Induced cycles in triangle graphs

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Abstract

The triangle graph of a graph G, denoted by $\mathcal{T}(G)$, is the graph whose vertices represent the triangles (K_3 subgraphs) of G, and two vertices of $\mathcal{T}(G)$ are adjacent if and only if the corresponding triangles share an edge. In this paper, we characterize graphs whose triangle graph is a cycle and then extend the result to obtain a characterization of C_n-free triangle graphs. As a consequence, we give a forbidden subgraph characterization of graphs G for which $\mathcal{T}(G)$ is a tree, a chordal graph, or a perfect graph. For the class of graphs whose triangle graph is perfect, we verify a conjecture of the third author concerning packing and covering of triangles.

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1 Introduction

In a simple undirected graph G, a *triangle* is a complete subgraph on three vertices. The *triangle graph* of G, denoted by $\mathcal{T}(G)$, is the graph whose vertices represent the triangles of G, and two vertices of $\mathcal{T}(G)$ are adjacent if and only if the corresponding triangles of G share an edge. This notion was introduced independently several times under different names and in different contexts [16, 22, 8, 4]. One fundamental motivation is its obvious relation to the important class of line graphs.

In a more general setting, for a $k \geq 1$, the k-line graph $L_k(G)$ of G is a graph which has vertices corresponding to the K_k subgraphs of G, and two vertices are adjacent in $L_k(G)$ if the represented K_k subgraphs of G have $k-1$ vertices in common. Hence, 2-line graph means line graph in the usual sense, whilst 3-line graph is just the triangle graph, which is our current subject.

Beineke's classic result [5] gave a characterization of 2-line graphs in terms of nine forbidden subgraphs. This implies that 2-line graphs can be recognized in polynomial time. In contrast to this, as proved very recently in [2], the recognition problem of triangle graphs (and also, that of k-line graphs for each $k \geq 3$) is NP-complete. In the same paper [2], a necessary and sufficient condition is given for nontrivial connected graphs G and H to ensure that their Cartesian product $G \Box H$ is a triangle graph.

Further related results have been obtained by Laskar, Mulder and Novick [11]. They prove that for an 'edge-triangular' and 'path-neighborhood' graph G (that is when the open neighborhood of v induces a non-trivial path for each vertex $v \in V(G)$), the triangle graph $\mathcal{T}(G)$ is a tree if and only if G is maximal outerplanar. Also, they raise the characterization problem of a path-neighborhood graph G for which $\mathcal{T}(G)$ is a cycle ([11, Problem 3]). As an immediate consequence of our Theorem 4, we will answer this question; moreover we will give a forbidden subgraph characterization of graphs whose triangle graph is a tree.

Triangle graphs were studied from several further aspects; see e.g. [3, 4, 8, 12, 13, 14, 17, 18, 19].

1.1 Standard definitions

Given a graph F, a graph G is called *F-free* if no *induced* subgraph of G is isomorphic to F. When $\mathcal{F}$ is a set of graphs, G is *$\mathcal{F}$-free* if it is F-free for all $F \in \mathcal{F}$.

On the other hand, when we say that a graph F is a *forbidden subgraph* for a class $\mathcal{G}$ of graphs, it means that no $G \in \mathcal{G}$ may contain *any* subgraph isomorphic to F.

As usual, the complement of a graph G is denoted by $\overline{G}$. The nth power of a graph G is the graph G^n whose vertex set is $V(G^n) = V(G)$ and two vertices are adjacent in G^n if and only if their distance is at most n in G. Moreover, given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we use the notation $G_1 \vee G_2$ for the *join* of G_1 and G_2, that is a graph with one copy of G_1 and G_2 each, being vertex-disjoint, and all the vertices of G_1 are made adjacent with all the vertices of G_2. In particular, the *n-wheel* W_n ($n \geq 3$) is a graph $K_1 \vee C_n$ (where, as usual, K_n and C_n denote the n-vertex complete graph and the n-cycle, respectively). An *odd wheel* is a graph W_n where $n \geq 3$ is odd; and an *odd hole* in a graph is an *induced* n-cycle of odd length $n \geq 5$, whereas an *odd anti-hole* is the complement of an odd hole.

While an *acyclic* graph does not contain any cycles, a *chordal graph* is a graph which does not contain *induced* n-cycles for $n \geq 4$. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color. A set of vertices is *independent* if all pairs of its vertices are non-adjacent. The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent vertex set in G. A *clique* is a complete subgraph maximal under inclusion (i.e., in our terminology different cliques in the same graph may have different size). The *clique number* $\omega(G)$ is the maximum number of vertices of a clique in G. The *clique covering number* $\theta(G)$ is the minimum cardinality of a set of cliques that covers all vertices of G. A graph G is *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' of G.

As usual, the *open neighborhood* $N(v)$ of v is the set of neighbors of v, whilst its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. In a less usual way, we also refer to the subgraphs induced by them as $N(v)$ and $N[v]$, respectively.

Throughout this paper, the notation $K_n - G$ will refer to the graph obtained from the complete graph K_n by deleting the *edge set* of a subgraph isomorphic to G. In this way, for instance, $K_4 - K_3$ means the claw $K_{1,3}$.

1.2 New definitions and terminology

In this paper, we use the following special terminology for some types of graphs.

- The *elementary types* are:

(a) the wheel W_4,

(b) the square C_n^2 of a cycle of length $n \geq 7$.

- The *supplementary types* are the following ones. (For illustration, see Fig. 1.)

  (A) $S_A = (V_A, E_A)$, where $V_A = \{v_i, u_i \mid 1 \leq i \leq 4\}$ and

  $$E_A = \{v_i v_{i+1} \mid 1 \leq i \leq 4\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid 1 \leq i \leq 4\}$$

  (subscript addition taken modulo 4).

  (B) $S_B = (V_B, E_B)$, where $V_B = \{v_i \mid 1 \leq i \leq 5\} \cup \{u_1, u_2, u_3\}$ and

  $$E_B = \{v_i v_{i+1} \mid 1 \leq i \leq 5\} \cup \{v_3 v_5, v_4 v_1\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid 1 \leq i \leq 3\}$$

  (subscript addition taken modulo 5).

  (C) $S_C = (V_C, E_C)$, where $V_C = \{v_i \mid 1 \leq i \leq 6\} \cup \{u_1, u_2\}$ and

  $$E_C = \{v_i v_{i+1} \mid 1 \leq i \leq 6\} \cup \{v_2 v_4, v_3 v_5, v_4 v_6, v_5 v_1\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid i = 1, 2\}$$

  (subscript addition taken modulo 6).

  (D) $S_D = (V_D, E_D)$, where $V_D = \{v_i \mid 1 \leq i \leq 6\} \cup \{u_1, u_4\}$ and

  $$E_D = \{v_i v_{i+1} \mid 1 \leq i \leq 6\} \cup \{v_1 v_3, v_2 v_4, v_4 v_6, v_5 v_1\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid i = 1, 4\}$$

  (subscript addition taken modulo 6).

We also define two operations as follows.

- Suppose that $e = xy$ is an edge contained in exactly one triangle xyz, whilst xz and zy belong to more than one triangle. An *edge splitting* of e means replacing e with the 3-path xwy (where w is a new vertex) and inserting the further edge wz.

