The Disjoint Domination Game *

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Abstract

We introduce and study a Maker-Breaker type game in which the issue is to create or avoid two disjoint dominating sets in graphs without isolated vertices. We prove that the maker has a winning strategy on all *connected* graphs if the game is started by the breaker. This implies the same in the $(2:1)$ biased game also in the maker-start game. It remains open to characterize the maker-win graphs in the maker-start non-biased game, and to analyze the $(a:b)$ biased game for $(a:b) \neq (2:1)$. For a more restricted variant of the non-biased game we prove that the maker can win on every graph without isolated vertices.

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1 Introduction

It is well known that every graph without isolated vertices contains two dominating sets which are disjoint. In this paper we introduce a combinatorial game in which one of the two players aims at constructing two disjoint dominating sets, while the other player wants to prevent this. We investigate who has a winning strategy if the graph of the game has a certain structural property.

1.1 Terminology and preliminaries

Throughout this paper, we consider finite, simple graphs *without isolated vertices*. For such a graph $G = (V, E)$ and for a vertex $v \in V$, the open neighborhood $N(v)$ of v is the set of all vertices adjacent to v, and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. Each vertex dominates itself and its neighbors, moreover a set $S \subseteq V$ dominates the vertices contained in $N[S] = \bigcup_{v \in S} N[v]$. A *dominating set* of G is just a subset D of V which dominates all vertices of the graph; that is, $N[D] = V$. The minimum cardinality of a dominating set is termed the domination number of G.

Domination is a well-studied subject in graph theory, with many related applications. A general overview can be found in [12]. On the other hand, domination in hypergraphs (set systems) is a relatively new area; see [13] and [8] for results and references.

Recently, Brešar, Klavžar and Rall [3] introduced the concept of the *domination game*. It is played on a graph G by two players, named Dominator and Staller. They take turns choosing a vertex from V such that at least one new vertex must be dominated in each turn. The game ends when no more legal moves can be taken. In this game Dominator's aim is to finish the game with a small dominating set, while Staller aims to delay the end of the game. The game domination number is the number of turns in the game when the first turn is Dominator's move and both players play optimally. For detailed description and results on this subject, see the papers [1, 2, 3, 4, 5, 6, 7, 9, 14, 15]. Let us mention further that a version of this game for the total dominating sets (where $\bigcup_{v \in D} N(v) = V$ is required for the *open* neighborhoods) was also introduced in [10, 11].

1.2 Disjoint Domination Game

We define the Disjoint Domination Game (DDG, for short) as a two-player game, where the players are named Dom and Sepy—these are the shortened forms of Dominator and Separator. For the game, we have an isolate-free graph G and a color palette $\mathcal{C} = \{p, b\} = \{purple,\ blue\}$. In the game, Dom and Sepy take turns choosing a vertex and assigning it with a color from $\mathcal{C}$. At any stage of the game, V_p and V_b denote the set of vertices colored with p and b, respectively. Moreover, for $c \in \mathcal{C}$ we denote by $\overline{c}$ the complementary color for which $\{c, \overline{c}\} = \mathcal{C}$ holds. Thus, $\overline{b} = p$ and $\overline{p} = b$.

The choice of a vertex v and its coloring with a color $c \in \mathcal{C}$ is a legal (or feasible) move in the game if and only if

(i) v has not been chosen and assigned with a color up to this point, that is $v \notin V_p \cup V_b$; and

(ii) there exists a vertex $u \in N[v]$ which has not been dominated in color c, that is $N[u] \cap V_c = \emptyset$.

Note that each player must select a vertex on his turn whenever a legal move is available. We shall discuss the situation in Section 4 for the game where some player may pass.

The game terminates when one of the following two situations is reached:

$\langle s^* \rangle$ some vertex has a monochromatic closed neighborhood

$\langle d^* \rangle$ both V_p and V_b are dominating sets

The winner is Sepy if $\langle s^* \rangle$ is reached, and Dom wins if $\langle d^* \rangle$ is reached.

