A CHARACTERIZATION OF CENTRAL ELEMENTS IN C^*-ALGEBRAS

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Abstract. In the paper [8] Wu proved that if the exponential function on the set of all positive elements of a C^*-algebra is monotone in the usual partial order, then the algebra in question is necessary commutative. In this note we present a local version of that result and obtain a characterization of central elements in C^*-algebras in terms of the order.

1. Introduction

Let $\mathcal{A}$ be a (unital) C^*-algebra and denote by $\mathcal{A}_s$ the space of all of its self-adjoint elements. An element $x \in \mathcal{A}_s$ is called positive, $x \geq 0$, if its spectrum $\sigma(x)$ lies in the non-negative part of the real line. The set of all positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}_+$. The usual partial order $\leq$ on $\mathcal{A}_s$ is then defined in the following way: for any $x, y \in \mathcal{A}_s$ we write $x \leq y$ iff $y - x \in \mathcal{A}_+$.

There are some classical results in the literature which characterize the commutativity of C^*-algebras in terms of certain properties of the order. For example, a result of Sherman [7] says that a C^*-algebra $\mathcal{A}$ is commutative if and only if $\mathcal{A}_s$ is a lattice (cf. [1]). Another famous result is due to Ogasawara [4] telling that squaring is monotone on $\mathcal{A}_+$ if and only if $\mathcal{A}$ is commutative. The slightly more general result [5, 1.3.9. Proposition] shows that if the power function $t \mapsto t^\beta$ with $\beta > 1$ is monotone with respect to the usual order on $\mathcal{A}_+$ (meaning that $x, y \in \mathcal{A}_+$, $x \leq y$ implies $x^\beta \leq y^\beta$), then the algebra $\mathcal{A}$ is necessarily commutative. In [8] Wu presented a similar statement saying that the same conclusion holds if the power function is replaced by the exponential function.

In this note we present a local version of Wu's result. Namely, we show that the "points of monotonicity" of the exponential function on $\mathcal{A}_s$ necessarily belong to the center of $\mathcal{A}$. Having proven that we get Wu's result as an apparent consequence.

2. The result

Our result read as follows.

Theorem 1. *Let $\mathcal{A}$ be a C^*-algebra and $x \in \mathcal{A}_s$. The following assertions are equivalent:*

(i) *we have $e^x \leq e^y$ for every $y \in \mathcal{A}_s$ with $x \leq y$;*

(ii) *$\int_0^1 e^{tx} z e^{(1-t)x} d\,t \in \mathcal{A}_+$ holds for all $z \in \mathcal{A}_+$;*

(iii) *x is a central element of $\mathcal{A}$.*

For the proof we need the following auxiliary lemma.

Lemma 2. *Let H be a complex Hilbert space and denote by $B(H)$ the algebra of all bounded linear operators on H. Let $T \in B(H)$ be self-adjoint. Assume that $0 = \min \sigma(T)$ and $r = \max \sigma(T)$. For every $\epsilon > 0$ we can choose orthogonal unit vectors $\xi, \eta \in H$ such that for any*

2010 *Mathematics Subject Classification.* Primary: 46L05.

Key words and phrases. C^*-algebra, central element, order, exponential function.

The author was supported by the "Lendület" Program (LP2012-46/2012) of the Hungarian Academy of Sciences and by the Hungarian Scientific Research Fund (OTKA) Reg. No. K115383.

$A \in B(H)$ with the properties $\|A\| \leq \sqrt{2}$, $A\xi = A\eta$, $\|A\xi\| = \|A\eta\| = 1$, the positivity of the operator

$$\int_0^1 \exp(tT)A^*A\exp((1-t)T)d\,t$$

implies the inequality

$$\left(\frac{e^r-1}{r}\right)^2 \leq (1+2\epsilon)(e^r+2\epsilon).$$

Proof. It is apparent that for any pair $f, g : [0,1] \to B(H)$ of continuous functions, the transformation

$$X \mapsto \int_0^1 f(t)Xg(t)d\,t$$

is a bounded linear map on $B(H)$ and its norm is majorized by the product of the supremum norms of f and g. It follows that the above integral depends continuously on the functions f and g, meaning that if $f_n, g_n : [0,1] \to B(H)$ are sequences of continuous functions uniformly converging to f and g, respectively, then the corresponding sequence

$$X \mapsto \int_0^1 f_n(t)Xg_n(t)d\,t$$

of bounded linear maps on $B(H)$ converges to the map

$$X \mapsto \int_0^1 f(t)Xg(t)d\,t$$

in the operator norm.