- Let u and v be two vertices at distance at least 4 apart. The *vertex sticking* of u and v means removing u and v and introducing a new vertex w adjacent to the entire $N(u) \cup N(v)$.[1]

The inverses of these operations can also be introduced in a natural way.

[1]'Vertex sticking' and its inverse operation were also introduced in [11].

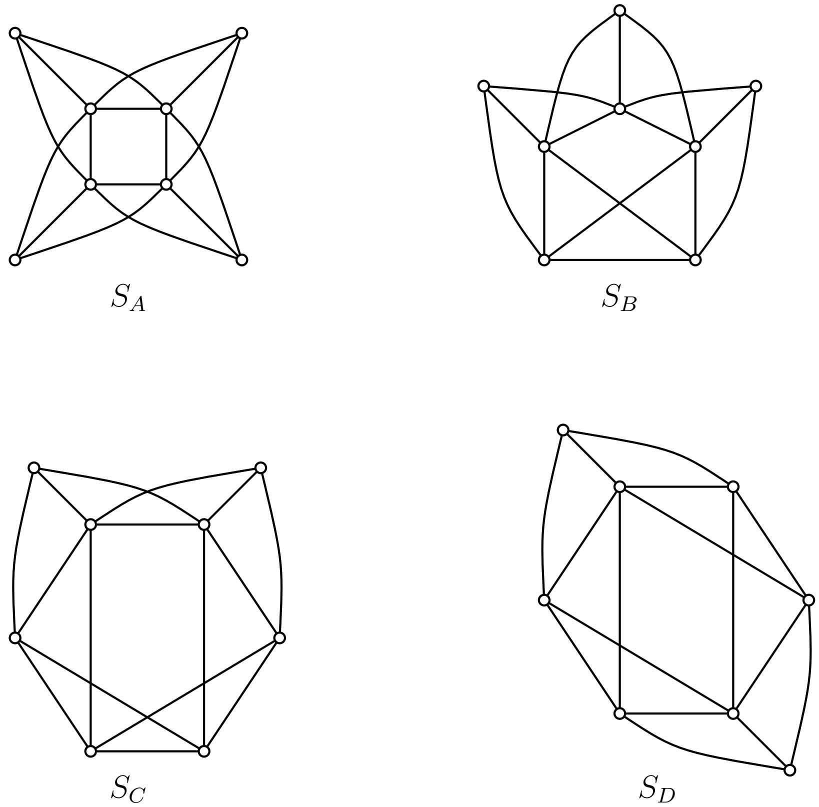

Figure 1: The four graphs of supplementary type

- Suppose that xwz and ywz are two triangles in the following position: w has degree 3, z is the unique common neighbor of w and x and also of w and y (in particular, x and y are not adjacent), and w and z are the only common neighbors of x and y. The *inverse edge splitting* at w means deleting w and its three incident edges, and inserting the new edge xy.

- Let w be a vertex whose neighborhood $N(w)$ is disconnected. The *inverse vertex sticking* at w means deleting w and its incident edges, and inserting two new vertices u and v in such a way that $N(u) \cup N(v) = N(w)$, $N(u) \cap N(v) = \emptyset$, and each component of $N(w)$ is either inside $N(u)$ or inside $N(v)$.

1.3 Our results

In Section 2, we characterize the graphs whose triangle graph is a cycle (Theorem 4) and then conclude a characterization of path-neighborhood graphs G with $\mathcal{T}(G) \cong C_n$ for some n. The latter one (Corollary 5) solves a problem raised in [11].

In Section 3, we prove a forbidden subgraph characterization of graphs G with C_n-free triangle graphs for any specified $n \geq 3$ (Theorem 13). Applying this result, we give necessary and sufficient conditions for graphs G whose $\mathcal{T}(G)$ is a tree, a chordal graph, and a perfect graph, respectively. In a sense these results form a hierarchy since every tree is chordal, and every chordal graph is perfect [7].

In Subsection 3.3 we consider the following old conjecture (usually referred to as "Tuza's Conjecture") of the third author regarding packing and covering the triangles of a graph. It was formulated in 1981 [20].

Conjecture 1 *If a graph $G = (V, E)$ does not contain more than t mutually edge-disjoint triangles, then there exists $E' \subseteq E$ such that $|E'| \leq 2t$ and each triangle of G has at least one edge in E'.*

We prove that Conjecture 1 holds for graphs whose triangle graph is perfect (Theorem 18).

2 Graphs whose triangle graph is a cycle

In this section, we give a characterization of graphs whose triangle graph is a cycle. We assume that every edge of $G = (V, E)$ is contained in a triangle, and there are no isolated vertices. Before stating the theorem, let us prove that the required property is invariant under the two operations introduced in Section 1.2.

Lemma 2 *For a graph G, let G' be a graph obtained from G by splitting an edge or sticking two vertices. Then $\mathcal{T}(G')$ is a cycle if and only if $\mathcal{T}(G)$ is a cycle.*

Proof. Let first G' be the graph obtained from G by splitting an edge $e = uv$ to the path uwv, where e belongs to exactly one triangle uvx. Let $t_1, t_2, \ldots, t_n$ be the triangles in G. Without loss of generality we may assume that $t_1 = uvx$, t_2 contains the edge vx and t_n contains the edge ux. Hence, by the edge splitting, two neighboring triangles uwx and wvx arose, where uwx and t_n, moreover wvx and t_2 also have a common edge. Then, clearly, $t_1, t_2, \ldots, t_n$ is an n-cycle in

$\mathcal{T}(G)$ if and only if the triangles $uwx, wvx, t_2, \ldots, t_n$ of G' form an $(n+1)$-cycle in $\mathcal{T}(G')$ in this cyclic order. Therefore, $\mathcal{T}(G) \cong C_n$ if and only if $\mathcal{T}(G') \cong C_{n+1}$.

Let now G' be the graph obtained from G by sticking two vertices u and v to a new vertex w, where the distance between u and v is at least four. If a triangle t of G contains neither u nor v, then t is a triangle in G'. If t is a triangle which contains u, say $t = ux_1x_2$, then wx_1x_2 is a triangle in G'. The same holds for triangles containing v. Since u and v are at distance at least 4 apart, no triangle of G is damaged and no new triangle can arise when u and v are stuck. Moreover, two triangles share an edge in G if and only if the corresponding triangles have a common edge in G'. Therefore, $\mathcal{T}(G) \cong \mathcal{T}(G')$. $\square$

Observe that splitting an edge increases the number of triangles by exactly one, whilst sticking two vertices far enough does not change the number of triangles and the number of edges in a graph but increases edge density. These observations have the following simple but important consequence.

Corollary 3 (Finite Reduction Lemma) *For every fixed n there is a finite s_n such that, starting from* any *graph whose triangle graph is a cycle, after s_n applications of edge splitting and vertex sticking in any feasible order, the length of the cycle $\mathcal{T}(G)$ of the graph G obtained surely exceeds n. Equivalently, starting from any G whose triangle graph is a cycle, inverse edge splitting and inverse vertex sticking can be applied only finitely many times.*

If an edge e belongs to exactly one triangle, we call e a *private edge* (of that triangle); and if e is contained in exactly two triangles, it is a *doubly covered edge*.

Theorem 4 *Let G be a graph which contains no isolated vertices and whose every edge is contained in at least one triangle. Then, $\mathcal{T}(G) \cong C_n$ for some $n \geq 3$ if and only if*

(i) $G \cong K_5 - K_3$, or

(ii) G is one of the elementary types or supplementary types, or

(iii) G can be obtained from one of the elementary types or supplementary types by a sequence of edge splittings and vertex stickings.