In other words, the aim of Dom is to obtain two disjoint dominating sets V_p and V_b at the end of the game, whilst Sepy would like to prevent him from reaching this situation.

It follows from the definition of dominating set that both players cannot win simultaneously. First of all we prove that the game always ends with a win of one of them.

Lemma 1 *As long as neither $\langle s^* \rangle$ nor $\langle d^* \rangle$ is reached, the next player has at least one feasible move.*

Proof. Assume that neither $\langle s^* \rangle$ nor $\langle d^* \rangle$ has been reached. Then, we have a vertex v which is not dominated by V_c (for some $c \in \mathcal{C}$) or equivalently,

$N[v] \cap V_c = \emptyset$. As $N[v]$ is not monochromatic in $\overline{c}$, there exists a vertex $u \in N[v]$ which has not been colored up to this moment. Thus, selecting u and assigning c to it is a feasible move because then v becomes dominated in c. $\square$

In Sections 2 and 3 we study the Sepy-start and the Dom-start versions of the Disjoint Domination Game. Especially, we prove that for every isolate-free graph G, Dom has a winning strategy whenever Sepy starts the game and the graph is connected, but in the Dom-start version it depends on the given G which player has a winning strategy. We also touch the biased version of the game in the short Section 5.

In Section 6, we introduce a variant called Bicolored Domination Game (or shortly BDG). While in the Disjoint Domination Game both players are allowed to use both colors, in BDG each player has his private color not usable by the other player. For this variant, the definition of $\langle d^* \rangle$ when Dom wins, is slightly modified as follows.

$\langle d^{**} \rangle$ For a color $c \in \mathcal{C}$, V_c dominates all vertices of G, and no vertex v has its closed neighborhood entirely contained in V_c.

In Section 6, we give an explanation for this change. The main result of that section is that Dom has a winning strategy on all graphs, no matter who starts the Bicolored Domination Game.

We close the Introduction with a lemma which is valid in both the Disjoint and the Bicolored Domination Game.

Lemma 2 *If a vertex v of an isolate-free graph G is chosen in a turn, then $N[v]$ cannot be monochromatic after a legal coloring of v.*

Proof. By the condition (ii), if v is colored with c then at least one $u \in N[v]$ has not been dominated by c up to this moment. If $u = v$, then no vertex from $N(v) \neq \emptyset$ has color c; and if $u \neq v$, then either u has not been selected yet or it has been colored with $\overline{c}$. In either case, $N[v]$ is not monochromatic after the move. $\square$

2 The Sepy-start Disjoint Domination Game

For this game, in this section we prove that there exists a winning strategy for Dom which works on each connected graph.

Theorem 3 *If G is connected, then Dom has a winning strategy for the Sepy-start Disjoint Domination Game on G.*

Proof. After each move of Sepy, Dom applies the following strategy.

Opposite Neighbor Strategy (ONS)

- Suppose that Sepy selected a vertex v and colored it with c. Then Dom chooses a vertex according to the following rules.

(ONS1) Dom selects a neighbor u of v which can be colored with $\overline{c}$.

(ONS2) If (ONS1) cannot be applied, Dom selects a vertex u which can be colored with $c' \in \mathcal{C}$, moreover u has a neighbor of color $\overline{c'}$.

First, we observe the following consequences of the definition above.

Lemma 4 *As long as Dom does not violate* ONS*,*

(i) after each move of Dom, every colored vertex has a neighbor assigned with the complementary color;

(ii) the game cannot terminate with $\langle s^* \rangle$.

