}))$. We see that $\psi$ is unital with range in $\mathcal{M}(\mathbb{K})$ and $\psi(PD_n(\mathbb{K})) \subset PD_n(\mathbb{K})$. Moreover, since the map $V \mapsto A_0VA_0$ $(V \in M_n(\mathbb{K}))$ is clearly linear and leaves the determinant invariant, we arrive at the equality

$$\det(\alpha\psi(A) + \psi(B)) = \det(\alpha A + B) \quad (A, B \in \mathcal{M}(\mathbb{K})). \tag{5}$$

Observe that $\psi$ preserves det.

Now we are going to verify that (5) is valid in the case where $A, B \in PD_n(\mathbb{K})$ and $\alpha$ is replaced by any positive number. In order to do so, we need to show that $\psi|_{PD_n(\mathbb{K})}$ is positive homogeneous. To obtain the latter conclusion, Minkowski's determinant inequality (see, e.g. [6, Theorem 7.8.8]) will be used. It states that for any matrices $A, B \in PD_n(\mathbb{K})$ one has $\sqrt[n]{\det(A+B)} \geq \sqrt[n]{\det A} + \sqrt[n]{\det B}$ and equality holds in this relation if and only if $A$ is a positive scalar multiple of $B$. Now let $a > 0$ be a number and $A \in PD_n(\mathbb{K})$ be a matrix. Using the invariance properties of $\psi$ and the latter assertion, we have

$$\sqrt[n]{\det(\alpha A + (aA))} = \alpha \sqrt[n]{\det A} + \sqrt[n]{\det aA}$$

yielding that

$$\sqrt[n]{\det(\alpha\psi(A) + \psi(aA))} = \alpha \sqrt[n]{\det \psi(A)} + \sqrt[n]{\det \psi(aA)} = \sqrt[n]{\det \alpha\psi(A)} + \sqrt[n]{\det \psi(aA)}$$

which implies the existence of a scalar $\lambda > 0$ satisfying $\psi(aA) = \lambda\psi(A)$. Since $\psi$ preserves det, we then obtain that

$$a^n \det A = \det aA = \det \lambda\psi(A) = \lambda^n \det A,$$

which gives us that $a = \lambda$ and thus $\psi(aA) = a\psi(A)$. Using the equality (5) and the latter one, we deduce that

$$\det(tA + B) = \det\left(\alpha\left(\frac{t}{\alpha}A\right) + B\right) = \det\left(\alpha\left(\frac{t}{\alpha}\psi(A)\right) + \psi(B)\right) = \det(t\psi(A) + \psi(B)) \quad (A, B \in PD_n(\mathbb{K});\ t > 0).$$

Observe that for each pair of matrices $V, W \in M_n(\mathbb{K})$, the map $p_{V,W} \colon \mathbb{K} \to \mathbb{K}$ defined by

$$p_{V,W}(x) = \det(xV + W) \quad (\forall x \in \mathbb{K})$$

is a polynomial. Referring to the last chain of equalities, we deduce that $p_{\psi(A),\psi(B)},\ p_{A,B}$ agree on an infinite set, so $p_{\psi(A),\psi(B)} = p_{A,B}$, thus

$$p'_{\psi(A),\psi(B)}(0) = p'_{A,B}(0) \quad (A, B \in PD_n(\mathbb{K})). \tag{6}$$

In what follows, we are going to use Jacobi's formula involving the adjugate adj and the trace tr of square matrices. It states that if $F \colon \mathbb{R} \to M_n(\mathbb{K})$ is a differentiable map, then so is $\det \circ F$ and for any $t \in \mathbb{R}$ the relation $(\det \circ F)'(t) = \operatorname{tr} \operatorname{adj}(F(t))F'(t)$ holds. Then we obtain that

$$p'_{V,W}(0) = \operatorname{tr} \operatorname{adj}(W)V \quad (V, W \in M_n(\mathbb{K}))$$

which, together with (6), implies that

$$\operatorname{tr} \operatorname{adj}(\psi(A))\psi(B) = \operatorname{tr} \operatorname{adj}(A)B \quad (A, B \in PD_n(\mathbb{K})). \tag{7}$$

Next, applying the positive homogeneity of the unital map $\psi|_{PD_n(\mathbb{K})}$, we get that it fixes the positive scalar multiples of $I$. Now let