Moreover, graphs whose triangle graph is an odd hole are characterized by (ii) and (iii), with properly chosen parity of the number of edge splittings.

Proof. Clearly, $\mathcal{T}(K_5 - K_3) \cong C_3$, $\mathcal{T}(W_4) \cong C_4$, and $\mathcal{T}(C_n^2) \cong C_n$ if $n \geq 7$, moreover the triangle graphs of S_A, S_B, S_C and S_D are isomorphic to C_8. By Lemma 2, also the triangle graphs of graphs satisfying (iii) are cycles.

Now, assume that for a graph $G = (V, E)$, fulfilling the conditions of the theorem, its triangle graph $\mathcal{T}(G)$ is a cycle C_n. If an edge $e \in E$ is contained in more than two triangles, then those triangles induce a complete subgraph of order at least 3 in the triangle graph. Hence, if the latter is a cycle, then there cannot be more than three triangles, thus $\mathcal{T}(G) = C_3$ and $G \cong K_5 - K_3$.

On the other hand, if a triangle has no private edge then the degree of the corresponding vertex in $\mathcal{T}(G)$ will be at least three, which contradicts the assumption that $\mathcal{T}(G)$ is a cycle.

From now on, assume that $G \not\cong K_5 - K_3$ and consequently, each triangle has precisely one private edge and exactly two doubly covered edges. Moreover, we will suppose that the inverse operations of edge splitting and vertex sticking cannot be applied to G. (Due to the Finite Reduction Lemma above, we may assume this without loss of generality.) It will suffice to prove that such non-reducible graphs necessarily belong to an elementary or a supplementary type.

Claim 1. For every vertex $v \in V$ the neighborhood $N(v)$ is a path or a cycle; and in the latter case we have $G \cong W_4$.

Proof. First, assume for a contradiction that v is a vertex such that $N(v)$ is not connected. Let N_1 be a component of $N(v)$, and set $N_2 = N(v) - N_1$. Let G' be the graph obtained from G by deleting v and introducing two new vertices v_1 and v_2 adjacent to the vertices in N_1 and N_2, respectively. In G' the distance between v_1 and v_2 is at least 4. Therefore, the inverse operation of vertex splitting can be applied to G, which is a contradiction. Consequently, each set $N(v)$ induces a connected graph.

If a vertex $u \in N(v)$ had more than two neighbors in $N(v)$, then the edge vu would belong to more than two triangles, which contradicts our present assumption. Hence, for every $v \in V$ and every $u \in N(v)$, the degree $\deg_{N(v)}(u)$ is at most two. Since $N(v)$ is connected, this implies that $N(v)$ must be a path or a cycle. If $N(v) \cong C_n$, then $N[v] \cong W_n$ and $\mathcal{T}(W_n) \cong C_n$. Therefore, $G \cong W_n$, and since it is assumed that G cannot be reduced by inverse edge splitting, $G \cong W_4$ must be valid. $\Diamond$

Therefore, consider only the case where $G \not\cong W_4$ and $N(v)$ is a path for every $v \in V$. Partition the edge set of G as $E = F \cup H$ where F is the set of doubly covered edges and H is the set of private edges of triangles.

If $t_1, \ldots, t_n$ are the triangles in the assumed cyclic order, then we use the notation $F = \{f_1, \ldots, f_n\}$ where $f_i = E(t_i) \cap E(t_{i+1})$, and denote by h_i the private edge of t_i. Hence $E(t_i) = \{f_{i-1}, f_i, h_i\}$ (subscript addition is considered modulo n). The graphs $G_F = (V, F)$ and $G_H = (V, H)$ contain no triangles. It is worth noting that the original graph G can be obtained from G_F if, for every $1 \le i \le n$, the non-common ends of f_i and f_{i+1} are connected by an edge (which is just the private edge h_{i+1} in G).

Claim 2. The graph G_F of doubly covered edges is a *hairy cycle* (a cycle with any number of pendant vertices attached to its vertices).

Proof. Let every vertex $v \in V$ be associated with the set $I(v)$ of indices of doubly covered edges incident to v:

$$i \in I(v) \qquad \Longleftrightarrow \qquad f_i \text{ is incident to } v.$$

By definition, every index $1 \le i \le n$ is contained in exactly two sets $I(v)$. Since F contains exactly two edges from each triangle, every vertex of G is incident with at least one edge of F and hence, no $I(v)$ is empty. The fact that f_i and f_{i+1} share a vertex for every $1 \le i \le n$, implies that for every i there exists a vertex v such that $\{i, i+1\} \subseteq I(v)$. The connectivity of G_F also follows.

Now, consider a vertex $v \in V$, say of degree d. By Claim 1, its neighborhood $N(v)$ induces a path $P_d = u_1 u_2 \ldots u_d$ in G. Any two consecutive vertices of P_d together with v form a triangle. Hence, the doubly covered edges incident to v are exactly $vu_2, vu_3, \ldots, vu_{d-1}$, and the $d-1$ triangles just mentioned correspond to consecutive vertices in the cycle $\mathcal{T}(G) \cong C_n$. It also follows that the index set $I(v)$ of v contains $d-2$ consecutive integers (viewing 1 as the successor of n).

By these facts we obtain that the set of vertices which have at least two incident doubly covered edges induce a cycle in G_F. This cycle will be referred to as $C^* = v_1 v_2 \ldots v_k$, where the vertices are indexed according to the cyclic order. It contains exactly those vertices v_i for which $|I(v_i)| \ge 2$ and the two edges of C^* incident to v_i are exactly the doubly covered edges with smallest and largest indices from $I(v_i)$ (where 'smallest' and 'largest' are meant along a fixed cyclic order of $1, 2, \ldots, n$).

If a vertex v_i from C^* is incident to an edge $f_\ell = v_i u$ which does not belong to C^*, then u cannot be incident to any other edge of G_F, since both $f_{\ell-1}$ and $f_{\ell+1}$ are also incident to v_i, and vertex u, too, must be incident to edges with consecutive indices without a gap. Thus, all vertices and edges not contained in the cycle C^* are pendant vertices and edges in G_F. Consequently, G_F is a hairy cycle. $\Diamond$

From now on, when the inverse operation of edge splitting is applied in G to a vertex w which is pendant in G_F, we say that w is *eliminated*. Let us emphasize that we excluded this situation by assumption at the very beginning; hence, several proofs below will apply the fact that it is impossible to identify a vertex which can be eliminated.

Claim 3. In G_F each vertex is incident with at most one pendant edge.

Proof. Assume for a contradiction that a vertex v_i has at least two pendant neighbors. Then, there exist two pendant neighbors u and w such that $f_j = v_i u$ and $f_{j+1} = v_i w$ for some j. Also, $f_{j-1} = v_i v'$ and $f_{j+2} = v_i v''$ have to be incident to v_i (otherwise u or w would be a vertex from the cycle). Under these assumptions vertex w could be eliminated, since $v''w$ and uw are private edges, moreover the only possible common neighbor, $x \neq v_i$, of v'' and u in G might be v', but since $G \ncong W_4$, it cannot be. This contradiction proves the claim. $\lozenge$

Claim 4. If there exists a vertex v_i incident to a pendant edge $f_j = v_i u$ in G_F then the length k of cycle C^* of G_F is at most 6.