Proof. Remark that if rule ONS1 cannot be applied, then the vertex v, colored with c by Sepy in the last step, already has a neighbor of color $\overline{c}$. Then, part (i) of the lemma immediately follows from the definition of ONS. Particularly, no choice of Dom can make any closed neighborhood $N[x]$ monochromatic. On the other hand, by Lemma 2 and part (i), when Sepy selects a vertex v, this cannot result in a monochromatic closed neighborhood either. Thus, $\langle s^* \rangle$ cannot be reached unless some move of Dom violates ONS. $\diamondsuit$

Therefore, it suffices to prove that Dom can apply ONS in each turn.

Lemma 5 *After each step of Sepy, Dom can apply* ONS *as long as neither $\langle s^* \rangle$ nor $\langle d^* \rangle$ is reached.*

Proof. We will prove that there exists a move complying with $\mathsf{ONS}2$. Let D_0, D_1, and $D_{\geq 1}$ denote the set of vertices which are dominated by 0, precisely 1, and at least 1 color from $\mathcal{C}$, respectively. Since $\langle d^* \rangle$ has not been reached, $D_0 \cup D_1 \neq \emptyset$; and $D_{\geq 1} \neq \emptyset$ holds already after the very first move.

First consider the case $D_0 \neq \emptyset$. As G is connected, there exists a vertex $u \in D_{\geq 1}$ with a neighbor $u' \in D_0$. Thus, u must be an uncolored vertex which has a neighbor u'' colored with a $c \in \mathcal{C}$. Then, choosing u and coloring it with $\overline{c}$ is a legal move for Dom (this dominates u' with color $\overline{c}$) and also corresponds to $\mathsf{ONS}2$.

In the other case we have $D_0 = \emptyset$, which implies $D_1 \neq \emptyset$. Hence, there is a vertex u dominated with a color c, but not dominated with $\overline{c}$. If $u \notin V_c$, it is uncolored and Dom may choose u and assign it with color $\overline{c}$. This satisfies the requirements. Now, suppose that $u \in V_c$. As u is not dominated by $\overline{c}$ and $\langle s^* \rangle$ is not reached, u has an uncolored neighbor u'. Selecting u' and coloring it with $\overline{c}$ is a legal move complying with $\mathsf{ONS}2$. $\Diamond$

Lemma 4 and Lemma 5 together mean that the game terminates with $\langle d^* \rangle$ if Dom applies the Opposite Neighbor Strategy. $\square$

3 The Dom-start game

Contrary to the Sepy-start game, some graphs admit a winning strategy for Sepy in the Dom-start game.

Proposition 6 *For every $n \geq 8$, Sepy has a winning strategy for the Dom-start Disjoint Domination Game on C_n.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices in cyclic order. Without loss of generality, we assume that Dom selects v_1 and assigns it with purple. Then, Sepy can color v_2 with purple. If Dom's next choice is v_3, v_4 or v_5 (assigning it with either color), then Sepy colors v_n with purple. This is a legal choice as $n \geq 8$ ensures that v_{n-1} was not dominated (with any color) before this turn. Thus, $N[v_1]$ becomes monochromatic and the game terminates with $\langle s^* \rangle$. Similarly, $\langle s^* \rangle$ is reached if Dom selects a vertex different from v_3, v_4 and v_5. In this case, Sepy's next move is coloring v_3 with purple. $\square$

Proposition 7 *Let G be a graph of minimum degree at least 2, and let G^{+2} be the graph obtained by subdividing each edge of G into a path of length 3. Then Sepy can win the Dom-start Disjoint Domination Game on G^{+2}.*

Proof. We show that these graphs offer Sepy a local strategy to reach $\langle s^* \rangle$ after a small number of steps. There are two kinds of first moves for Dom: to select an original vertex of G or a subdivision vertex.

Case 1: The first move is a subdivision vertex.

Say, an edge wz of G has been subdivided to a path $wxyz$, and Dom assigned color c to vertex x in the first move. Then assigning c to y, Sepy creates a double threat: putting color c further on any of w and z terminates the game with $\langle s^* \rangle$. Dom may color only one of w and z in one step, therefore the only way to delay the winning of Sepy would be to make c infeasible on both w and z. This situation would occur precisely if the entire $N(w) \cup N(z)$ became dominated by V_c after the second move of Dom. But the neighbors of w and z on the subdivided edges do not have any common neighbors, therefore c remains feasible for w or z after the move of Dom.