It is easy to see that if (T_k) is a sequence in $B(H)$ which converges in norm to T, then the sequence $t \mapsto \exp(tT_k)$ of operator valued functions converges to $t \mapsto \exp(tT)$ uniformly in $t \in [0,1]$. It follows that given $T \in B(H)$, for every real number $\epsilon > 0$ we have a real number $\delta > 0$ such that

$$\sup_{\|X\|\leq 1} \left\| \int_0^1 \exp(tT)X\exp((1-t)T)d\,t - \int_0^1 \exp(tT')X\exp((1-t)T')d\,t \right\| \leq \epsilon \tag{1}$$

holds whenever $T' \in B(H)$ with $\|T - T'\| \leq \delta$. Obviously, we may assume that $2\delta < r$. Consider a continuous function $h : [0,r] \to [0,r]$ which is zero on the interval $[0,\delta]$, it equals r on $[r-\delta, r]$ and its distance to the identity function on $[0,r]$ in the supremum norm is not greater than δ. Then we have $\|T - h(T)\| \leq \delta$ and hence we obtain from (1) that

$$\begin{aligned} \left| \int_0^1 \langle \exp(tT)A^*A\exp((1-t)T)\zeta,\zeta\rangle d\,t - \int_0^1 \langle \exp(th(T))A^*A\exp((1-t)h(T))\zeta,\zeta\rangle d\,t \right| \\ \leq \epsilon\|A\|^2\|\zeta\|^2 \end{aligned} \tag{2}$$

holds for every operator $A \in B(H)$ and vector $\zeta \in H$. Observe that, by elementary change of variables, for any self-adjoint operator $S \in B(H)$ we obtain that

$$\int_0^1 \exp(tS)A^*A\exp((1-t)S)d\,t = \int_0^1 \exp((1-t)S)A^*A\exp(tS)d\,t, \quad A \in B(H)$$

implying that the values of these integrals are self-adjoint operators. Therefore, if

$$\int_0^1 \exp(tT)A^*A\exp((1-t)T)d\,t \geq 0, \tag{3}$$

then it follows from (2) that

$$\begin{aligned}0 &\le \int_0^1 \langle \exp(tT)A^*A\exp((1-t)T)\zeta,\zeta\rangle dt\\ &\le \int_0^1 \langle \exp(th(T))A^*A\exp((1-t)h(T))\zeta,\zeta\rangle dt + \epsilon\|A\|^2\|\zeta\|^2\\ &= \int_0^1 \langle A\exp((1-t)h(T))\zeta, A\exp(th(T))\zeta\rangle dt + \epsilon\|A\|^2\|\zeta\|^2.\end{aligned}$$

Denote by E the spectral measure of T on the Borel subsets of $[0,r]$. Pick a unit vector ξ from the range of $E([0,\delta])$ and another one η from the range of $E([r-\delta,r])$. Clearly, we have that ξ is orthogonal to η. Let s be an arbitrary real number and set $\zeta = s\xi+\eta$. We compute

$$A\exp((1-t)h(T))\zeta = A(s\xi + e^{(1-t)r}\eta), \quad A\exp(th(T))\zeta = A(s\xi+e^{tr}\eta)$$

and hence, for any $A\in B(H)$ satisfying (3), $\|A\|\le\sqrt{2}$, $A\xi = A\eta$, $\|A\xi\| = \|A\eta\| = 1$ we obtain

$$0 \le \int_0^1 (s+e^{(1-t)r})(s+e^{tr})dt + \epsilon 2(s^2+1)$$

for every real number s. This implies

$$0\le s^2 + e^r + 2s\frac{e^r-1}{r} + \epsilon 2(s^2+1)$$

for every real number s. Examining the discriminant of the corresponding quadratic equation, we obtain

$$4\left(\frac{e^r-1}{r}\right)^2 - 4(1+2\epsilon)(e^r+2\epsilon)\le 0$$

which gives us the statement of the lemma. □

After this we are now in a position to prove the theorem.