Proof. The graph G_F contains no triangle, hence $v_{i-1}v_{i+1} \notin F$. Moreover, since $f_{j-1} = v_{i-1}v_i$ and $f_{j+1} = v_i v_{i+1}$ are not consecutive doubly covered edges, $v_{i-1}v_{i+1}$ cannot be a private edge of G and so, $v_{i-1}v_{i+1} \notin E$. If the only common neighbor of v_{i-1} and v_{i+1} were v_i, then u could be eliminated and replaced by the edge $v_{i-1}v_{i+1}$. As u cannot be eliminated, vertices v_{i-1} and v_{i+1} have some common neighbor $x \neq v_i$ in G. Now, assume that the cycle of doubly covered edges is of length $k > 6$. Then, every vertex of C^* is at distance greater than two apart from at least one of v_{i-1} and v_{i+1} in G_F, and the same is true for the possible pendant vertices of G_F. Thus, no doubly covered edges and no private edges form a second triangle with $v_{i-1}v_{i+1}$ in G, and u can be eliminated if $k > 6$, which contradicts the present conditions. Thus, $k \leq 6$ follows. $\lozenge$

If there exist no pendant edges in G_F and $k \geq 7$, then G belongs to the elementary type (b). Assume that this is not the case and $k \leq 6$. Since G_F contains no triangle, the length k of C^* is either 4 or 5 or 6.

Claim 5. If $k = 4$ then $G \cong S_A$.

Proof. No chord of the four-cycle belongs to the graph G, since it would be a doubly covered edge and G_F cannot contain triangles. Avoiding such a case, G_F has to contain precisely one pendant edge on each vertex of C_4. Supplementing G_F with the private edges between the non-common ends of consecutive double edges, the graph $G \cong S_A$ is obtained. $\lozenge$

Claim 6. If $k = 5$ then $G \cong S_B$.

Proof. Let the five-cycle be $v_1v_2v_3v_4v_5$. If one of these vertices, say v_i has no pendant edge in G_F, then $v_{i-1}v_{i+1} \in H$. This edge cannot belong to any other triangle in G; hence, neither $v_{i-2}v_{i+1}$ nor $v_{i-1}v_{i+2}$ can be an edge in G. This means that both v_{i+2} and v_{i-2} must have pendant edges in G_F. Similarly, if there is no pendant edge on v_{i+1}, then there is one on vertex v_{i-1}. This proves that if G is complying with our assumption, there exist three consecutive vertices on the five-cycle of G_F such that each of them has a pendant edge.

On the other hand, if four consecutive vertices, say v_i, v_{i+1}, v_{i+2} and v_{i+3} have pendant edges, the pendant vertex adjacent to v_i can be eliminated. Consequently, if G_F has a cycle of length 5, then exactly three of its consecutive vertices have pendant edges, and $G \cong S_B$ holds. $\diamondsuit$

Claim 7. If $k = 6$ then $G \cong S_C$ or $G \cong S_D$.

Proof. Consider a six-cycle $v_1v_2v_3v_4v_5v_6$. If none of the vertices v_1, v_3 and v_5 (or v_2, v_4 and v_6) has a pendant edge, then a forbidden triangle $v_2v_4v_6$ ($v_1v_3v_5$) would arise in H. Hence, two consecutive or two opposite vertices of the six-cycle surely have pendant edges. On the other hand, if two of the vertices v_1, v_3 and v_5 (or two of v_2, v_4 and v_6) have pendant edges, then one of the corresponding pendant vertices can be eliminated. Hence, under the given conditions, either two consecutive or two opposite vertices have pendant edges and $G \cong S_C$ or $G \cong S_D$ is obtained, respectively. $\diamondsuit$

The above cases cover all possibilities, therefore the theorem follows. $\square$

Note that $\mathcal{T}(G)$ is an odd hole if and only if

- G can be obtained from W_4, or from one of the four supplementary types, or from C_n^2 where $n \geq 8$ is even, by a sequence of an odd number of edge splittings and an arbitrarily number of vertex stickings; or

- G can be obtained from C_n^2 where $n \geq 7$ is odd, by a sequence of an even number of edge splittings and an arbitrarily number of vertex stickings.

As a vertex sticking clearly creates a graph which is not path-neighborhood, and then applying any further vertex stickings or edge splittings this property does not change, we infer the following characterization of path-neighborhood graphs whose triangle graph is a cycle. This gives a solution for Problem 3(a) posed by Laskar, Mulder and Novick in [11].

Corollary 5 *Let G be a graph and assume that for every vertex $v \in V(G)$ the open neighborhood $N(v)$ induces a path on at least two vertices. Then, $\mathcal{T}(G) \cong C_n$ for some $n \geq 3$ if and only if*

(i) *$G \cong C_k^2$ for a $k \geq 7$; or*

(ii) *$G \cong S_A$ or S_B or S_C or S_D; or*

(iii) *G can be obtained by a sequence of edge splittings from a graph which satisfies (i) or (ii).*

Remark 6 *It is worth investigating the status of graphs C_n^2 for $n \leq 6$.*

(i) The square C_4^2 of the four-cycle is K_4, and also its triangle graph is K_4. But if we double the two diagonals added to C_4 for C_4^2, and apply four edge splittings on these two pairs of parallel edges, we obtain S_A from the supplementary type (A). In fact, this does not correspond precisely to the definition of edge splitting, but we can view the case as if one of the edges v_1v_3 belonged to triangle $v_1v_2v_3$, and its 'twin' edge to $v_1v_4v_3$. The two edges v_2v_4 can be treated similarly.

(ii) For the five-cycle $C_5 = v_1 \dots v_5$, the square graph C_5^2 is a complete K_5, and its triangle graph is the complement of the Petersen graph, hence not at all a cycle. But if we consider only the five triangles of the form $v_iv_{i+1}v_{i+2}$ (where $1 \leq i \leq 5$), and the remaining triangles of the form $v_iv_{i+1}v_{i+3}$ are "damaged" by properly chosen edge splittings, we obtain a graph G whose triangle graph is a cycle. If such a G cannot be reduced by the inverse of edge splitting, then it is isomorphic to S_B.

(iii) A similar observation can be made for $n = 6$, where at least two edge splittings have to be taken in C_6^2 to achieve that the triangle graph becomes a cycle. In the minimal (non-reducible) configurations exactly two edge splittings are needed, either neighboring or opposite. This yields a graph isomorphic either to S_C or to S_D.

3 Forbidden subgraph characterizations

In Subsection 3.1 we prove the main result of this section, which is a forbidden subgraph characterization of graphs whose triangle graph does not contain an induced cycle of a given length $n \geq 3$. This problem is in close connection with the problem we solved in Theorem 4. Clearly, if the triangle graph $\mathcal{T}(G)$ is C_n-free, then G contains no subgraph H with $\mathcal{T}(H) \cong C_n$. On the other hand, avoiding these subgraphs is not sufficient for $\mathcal{T}(G)$ to be C_n-free. For instance, none of the subgraphs of C_6^2 has a triangle graph isomorphic to the 6-cycle, but $\mathcal{T}(C_6^2)$ contains an induced C_6; moreover, no subgraph of K_4 has a triangle graph isomorphic to a cycle, but $\mathcal{T}(K_4) \cong K_4$ contains 3-cycles.