Case 2: The first move is an original vertex.

Suppose that Dom first selects a vertex w, whose neighbors in G are $z_1, \ldots, z_d$, and assigned color c to it. Denote by $wx_iy_iz_i$ the subdivision path of the edge wz_i ($i = 1, \ldots, d$). The strategy of Sepy is to color all the vertices x_i with c one by one. Each such move creates a threat on x_i, because $\langle s^* \rangle$ will occur once Sepy assigns c to y_i. The only way for Dom to prevent this is to color a vertex in $N[z_i]$. As these sets $N[z_i]$ are pairwise disjoint, Dom needs a distinct move for each, and hence Sepy can make the entire $N[w]$ monochromatic and achieve $\langle s^* \rangle$. □

On the other hand, in every DDG played on a complete graph K_n or on a path P_n ($n \geq 2$) Dom can win, even if he begins the game. These examples are special cases of the following theorem.

Theorem 8 *Let $G = (V, E)$ be a connected graph, which has two different vertices u and v satisfying $N[u] \subseteq N[v]$. Then, Dom has a winning strategy for the Dom-start Disjoint Domination Game played on G.*

Proof. (sketch) We say that a vertex v is safe (concerning DDG) if no matter how the game is continued, $N[v]$ cannot be entirely monochromatic.

The winning strategy of Dom is as follows.

- In the first turn, Dom selects a vertex v which has a neighbor u with $N[u] \subseteq N[v]$. If v gets color c, then u cannot be colored with c in any later turns. Hence, $N[v]$ cannot become monochromatic, this is a safe vertex.

- In the later turns, Dom applies ONS.

The details of the proof are similar to those of Lemma 4 and Lemma 5. After each turn of Dom, every vertex in $V_b \cup V_p$ is safe. Moreover, no move of Sepy creates a monochromatic closed neighborhood. If ONS has not been violated and the game has not ended, Dom can apply ONS in his next turn. $\square$

4 Passing allowed

In the standard versions of domination games both players have to move in each turn until no legal move is possible. It is well known, however, that in some games there exist situations where it really does matter if the next player is not allowed to skip the move.

By definition we exclude the possibility of passing in the very first move, because it would immediately change the character of the game by switching between Dom-start and Sepy-start. Another reason for this restriction is that if passing was allowed for both players at any time, then in a Sepy-win graph in the Dom-start game first Dom should pass to avoid losing, but then also Sepy should pass because otherwise Dom can surely win, as we shall see below; hence the game would end up with a trivial draw. There exist further situations, too, where the possibility of double passing would cause unwanted anomalies. For this reason we restrict our attention to games in which just one specified player — or none of the players — is allowed to pass.

Passing may or may not help a player. Namely, we shall see in Section 6 that the possibility of passing has no effect on the outcome of the Bicolored Domination Game. On the other hand, 'virtual passing' may be a useful concept for designing strategies in 'biased games' discussed in the next Section 5. Moreover, by comparing the results of the previous two sections we can see that passing may help a lot for Dom, as expressed in the following variant of Theorem 3.

Theorem 3′ *If Dom is allowed to pass but Sepy isn't, then Dom has a winning strategy in the Sepy-start Disjoint Domination Game on every graph. Also, Dom has a winning strategy in the Dom-start Disjoint Domination Game on every graph containing at least one Dom-win component*

Proof. For connected graphs the assertion clearly is valid by Theorem 3. If G is disconnected, then Dom can win with the following strategy.

- If Dom should start the game, then start with the first move of a winning strategy in a Dom-win component.
- Afterwards, or if Sepy starts the game, always play in the same connected component where Sepy made his latest move.
- In that component apply the Opposite Neighbor Strategy.
- If Sepy moved and there is no legal move in that component anymore, then pass.