$A \in \mathcal{M}(\mathbb{K})$, $t > 0$ be arbitrary elements. Plugging $B = (\alpha t)I$ in the equality (5) and applying the latter observation, we infer $\det(\psi(A) + tI) = \det(A + tI)$ which, similarly to the beginning of the preceding paragraph, yields that $p_{\psi(A),I} = p_{A,I}$. This means that $\psi$ leaves characteristic polynomials invariant, thus it preserves the eigenvalues together with their multiplicities implying that

$$\operatorname{tr} \psi(A) = \operatorname{tr} A. \tag{8}$$

Denoting by $P_1(\mathbb{K})$ the collection of all projections in the $C^*$-algebra $M_n(\mathbb{K})$ of rank 1, observe that $A \in I + P_1(\mathbb{K})$ exactly when $\sigma(A) = \{1, 2\}$ and the multiplicity of the eigenvalue 2 is 1. We infer that $\psi$ maps $I + P_1(\mathbb{K})$ into itself. Let $B \in I + P_1(\mathbb{K})$ be an element. Then one can check that $\operatorname{adj}(B) = (\det B)B^{-1} = 3I - B$, thus we deduce that

$$\operatorname{tr} \operatorname{adj}(B)A = 3 \operatorname{tr} A - \operatorname{tr} AB = 2 \operatorname{tr} A - \operatorname{tr} A(B - I). \tag{9}$$

If $A \in PD_n(\mathbb{K})$, then by the invariance of $I + P_1(\mathbb{K})$ under $\psi$ and by the equalities (7), (9), we arrive at the relation

$$3 \operatorname{tr} \psi(A) - \operatorname{tr} \psi(A)\psi(B) = 3 \operatorname{tr} A - \operatorname{tr} AB$$

which, together with the equality (8), gives us that $\operatorname{tr} \psi(A)\psi(B) = \operatorname{tr} AB$.

We proceed with defining a transformation $\Psi \colon P_1(\mathbb{K}) \to M_n(\mathbb{K})$ by

$$\Psi(P) = \psi(I + P) - I \quad (P \in P_1(\mathbb{K})).$$

The inclusion $\psi(I + P_1(\mathbb{K})) \subset I + P_1(\mathbb{K})$ shows that $\Psi$ maps $P_1(\mathbb{K})$ into itself. Furthermore, the equality (8) and the last one in the previous paragraph yield that all elements $P, Q \in P_1(\mathbb{K})$ satisfy

$$\operatorname{tr} \Psi(P)\Psi(Q) = \operatorname{tr} \psi(I + P)\psi(I + Q) - \operatorname{tr} \psi(I + P) - \operatorname{tr} \psi(I + Q) + n =$$
$$\operatorname{tr}(I + P)(I + Q) - \operatorname{tr}(I + P) - \operatorname{tr}(I + Q) + n = \operatorname{tr} PQ.$$

This gives us that $\Psi$ leaves the trace of the product invariant. Then the non-surjective version of a famous theorem of Wigner (see, e.g. [8, Theorem 2.1.4.]) applies and we easily obtain the following conclusion. In the case $\mathbb{K} = \mathbb{R}$, resp. $\mathbb{K} = \mathbb{C}$ there is an orthogonal, resp. a unitary matrix $U \in M_n(\mathbb{K})$ such that $\Psi$ is either the map $P \mapsto UPU^*$ ($P \in P_1(\mathbb{K})$) or the transformation $P \mapsto UP^tU^*$ ($P \in P_1(\mathbb{K})$). This statement immediately implies that $\psi|_{I+P_1(\mathbb{K})}$ is of one of these forms and since $\psi|_{PD_n(\mathbb{K})}$ is positive homogeneous, we find that either

$$\psi(\lambda A) = U(\lambda A)U^* \quad (A \in I + P_1(\mathbb{K}),\ \lambda > 0) \tag{10}$$

or

$$\psi(\lambda A) = U(\lambda A)^tU^* \quad (A \in I + P_1(\mathbb{K}),\ \lambda > 0). \tag{11}$$

In what follows, we define a map $\tilde{\psi}\colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(\mathbb{K})$. In the case where (10) holds, we give $\tilde{\psi}$ by the formula $\tilde{\psi}(A) = U^*\psi(A)U$, otherwise it is introduced by the equality

$$\tilde{\psi}(A) = (U^*\psi(A)U)^t \quad (A \in \mathcal{M}(\mathbb{K})).$$

Next, we show that $\tilde{\psi}$ is the identity map. To see this, pick arbitrary elements $A \in \mathcal{M}(\mathbb{K})$, $X \in I + P_1(\mathbb{K})$ and $t \in ]0, \infty[$. Then referring to the relations (5), (10), (11) and to the properties of det, we arrive at the equality

$$\det\left(\alpha\tilde{\psi}(A) + \frac{\alpha}{t}X\right) = \det\left(\alpha\psi(A) + \psi\left(\frac{\alpha}{t}X\right)\right) = \det\left(\alpha A + \frac{\alpha}{t}X\right).$$