In Subsection 3.2 we establish some immediate consequences of the main result. This contains a forbidden subgraph characterization of graphs whose triangle graph is a tree, a chordal graph, or a perfect graph. Further, in Subsection 3.3 we determine a graph class on which Conjecture 1 holds.

Let us recall that C_n-free triangle graph means that no *induced* subgraph of $\mathcal{T}(G)$ is isomorphic to C_n; whilst forbidden subgraphs given for G are meant that they must not occur as *non-induced* subgraphs either.

3.1 Graphs with C_n-free triangle graphs

In this subsection *"minimal forbidden graph for C_n"* is meant as a graph G whose triangle graph contains an induced cycle C_n but this does not hold for any proper subgraph of G. If G is a minimal forbidden graph for C_n, the triangles $T_1, \ldots, T_n$ in G, which correspond to the vertices $t_1, \ldots, t_n$ inducing a specified cycle in $\mathcal{T}(G)$, are called *cycle-triangles*, while the other triangles of G are called *additional triangles*. By definition, in a minimal forbidden graph, every edge is contained in at least one cycle-triangle. The edge $e \in E(G)$ will be called *private edge* if it belongs to exactly one cycle-triangle, and those contained in exactly two cycle-triangles will be called *doubly covered*. That is, when an edge is declared to be private or doubly covered, additional triangles are disregarded.

Proposition 7 *Let $n \geq 3$ be an integer. Assume that G is a minimal forbidden graph for C_n. Then, either G is isomorphic to $K_5 - K_3$, or else no edge of G is covered by more than two cycle-triangles and furthermore G contains exactly $2n$ edges, from which n edges are private and n edges are doubly covered.*

Proof. Three cycle-triangles sharing a common edge correspond to a 3-cycle in the triangle graph. Hence, if an edge of G is covered by three cycle-triangles, then $n = 3$. Moreover, by the minimality assumption, $G \cong K_5 - K_3$ must hold and G has no additional triangles.

Now, assume that $G \ncong K_5 - K_3$ and G is minimal for C_n. Fixing an induced n-cycle in $\mathcal{T}(G)$, each cycle-triangle in G has exactly two neighboring cycle-triangles, and hence exactly two doubly covered edges. By minimality, each edge of G belongs to at least one cycle-triangle. Therefore, $E(G)$ consists of exactly n private and n doubly covered edges. $\square$

Corollary 8 *Let G be a graph which is not isomorphic to $K_5 - K_3$. Then, G is a minimal forbidden graph for C_n (for a specified $n \geq 3$) if and only if $\mathcal{T}(G)$ contains an induced n-cycle and G has exactly $2n$ edges.*

Proof. If $\mathcal{T}(G)$ contains an induced n-cycle, then either G or some proper subgraph of it must be minimal forbidden for C_n. Since each proper subgraph has fewer than $2n$ edges, Proposition 7 implies that G itself is a minimal forbidden graph. The other direction follows immediately from Proposition 7. $\square$

Corollary 9 *There exists no graph which is minimal forbidden for both C_n and C_m if $n \neq m$.*

The operations edge splitting and vertex sticking will be meant in the same way as introduced in the previous section, but the conditions of their applicability are relaxed — and indicated with the adjective 'weak' — as described next. Recall that throughout this section the position of an n-cycle in $\mathcal{T}(G)$ is assumed to be specified, in order to distinguish between cycle-triangles and additional triangles. We also emphasize that these operations cannot be applied for a graph where the fixed n-cycle in $\mathcal{T}(G)$ corresponds to three triangles incident to a common edge. Particularly, we assume $G \not\cong K_5 - K_3$.

- *Weak edge splitting* can be applied for any private edge e. If this edge $e = uv$ belongs to the cycle-triangle uvx, we introduce a new vertex w and change the edge set from E to $E \setminus \{uv\} \cup \{uw, wv, wx\}$. This transforms each additional triangle (if exits) incident with e to a cycle of length 4. The new vertex w is of degree 3, and the two edges uw and wv originated from e are two incident private edges in the graph obtained. Particularly, if no additional triangles are incident with e, a weak edge splitting applied to e is also called *strong edge splitting*.

- *Weak vertex sticking* can be applied for any two vertices at distance at least 3 apart. If this distance is at least 4, the triangle graph remains unchanged and the operation is also a *strong vertex sticking*, and corresponds to 'vertex sticking' introduced in Section 1.2. A weak vertex sticking, when applied for vertices at distance 3, creates some new additional triangle(s), but a strong vertex sticking cannot cause change in the triangle graph.

For instance, strong edge splitting cannot be applied for K_4, but a weak edge splitting can be applied for any of its edges and results in a wheel W_4. In C_{10}^2, no two vertices are at distance 4 or more, so strong vertex sticking cannot be applied, but two opposite vertices of the cycle can be stuck in the weak sense. In this case, six additional triangles arise.

These operations have their inverses in a natural way. Before investigating the conditions of their applicability, let us describe their effect on minimal forbidden graphs.

Proposition 10

(i) *If G' is obtained from G by a weak edge splitting (or, equivalently, if G is obtained from G' by an inverse weak edge splitting), then G' is a minimal forbidden graph for C_{n+1} if and only if G is minimal forbidden for C_n.*

(ii) *If G'' is obtained from G by a weak vertex sticking (or, equivalently, if G is obtained from G'' by an inverse weak vertex sticking), then G'' is a minimal forbidden graph for C_n if and only if G is minimal forbidden for C_n.*

Proof. (i) It is clear from the definition of weak edge splitting that the fixed n-cycle of $\mathcal{T}(G)$ is transformed into an induced $(n+1)$-cycle of $\mathcal{T}(G')$ and vice versa. Moreover, $|E(G)| = 2n$ if and only if $|E(G')| = 2n + 2$. Hence, the statement follows by Corollary 8.

(ii) If G'' is obtained from G by a weak vertex sticking, then $\mathcal{T}(G)$ contains an induced n-cycle if and only if $\mathcal{T}(G'')$ contains an induced n-cycle. Additionally, $|E(G)| = 2n$ holds if and only if $|E(G'')| = 2n$. Similarly to the previous case, Corollary 8 implies the statement. $\square$

Next we prove necessary and sufficient conditions under which the inverse operations can be applied. Let us introduce the following notion. For a graph G and for a fixed cycle in the triangle graph $\mathcal{T}(G)$, the *cycle-triangle neighborhood* $N^*(v)$ of a vertex $v \in V(G)$ is obtained by taking the vertices and edges of the cycle-triangles incident to v and then removing vertex v and the incident edges.

Proposition 11 *Given a graph G, with a fixed n-cycle in its triangle graph such that each edge of G is contained in at least one cycle-triangle, the following statements hold:*

(i) *An inverse weak edge splitting which eliminates vertex w exists if and only if w has degree 3 and there are two neighbors u and v of w such that $uv \notin E(G)$.*

(*ii*) *An inverse strong edge splitting which eliminates vertex* w *exists if and only if* w *has degree* 3*, moreover for the three neighbors* u*,* v*, and* x *of* w*, we have* $uv \notin E(G)$*, and* w *is the only common neighbor of* u *and* v *besides* x*.*

(*iii*) *An inverse weak vertex sticking can be applied for a vertex* v *if and only if its cycle-triangle neighborhood* $N^*(v)$ *is disconnected.*

(*iv*) *An inverse strong vertex sticking can be applied for a vertex* v *if and only if its neighborhood* $N(v)$ *is disconnected.*

Proof. By definition, an edge splitting always creates a vertex of degree 3. If this operation was applied for the private edge $e = uv$ of the cycle triangle xuv, then in the obtained graph G, uv is not an edge. Moreover, if a strong edge splitting was applied, uv is not contained in any triangles different from xuv. This proves that the conditions given in (*i*) and (*ii*) are necessary for the applicability of inverse weak and strong edge splittings.