With the first and the last rules of this strategy Dom can force Sepy to open each connected component which would be Sepy-win in the Dom-start game. Consequently, Dom can win by applying ONS according to Theorem 3. $\square$

On the other hand, it turns out that passing has no benefit for Sepy if he begins the game.

Theorem 3″ *If G is connected, then Dom has a winning strategy for the Sepy-start Disjoint Domination Game on G, even when Sepy is allowed to pass at any time except in the first move.*

Proof. Dom can win in a similar way as in the original game when passing was not allowed. In fact he never needs to pass, as shown by the following scheme.

Opposite-to-Previous Strategy (OPS)

- Suppose that in the latest turn a vertex v was selected and colored with c. Then, Dom chooses a vertex according to the following rules.

($\mathsf{OPS}1$) Dom selects a neighbor u of v which can be colored with $\overline{c}$.

($\mathsf{OPS}2$) If ($\mathsf{OPS}1$) cannot be applied, Dom selects a vertex u which can be colored with $c' \in \mathcal{C}$, moreover u has a neighbor of color $\overline{c'}$.

It can be verified along the lines of the proof of Theorem 3 that this strategy is feasible and Dom wins if he applies it. $\square$

5 Biased games

In a *biased* or *asymmetric* game the players may make more than one move at a time. Such games are parameterized with two positive integers indicating the numbers of moves of the players per turn.

Adopting this notion to the Disjoint Domination Game, let d, s be two fixed positive integers. We use the shorthand $(d : s)$*-game* for the game where Dom sequentially selects and colors exactly d vertices in each turn (except near the end of the game when only fewer than d possibilities remain), and Sepy colors at most s vertices per turn (and may pass if he wishes to do so). Note that d always refers to Dom and s always refers to Sepy, no matter who starts the game. The general requirement to dominate new vertices in the colors of the successively selected vertices is kept also in the $(d : s)$-game. Hence the case $d = s = 1$ precisely means the Disjoint Domination Game where Sepy is allowed to pass.

Here we only consider the case $s = 1$; some short comments on larger values will be given in the concluding section. First, we prove the following lemma:

Lemma 9 *If Dom has the possibility to achieve $\langle d^* \rangle$ by playing just one next vertex inside a component C_i then no sequence of legal moves results in a monochromatic $N[v]$ inside C_i.*

Proof. Assume that the assignment of color c to $u \in V(C_i)$ would make the entire $V(C_i)$ dominated by both colors. Then, already without this action, each $w \in V(C_i)$ is dominated by $\overline{c}$. That is, $N[w] \cap V_{\overline{c}} \neq \emptyset$ and no matter which vertex or vertices are chosen (and assigned to c) later, no $N[w]$ will be monochromatic in color c in this component. On the other hand, in the continuation of the game, no $v \in V(C_i)$ can be colored with $\overline{c}$, which implies that no closed neighborhood can become monochromatic in $\overline{c}$ inside C_i. $\square$

The components in which no closed neighborhoods can become monochromatic in any continuation of the game will be called *safe components*.

Our main result in this section states that any $d > 1$ yields substantial advantage for Dom.

Theorem 10 *Dom has a winning strategy in the $(d : 1)$-game on every graph, for every $d \geq 2$.*

Proof. We advise Dom to play a variant of the Opposite-to-Previous Strategy with a simple modification.

1. If Dom starts or continues the game on a new component with at least two consecutive choices, color the first (any) vertex arbitrarily, and then color the next vertex according to OPS.

2. In the other cases, apply OPS itself, apart from one exceptional situation described below in the third rule of the strategy. Note that when Dom applies $\mathsf{OPS}2$, this current second rule might mean a choice of a vertex from another component, from which some vertices were played earlier.