Proof of Theorem 1. According to the bottom line on p. 148 in [6], the (Fréchet-) derivative of the exponential function $T\mapsto \exp T$ on $B(H)$ at the point T is the linear map

$$X\mapsto \int_0^1 \exp(tT)X\exp((1-t)T)dt.$$

This implies that the function $x\mapsto e^x$ on the C^*-algebra $\mathcal{A}$ is differentiable at x and its derivative is the linear map

$$z\mapsto \int_0^1 \exp(tx)z\exp((1-t)x)dt.$$

Now, assuming (i) we clearly obtain (ii).

Suppose (ii) holds. Select an irreducible representation π of $\mathcal{A}$ on a Hilbert space H. We apparently obtain that

$$\int_0^1 \exp(t\pi(x))\pi(z)^*\pi(z)\exp((1-t)\pi(x))dt \ge 0$$

holds for all $z\in\mathcal{A}$. Since $\pi(1) = I$, adding a real constant times the identity to x if necessary, we may assume that the operator $\pi(x)$ is positive, 0 belongs to its spectrum whose largest element is r. By our lemma, for every $\epsilon > 0$ we can choose orthogonal unit vectors $\xi, \eta \in H$ such that for any $A \in B(H)$ with the properties $\|A\| \leq \sqrt{2}$, $A\xi = A\eta$, $\|A\xi\| = \|A\eta\| = 1$, the positivity of the operator

$$\int_0^1 \exp(t\pi(x))A^*A\exp((1-t)\pi(x))dt$$

implies

$$\left(\frac{e^r - 1}{r}\right)^2 \leq (1+2\epsilon)(e^r + 2\epsilon).$$

Define $A = \nu \otimes (\xi + \eta)$ where $\nu \in H$ is any unit vector. Clearly we have $\|A\| = \sqrt{2}$, $A\xi = A\eta$, $\|A\xi\| = \|A\eta\| = 1$. Since π is an irreducible representation, by a sharper version of Kadison transitivity theorem (see [2], Exercise 5.7.41. (ii) on p. 379), we have that there is an element $z \in \mathcal{A}$ such that $\|\pi(z)\| \leq \sqrt{2}$ and $\pi(z)\xi = A\xi, \pi(z)\eta = A\eta$. It then follows that

$$\left(\frac{e^r - 1}{r}\right)^2 \leq (1+2\epsilon)(e^r + 2\epsilon).$$

But here $\epsilon > 0$ is arbitrary, consequently, we have

$$\left(\frac{e^r - 1}{r}\right)^2 \leq e^r.$$

It is easy to check that for a non-negative real number r this holds only if $r = 0$. Therefore, we have $\pi(x) = 0$. Since we may have added a constant multiple of the identity to x, this means for the original element x that $\pi(x) = \lambda I$ holds for some real number λ. We know that this is true for all irreducible representations π of $\mathcal{A}$ and claim that x is necessarily central. Indeed, if $a \in \mathcal{A}$ is an element and $xa - ax \neq 0$, then by [3, 10.2.4. Corollary] we have an irreducible representation π such that $0 \neq \pi(xa - ax) = \pi(x)\pi(a) - \pi(a)\pi(x)$, a clear contradiction.

Finally, the implication (iii)⇒(i) is apparent and hence the proof is complete. □

As an immediate corollary we obtain the following statement which is formally still stronger than Wu's original theorem.

Corollary 3. *Let $\mathcal{A}$ be a C^*-algebra such that the exponential function is monotone on a non-generate interval I of the real line meaning that I is of positive length and for any $x, y \in \mathcal{A}_s$ with $\sigma(x), \sigma(y) \subset I$ and $x \leq y$, we have $e^x \leq e^y$. Then $\mathcal{A}$ is commutative.*

Proof. Let I' be a non-generate compact interval in the interior of I. Select $x \in \mathcal{A}_s$ such that $\sigma(x) \subset I'$. Apparently, for any element $z \in \mathcal{A}_+$ we have $\sigma(x + tz) \subset I$ holds for every small enough real number $t > 0$. It follows that the directional derivative of the exponential function on $\mathcal{A}_s$ at x along z belongs to $\mathcal{A}_+$. By the implication (ii)⇒(iii) in the theorem we have x is central in $\mathcal{A}$. We then easily obtain the desired conclusion. □

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