To prove sufficiency, first observe that a vertex w which satisfies the conditions in (*i*) and (*ii*) does not belong to a K_4. It is also assumed that each edge is involved in a cycle-triangle. Then, since w has three neighbors, it is incident to exactly two cycle-triangles, which share an edge. This doubly covered edge must be xw, while the remaining two edges uw and vw are private edges in xuw and xvw, respectively. Then, by removing w and inserting the edge uv, the triangles xuw and xvw are replaced with xuv. Since the new edge uv is the private edge of xuv, a weak edge splitting can be applied to it and G is reconstructed. Under the conditions of (*ii*) no additional triangle is incident to uv and so G can be reconstructed by a strong edge splitting. These prove that the inverse operations can be applied to G under the conditions of (*i*) and (*ii*).

To prove necessity in (*iii*) and (*iv*), consider two vertices v_1 and v_2 to which a weak vertex sticking is applied. By definition, v_1 and v_2 have distance at least 3. Hence, there are no common vertices in $N^*(v_1)$ and $N^*(v_2)$. Recall that this transformation does not create new cycle-triangles. Therefore, sticking v_1 and v_2, the new vertex v will have a disconnected cycle-triangle neighborhood. If it is a strong vertex sticking, the distance of v_1 and v_2 is at least 4 and we cannot have edges between the vertices of $N(v_1)$ and $N(v_2)$. This implies that $N(v)$ will be disconnected.

For the other direction, assume that the condition given in (*iii*) holds. Let v be deleted, and let the vertices from one component of $N^*(v)$ be joined to a new vertex v_1 while the further vertices from $N^*(v)$ be joined to another new vertex v_2. Then, the cycle triangles do not change (apart from the fact that v

is replaced with v_1 or v_2). We observe that v_1 and v_2 do not have a common neighbor, hence their distance is at least 3. This proves that the original graph G can be reconstructed by applying a weak vertex sticking to v_1 and v_2. If the stronger condition from (iv) also holds for G, we will not have any edges between $N(v_1)$ and $N(v_2)$. Hence, the distance of v_1 and v_2 is at least 4, and G can be reconstructed by a strong vertex sticking. $\square$

Remark that an inverse weak edge splitting may create new additional triangles, an inverse weak vertex sticking may damage some of the additional triangles, while the inverse strong vertex sticking keeps the triangle graph the same.

Concerning the order in which these transformations can be applied, we prove the following property. (Although the third part could also be made more detailed, by performing strong vertex stickings before non-strong ones, we do not need this fact in the current context.)

Proposition 12 *Assume that graph F can be obtained from G by a sequence of weak edge splittings and weak vertex stickings. Then, F can also be obtained from G by performing the operations in the following order:*

1. *some (maybe zero) weak but not strong edge splittings,*
2. *some (maybe zero) strong edge splittings,*
3. *some (maybe zero) weak vertex stickings.*

Consequently, G can be obtained from F in the reverse order of the corresponding inverse operations.

Proof. First, observe that if the ith transformation O_i is a weak sticking of x and y, moreover the $(i+1)$st transformation O_{i+1} is the weak splitting of the edge e, then they can also be applied in the order O_{i+1}, O_i. Indeed, a vertex sticking does not create a new private edge, hence O_{i+1} can be performed before O_i. On the other hand, an edge splitting cannot decrease the distance of x and y and cannot create a new edge between two vertices which were present previously. Hence, the order O_{i+1}, O_i is feasible and gives the same result as O_i, O_{i+1}. Therefore, we can re-order the transformations in such a way that all edge splittings precede all vertex stickings.

A sequence of edge splittings can be unambiguously described by assigning a nonnegative integer $s(e)$ to each edge $e \in E(G)$, where $s(e)$ is the number of edge splittings applied to e and to the edges originated from e. Equivalently, this is the number of subdivision vertices we have on e at the end. This also shows that edge splittings can be performed in any order. If we want to start with weak but not strong edge splittings, we just take an edge with $s(e) \geq 1$ which is incident with an additional triangle and apply an edge splitting as long as such an edge exists. $\square$

Theorem 13 *Let $n \geq 3$ be a given integer. The triangle graph $\mathcal{T}(G)$ of a graph G does not contain an induced cycle of length n if and only if G has no subgraph which is isomorphic to any of the following forbidden ones.*

(*a*) *If $n = 3$, the forbidden subgraphs are K_4 and $K_5 - K_3$.*

(*b*) *If $n = 4$, the only forbidden subgraph is W_4.*

(*c*) *If $n = 5$, the forbidden subgraphs are W_5 and $C_5^2 \cong K_5$.*

(*d*) *If $n = 6$, the forbidden subgraphs are W_6, C_6^2, $K_6 - K_3$, $K_6 - P_4$, and the graph obtained from $C_5^2 \cong K_5$ by a weak edge splitting.*

(*e*) *If $n \geq 7$, the forbidden subgraphs are*

- (*i*) *W_n;*
- (*ii*) *graphs obtained from C_m^2 by $n - m$ weak edge splittings ($5 \leq m \leq n$);*
- (*iii*) *graphs obtained from $K_6 - K_3$ by $n - 6$ weak edge splittings;*
- (*iv*) *graphs obtained from $K_6 - P_4$ by $n - 6$ weak edge splittings;*
- (*v*) *graphs obtained from any graphs described in (ii)−(iv) by any number of weak vertex stickings.*

Proof. First, observe that three triangles, any two of which share an edge, are either three triangles having a fixed common edge, or they belong to a common K_4 subgraph. Hence, any 3-cycle in $\mathcal{T}(G)$ origins either from a $K_5 - K_3$ or from a K_4. Consequently, by minimality, if $n = 3$ then either $G \cong K_5 - K_3$ or $G \cong K_4$ holds. From now on, we consider a graph G which is not isomorphic to $K_5 - K_3$ and is minimal forbidden for a specified $n \geq 3$. By Proposition 7, there are exactly n private edges and n doubly covered edges in G.

By Proposition 10, weak edge splittings, vertex stickings and their inverse operations do not change the status of a graph being minimal forbidden for at least one cycle C_n. Hence, we may assume further that inverse weak vertex sticking and inverse weak edge splitting cannot be applied to G.

We have the following two cases concerning the additional triangles of G.

Case 1. Each additional triangle contains at least one private edge (or there is no additional triangle).

In this case, we apply a minimum number of weak edge splittings such that all the additional triangles of G are damaged. This yields a graph G' with $\mathcal{T}(G') \cong C_n$. We shall prove that neither inverse strong edge splitting nor inverse strong vertex sticking can be applied for G'.