3. If these rules yield a situation where some components $C_1, \ldots, C_i$ are completely dominated in both colors and the remaining ones $C_{i+1}, \ldots,$ C_k are completely undominated, moreover Dom can take only one further choice before the turn of Sepy, then instead of taking the preceding move in $C_1 \cup \cdots \cup C_i$ he chooses two vertices from the new component C_{i+1} in the way described by the first rule above. By Lemma 9, the non-completed component inside $C_1 \cup \cdots \cup C_i$ is safe.

By the proof of Lemma 5, this strategy is feasible and leads to the winning of Dom. $\square$

6 The Bicolored Domination Game

A natural variant of the Disjoint Domination Game is when Dom and Sepy have their private colors; that is, Dom may only use p and Sepy may only use b. Also in this case the game may terminate with $\langle s^* \rangle$ as above, in which case Sepy wins. However, recall that we had to modify the meaning of $\langle d^* \rangle$ slightly:

$\langle d^{**} \rangle$ For a color $c \in \mathcal{C}$, V_c dominates all vertices of G, and no vertex v has its closed neighborhood entirely contained in V_c.

The point is that if V_c dominates G, then the player of color c does not have any further feasible moves, while the other player may still have some; but on the other hand, forbidding $N[v] \subseteq V_c$ for all $v \in V$, the set $V \setminus V_c$ dominates

G and therefore letting the player of color $\overline{c}$ play as long as a feasible move is available, eventually two disjoint dominating sets are reached.

In this section we will show that Dom can win the Bicolored Domination Game on any isolate-free graph G. First, we prove this statement for the special case where G contains a perfect matching. Then, the Matching Strategy introduced here will be extended to obtain a winning strategy for Dom which works on all isolate-free graphs.

Proposition 11 *If G has a perfect matching, then Dom can win the Sepy-start and also the Dom-start Bicolored Domination Game on G.*

Proof. Suppose that G has $2n$ vertices, and let $\{v_1v'_1, v_2v'_2, \ldots, v_nv'_n\}$ be a perfect matching in G. The winning strategy of Dom is as follows.

Matching Strategy (MS)

- If Sepy selects a vertex from the matching edge $\{v_i, v'_i\}$, say he plays v_i, and playing the other vertex v'_i of this edge is a legal move for Dom, then Dom plays v'_i.
- Otherwise, Dom is free to make any legal choice from any pair $\{v_j, v'_j\}$ in which both vertices remained uncolored until that move.

It is clear that Dom can apply this strategy throughout the game, without making any $\{v_i, v'_i\}$ monochromatic at any time. To show that it is a winning strategy indeed, observe that the only reason why it is not legal for Dom to play v'_i as a response to Sepy's v_i is that both v_i and v'_i are already dominated in Dom's color. At the end of the game this property holds for all edges of the perfect matching. Since Dom selects at most one vertex from each matching edge, the game can never end with $\langle s^* \rangle$ but only with $\langle d^{**} \rangle$. $\square$

Theorem 12 *Dom can win the Bicolored Domination Game on every graph without isolated vertices, both in the Dom-start and Sepy-start cases.*

Proof. As a preprocessing for his strategy, Dom determines a maximum matching in $G = (V, E)$, denote it by $M = \{u_1v_1, u_2v_2, \ldots, u_mv_m\}$. We refer to the edges $e_i = u_iv_i$ as *matching-edges*, and their vertices as *matching-vertices*. In the complementary part $V \setminus M$ we call the vertices *external*. Note that all neighbors of an external vertex are matching-vertices.

Each external vertex is adjacent to some matching-vertex, because isolates are excluded. If some $x \in V \setminus M$ is adjacent to both u_i and v_i, then no other $x' \in V \setminus M$ is adjacent to any of u_i and v_i, by the maximality of M. So, such an e_i is a *triangle-edge*.