Hence one has $\det\left(t\tilde{\psi}(A) + X\right) = \det(tA + X)$, i.e., $p_{\tilde{\psi}(A),X}(t) = p_{A,X}(t)$ and since it holds for all $t > 0$, similarly to the second paragraph of the proof, we deduce that $\operatorname{tr}\operatorname{adj}(X)\tilde{\psi}(A) = \operatorname{tr}\operatorname{adj}(X)A$. According to (9), this gives us that

$$2\operatorname{tr}\tilde{\psi}(A) - \operatorname{tr}\tilde{\psi}(A)(X - I) = 2\operatorname{tr} A - \operatorname{tr} A(X - I).$$

Referring to the equality (8), we infer that $\operatorname{tr}\tilde{\psi}(A) = \operatorname{tr} A$ and then it follows that $\operatorname{tr}\tilde{\psi}(A)(X-I) = \operatorname{tr} A(X-I)$, i.e. $\operatorname{tr}\tilde{\psi}(A)P = \operatorname{tr} AP$ for any element $P \in P_1(\mathbb{K})$ (here we use the condition that $X$ is an arbitrary matrix in $I + P_1(\mathbb{K})$). One can check that for each unit column vector $x \in \mathbb{K}^n$, the element $x\overline{x}^t \in M_n(\mathbb{K})$ is in $P_1(\mathbb{K})$, therefore we obtain that $x^t\tilde{\psi}(A)^t\overline{x} = x^tA^t\overline{x}$ for all $n$-tuples $x \in \mathbb{K}^n$. It implies that the matrices of the latter quadratic forms are the same, i.e. $\tilde{\psi}(A) = A$ and since this holds for any element $A \in \mathcal{M}(\mathbb{K})$, we infer that $\psi(A) = UAU^*$ $(A \in \mathcal{M}(\mathbb{K}))$ or $\psi(A) = UA^tU^*$ $(A \in \mathcal{M}(\mathbb{K}))$. Therefore, transforming back to $\phi$, we find that $\phi(A) = MAM^*$ $(A \in \mathcal{M}(\mathbb{K}))$ or $\phi(A) = MA^tM^*$ $(A \in \mathcal{M}(\mathbb{K}))$, where $M = A_0^{-1}U$. It has been mentioned at the beginning of the proof that $\det A_0 = 1$, so we deduce that $|\det M| = 1$. Now the verification of Theorem 2 is complete. $\square$

**Remarks.** We complete the paper with the following comments. In Theorem 1, we have supposed that $\dim H = \infty$. However, it is a natural question whether that result is true in the finite dimensional case. In that case, a much more general statement is valid, namely [14, Theorem 1]. It tells us that if $\phi$ is a selfmap of $M_n(\mathbb{C})(= T(\mathbb{C}^n))$ satisfying (2) for all matrices $A, B \in M_n(\mathbb{C})$ and scalars $\alpha \in \mathbb{C}$, then the conclusion of our first result holds.

Concerning Theorem 2 one may ask that given numbers $\alpha, \beta \in \mathbb{C}$, can the structure of maps $\Phi\colon \mathcal{M}(\mathbb{K}) \to \mathcal{M}(\mathbb{K})$ preserving the quantity $\det(\alpha A + \beta B)$ $(A, B \in \mathcal{M}(\mathbb{K}))$ be described? In the case $\alpha\beta = 0$, the answer is negative, because then $\Phi$ has no regular form. Otherwise, observe that since $\det\colon M_n(\mathbb{K}) \to \mathbb{K}$ is homogeneous of degree $n$, we may and do assume that $\beta = 1$. By Theorem 2, the answer to the previous question is affirmative if $\alpha > 0$. In other cases, we do not know anything about the general form of $\Phi$. However, if $\mathbb{K} = \mathbb{C}$, $\alpha = -1$ and $\Phi$ is bijective, then its form can be determined. Indeed, in that case $\Phi$ is a bijection of $\mathcal{M}(\mathbb{K})$ which preserves the determinant and hence the invertibility of the difference. Thus [11, Theorems 3.15, 3.16] apply and we easily get that $\Phi$ is of one of the forms (3) or (4).

Finally, let $\phi$ be such a selfmap of $-\mathcal{M}(\mathbb{K})$, i.e., the set of all negative definite or negative semidefinite matrices in $M_n(\mathbb{K})$ which satisfies (2) for a number $\alpha > 0$ and for any elements $A, B \in -\mathcal{M}(\mathbb{K})$. Then applying Theorem 2 to the transformation $A \mapsto -\phi(-A)$ $(A \in \mathcal{M}(\mathbb{K}))$, we obtain that $\phi$ can be written in one of the forms (3) or (4).

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