By our assumption, inverse weak vertex sticking cannot be applied for G. Hence, by Proposition 11(iii), for every $v \in V(G)$, $N^*(v)$ is connected. Assume first that an edge splitting is applied for an edge $e = vu$ of G. Then, u is omitted from the neighborhood of v, but the new vertex w appears in $N^*(v)$ and has exactly the same neighbor there as u had. Therefore, $N^*(v)$ remains connected. Now, consider an edge splitting applied for a private edge xy from the cycle-triangle vxy. In $N^*(v)$, this means only the subdivision of the edge xy. This also keeps connectivity. As the third case, for any new vertex w which was introduced by an edge splitting, $N^*(w)$ is a path of order 3. Therefore, every vertex of G' has a connected cycle-triangle neighborhood, and by Proposition 11(iii), inverse weak (and also, strong) vertex sticking cannot be applied for G'.

Concerning the other operation, we supposed that inverse weak edge splitting cannot be applied for G. Then we applied minimum number of weak but not strong edge splittings to damage all the additional triangles. The minimality condition implies that each new vertex belongs to at least one induced 4-cycle originated from an additional triangle. Thus, every new vertex x has two neighbors x_1 and x_2 which share a neighbor y such that y is not adjacent to x. By Proposition 11(ii), inverse strong edge splitting cannot be applied for x. The second case is when a vertex w was present already in G and had degree 3. By Proposition 11(i), as inverse weak edge splitting cannot be applied for G, the neighbors of w are pairwise adjacent. Let u and v be the neighbors of w such that uw and vw are the private edges of triangles xuw and xvw, respectively. While minimum number of edge splittings were performed, no edge could be split twice. Hence, if weak edge splitting was applied for neither uw nor vw, then u and v either remain adjacent or have a common neighbor in G that is different from x and w (the latter case occurs when the edge uv was split). Then, Proposition 11(ii) implies that vertex w cannot be eliminated by an inverse strong edge splitting. Now, assume that at least one of the edges uw and vw, say uw was split by inserting a new edge xu'. By our minimality condition, the new vertex u' is contained in an induced 4-cycle. Since u' has only three neighbors u, x, and w, furthermore $xu, xw \in E(G')$, the induced 4-cycle contains u, u' and w plus one vertex which is different from x. This fourth vertex must be v, because w is also of degree 3. Again, the two neighbors of w, namely u' and v, have the common neighbor which is u. Thus, by Proposition 11(ii), w cannot be eliminated by an inverse strong edge splitting. Finally, we observe that the edge splittings performed in G do not decrease the degrees of the vertices. Hence, if a vertex has degree greater than 3 in G, it cannot be eliminated by an inverse strong edge splitting in G'.

Therefore, inverse strong edge splitting and inverse strong vertex sticking cannot be applied for G'. By Theorem 4, graph G' is isomorphic either to W_4, or to C_n^2 with $n \geq 7$, or to one of the supplementary types S_A, S_B, S_C, S_D. According to the way G' is derived, we see that either $G = G'$ or G can be reconstructed from G' by applying some number of inverse weak edge splittings, to be performed as long as at least one is possible, because it has been assumed that inverse weak edge splitting cannot be applied to G.

Checking all items from our list for G', we can observe the following.

- In W_4, we can apply inverse weak edge splitting exactly once. This yields $K_4 \cong W_3$.
- In C_n^2 (with $n \geq 7$) there are no vertices of degree 3. Thus, inverse weak edge splitting cannot be applied.
- In S_A we have four vertices of degree 3, and we can choose from four possible inverse weak edge splittings at the first step (all the four are isomorphic). After one is performed, only two further (isomorphic) possibilities remain. At the end, after two inverse edge splittings we obtain $K_6 - P_4$.
- In S_B three vertices have degree 3, and inverse weak edge splittings can be applied to all of them. These can be performed in any order, the result will be $C_5^2 \cong K_5$.
- For S_C and S_D we can apply inverse weak edge splitting twice. After performing them we have a C_6^2.

We conclude that in this case G must be isomorphic either to K_4, or to C_n^2 with $n \geq 5$, or to $K_6 - P_4$.

Case 2. There is at least one additional triangle uvw in G such that each of the edges uv, vw, uw is doubly covered.

By our assumption, no inverse weak vertex sticking can be applied for G. Hence, every vertex x has a connected cycle-triangle neighborhood. Since every triangle T_i has exactly one private edge and the two doubly covered edges correspond to $T_{i-1} \cap T_i$ and $T_i \cap T_{i+1}$, the triangles incident with x are consecutive triangles along the triangle cycle. Let us refer to this property as 'continuity'.

Since uv, vw and uw are doubly covered, the incident triangles can be given with their vertex sets in the form $T_1 = \{uva_1\}$, $T_2 = \{uva_2\}$, $T_i = \{vwb_1\}$, $T_{i+1} = \{vwb_2\}$, $T_j = \{wuc_1\}$ and $T_{j+1} = \{wuc_2\}$, where these six triangles are not assumed to be consecutive, but they are given in a cyclic order.

If $a_2 = b_1$, $b_2 = c_1$ and $c_2 = a_1$, we get the desired result $G \cong K_6 - K_3$. Now, suppose that $a_2 \neq b_1$, which is equivalent to $i \neq 3$. Since T_2 and T_i are incident with vertex v (but T_j is not), the continuity of triangles at v implies that T_3 has vertices va_2x, moreover a_2x must be the private edge. Now, ua_2 and a_2x are two incident private edges of consecutive triangles. The inverse weak edge splitting could not be applied for them only if ux is an edge in G. Then, the triangle T_ℓ having this edge ux belongs to triangles incident with u. Since T_1, T_2 are incident with u but T_3, T_i, T_{i+1} are not, $i + 2 \leq \ell$ follows. But in this case, the triangles incident with x cannot satisfy the continuity (since x belongs to T_3, might belong to T_i, but it is surely not contained in T_{i+1} and T_1). This contradiction proves that $a_2 = b_1$ and similarly, $b_2 = c_1$ and $c_2 = a_1$ must be valid, as well. Thus, $G \cong K_6 - K_3$ holds.

These cases together cover all possibilities, therefore the theorem is proved. □

3.2 Trees, chordal graphs and perfect graphs

Lemma 14 *$\mathcal{T}(G)$ is connected if and only if there does not exist a partition of $E(G)$ into two sets A and B such that each of A and B contains at least three edges which induce a triangle and each triangle in G is either in A or in B.*

Proof. Let $A \cup B = E(G)$ be an edge partition such that each triangle of G is contained in either A or B. Then in $\mathcal{T}(G)$ there cannot be any edges from the vertices representing the triangles inside A to those representing the triangles inside B. Thus, if there exist two triangles $T_A \subset A$ and $T_B \subset B$ in G, then $\mathcal{T}(G)$ has at least two components.

Conversely, assume that $\mathcal{T}(G)$ is disconnected. Let A be the collection of all edges of G corresponding to the triangles in one of the components of $\mathcal{T}(G)$, and let $B = E(G) - A$. This B also contains at least one triangle, since $\mathcal{T}(G)$ is disconnected. Now, $\{A, B\}$ is a partition of $E(G)$ such that each of A and B contains at least one triangle and each triangle is either in A or in B. □

Let us say that graph G is *triangle-connected* if $\mathcal{T}(G)$ is connected.

Now, the characterization of graphs whose triangle graph is a tree or a chordal graph follows immediately from Theorem 13.

Corollary 15 *For a graph G, its triangle graph $\mathcal{T}(G)$ is a tree if and only if G is triangle-connected and does not contain a subgraph which is isomorphic to one of the following graphs.*

- (a) *W_n, for $n \geq 3$;*
- (b) *$K_5 - K_3$;*
- (c) *C_n^2, for $n \geq 5$;*
- (d) *$K_6 - K_3$;*
- (e) *$K_6 - P_4$;*
- (f) *graphs obtained from any of the graphs described in $(c)-(e)$ by any number of weak edge splittings and weak vertex stickings.*

Corollary 16 *For a graph G, its triangle graph $\mathcal{T}(G)$ is chordal if and only if G does not contain a subgraph which is isomorphic to any of the following graphs:*

- (a) *W_n, for $n \geq 4$;*
- (b) *C_n^2, for $n \geq 5$;*
- (c) *$K_6 - K_3$;*
- (d) *$K_6 - P_4$;*
- (e) *graphs obtained from any graphs described in $(b) - (d)$ by any number of weak edge splittings and weak vertex stickings.*

Imposing parity conditions, we also obtain a characterization of graphs whose triangle graph is perfect.

Theorem 17 *For a graph G, its triangle graph $\mathcal{T}(G)$ is perfect if and only if G does not contain any subgraph which is isomorphic to one of the following graphs:*

- (a) *W_n, for an odd integer $n \geq 5$;*
- (b) *graphs obtained from C_n^2 by an even number of weak edge splittings for an odd $n \geq 5$;*

(c) *graphs obtained from C_n^2 by an odd number of weak edge splittings for an even $n \geq 6$;*

(d) *graphs obtained from $K_6 - K_3$ by an odd number of weak edge splittings;*

(e) *graphs obtained from $K_6 - P_4$ by an odd number of weak edge splittings;*

(f) *graphs obtained from the graphs described in $(b) - (e)$ by any number of weak vertex stickings.*

Proof. Since $\overline{K_2} \vee \overline{P_3}$ is forbidden for triangle graphs [13] and $\overline{C_n}$ contains it as an induced subgraph for all $n \geq 7$, we have that $\mathcal{T}(G)$ is $\overline{C_n}$-free for $n \geq 7$. Also, $\overline{C_5} = C_5$. Therefore, by the Strong Perfect Graph Theorem [6], $\mathcal{T}(G)$ is perfect if and only if has no induced odd hole. Moreover, $\mathcal{T}(G)$ contains an induced odd hole if and only if G has a subgraph from the types described in $(a) - (f)$. This completes the proof. □

3.3 Consequences for triangle packing and covering

Here we consider Conjecture 1 which was posed in [20]. To discuss it in a more detailed way, we need some definitions. We say that a family $\mathcal{F}$ of triangles in $G = (V, E)$ is *independent* if the members of $\mathcal{F}$ are pairwise edge-disjoint. An edge set $E' \subseteq E$ is a *$\mathcal{T}$-transversal* if every triangle of G contains at least one edge from E'. We denote by $\nu_\triangle(G)$ the maximum cardinality of an independent family of triangles in G, and by $\tau_\triangle(G)$ the minimum cardinality of a $\mathcal{T}$-transversal in G. With this notation, Conjecture 1 is equivalent to the statement $\tau_\Delta(G) \leq 2\nu_\Delta(G)$.

This inequality has been proved only for few classes of graphs; namely, for planar graphs, some subclasses of chordal graphs, graphs with n vertices and at least $\frac{7}{16}n^2$ edges [21], graphs without a subgraph homeomorphic to $K_{3,3}$ [10], graphs with chromatic number three [9], graphs in which every subgraph has average degree smaller than seven [15], odd-wheel-free graphs, and graphs admitting an edge 3-coloring in which each triangle receives three distinct colors on its edges [1]. (The latter class contains all graphs with chromatic number at most four, and also all graphs which have a homomorphism into the third power of an even cycle, C_{2k}^3 with $k \geq 5$.)

The case of equality $\tau_\Delta = \nu_\Delta$ has also been studied to some extent ([23, 1]). For instance, it was proved in [1] that $\tau_\Delta(G) = \nu_\Delta(G)$ is valid for K_4-free graphs G whose triangle graph is odd-hole free. Now, by Theorem 17, this class of graphs is determined in a more direct way by forbidden subgraph characterization. (Some redundancies could be eliminated; e.g., $K_6 - P_4$ and $K_6 - K_3$ contain K_4 which is forbidden, too. But after some appropriate edge splittings, the K_4 subgraphs disappear, hence the graphs listed in parts (d) and (e) of Theorem 17 cannot be totally omitted.)

Here we prove Conjecture 1 for graphs G whose triangle graph is perfect, but the K_4 subgraphs are not excluded from G. The condition that $\mathcal{T}(G)$ is perfect, can be replaced either by assuming that $\mathcal{T}(G)$ is odd-hole-free or by the forbidden subgraph characterization of Theorem 17.

Theorem 18 *If the triangle graph of a graph G is perfect, then $\tau_\Delta(G) \leq 2\nu_\Delta(G)$ holds.*

Proof. As discussed already in [1], the maximum number $\nu_\Delta(G)$ of independent triangles in G equals the independence number $\alpha(\mathcal{T}(G))$ of the triangle graph. Since the triangle graph is supposed to be perfect, its complement is also perfect and we have

$$\nu_\Delta(G) = \alpha(\mathcal{T}(G)) = \omega(\overline{\mathcal{T}(G)}) = \chi(\overline{\mathcal{T}(G)}) = \theta(\mathcal{T}(G)),$$

where $\theta(\mathcal{T}(G))$ is the minimum number of cliques in the triangle graph which together cover all vertices.

In a triangle graph $\mathcal{T}(G)$ we may have two types of cliques: (A) its vertices correspond to triangles of G all of which are incident with a fixed edge; (B) its vertices correspond to four triangles of a K_4 subgraph of G.

Having a minimum clique cover of $\mathcal{T}(G)$ at hand we can construct an edge cover for triangles of G. For every clique C_A of type A, we put the corresponding edge of G into the covering set. This edge covers all triangles corresponding to the vertices covered by C_A in $\mathcal{T}(G)$. Then, for every clique C_B of type B, we put two independent edges from the corresponding K_4 subgraph of G into the covering set. These two edges together cover all the four triangles. Since every vertex $t \in V(\mathcal{T}(G))$ is covered by a clique in $\mathcal{T}(G)$, every triangle of G is covered by at least one of the selected edges. Thus, the set of the selected edges (wchich covers all triangles of G) contains at most $2\theta(\mathcal{T}(G)) = 2\nu_\Delta(G)$ edges. Consequently, $\tau_\Delta(G) \leq 2\nu_\Delta(G)$ is valid. $\square$

Note added on 4$^{\text{th}}$ November, 2014. After the first appearance of this manuscript, Gregory Puleo kindly informed us that from his results in [15] the inequality $\tau_\Delta(G) \leq 2\nu_\Delta(G)$ follows for a class of graphs which is larger than the one in our Theorem 18. Namely, more generally than the graphs with perfect $\mathcal{T}(G)$, it suffices to assume that G has no subgraph isomorphic to W_n for any odd $n \geq 5$. That is, from the forbidden subgraphs listed in our Theorem 17, already the case (a) is sufficient to derive $\tau_\Delta \le 2\nu_\Delta$. As Puleo explains in his email, this follows by the properties of the so-called 'weak König–Egerváry graphs', proved in Section 4 of [15]. This extends Theorem 3 of [1] where the analogous result was proved for graphs without any odd wheels (i.e., excluding $W_3 \cong K_4$, too).

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