On the other hand, if $x \in V \setminus M$ is adjacent to u_i but not to v_i, then *all* $x' \in V \setminus M$ having a neighbor in e_i have precisely the same neighbor, u_i. We shall then call u_i a *center*, and e_i a *star-edge*. Note that if a matching-edge has at least one external neighbor then either it is a triangle-edge or its center is uniquely determined.

To win on G, in both the Dom-start and Sepy-start version, Dom applies the following rules.

1. **Center Rule.**

   If Dom selects a vertex from a star-edge, from which no vertex has previously been selected, then he selects the center.

2. **Incompleteness Rule.**

   Dom does not select more than one vertex from a matching-edge.

3. **Neighbor Rule.**

   After each step of Dom, V_p dominates V_b. Moreover, if Sepy selected a vertex of some matching edge e_i and the other vertex of e_i is feasible for Dom, then Dom plays that vertex. As a consequence, each blue vertex selected by Sepy has at least one purple neighbor selected by Dom.

4. **Matching Rule.**

   Dom selects a vertex outside M only if every feasible matching-vertex violates some of the rules above.

Lemma 13 *If Dom can keep all these rules during the whole game, then he wins.*

Proof. We have to prove that the game cannot terminate with $\langle s^* \rangle$, i.e. no closed neighborhood can become monochromatic.

It is clear by the Neighbor Rule and Lemma 2 that Sepy cannot create any $N[v]$ in blue. Suppose that Dom is forced to do so, say an entire $N[v]$ becomes purple when Dom selects a vertex x. Again by Lemma 2, $x \neq v$ certainly holds. Now, the vertex v with totally purple neighborhood cannot be a matching-vertex because its pair in M cannot be purple together with v, by the Incompleteness Rule. And it cannot be an external vertex either, because it would assume that Dom selected the external vertex v earlier than the feasible matching-vertex x, which would violate the Matching Rule. $\diamondsuit$

As the reader can observe, the Center Rule has not been used in the argument of the previous proof. In fact it is needed to ensure that Dom can keep all the other rules during the whole game.

Lemma 14 *As long as $\langle d^{**} \rangle$ has not been reached, after any move (or pass) of Sepy, Dom has a legal move respecting the four rules above.*

Proof. Consider a turn of Dom and assume that he obeyed all the four rules in all of his previous turns. We check the four requirements for Dom's current move one by one.

Center Rule. Let $e_i = u_i v_i$ be a star-edge with center u_i, and suppose that none of its ends has been selected yet. If an external neighbor of e_i is not dominated in purple, then Dom can play u_i. The same move is legal if u_i or v_i is not dominated in purple. The only bad case is if V_p already dominates $N[u_i]$, but v_i still has a neighbor, say x, which is not dominated in purple. Since v_i is not the center of e_i, this x must be a matching-vertex. It also means that no purple vertex has been selected from the matching-edge of x. But then Dom can select the center of that edge as a legal move if it is a star-edge, or any of x and its neighbor if it is a triangle-edge. Note that both vertices of the edge in question were previously non-selected, because if Sepy had selected the pair of x in an earlier move then Dom would have immediately responded with selecting x as a vertex non-dominated in purple, by the Neighbor Rule.

Incompleteness Rule. This rule would be violated only if Dom selects both u_i and v_i. Due to the Center Rule, this can happen only if v_i is adjacent to a matching-vertex y non-dominated in purple. But then the matching-edge containing y either is still completely non-selected and Dom can play its center, or the pair of y in that edge was selected by Sepy, which contradicts the Neighbor Rule.

Neighbor Rule. Suppose first that Sepy selected a vertex of e_i in his latest move. If the entire e_i is not yet dominated in purple, then the other vertex of e_i is feasible for Dom, and the rule is kept by choosing that vertex. In the other case the selection of Sepy is already dominated in purple, and the rule does not put any restriction on Dom.

Suppose next that Sepy selected an external vertex x. If x is not dominated in purple yet, then it must have an uncolored neighbor, which necessarily is either the center of a star-edge or belongs to a triangle-edge. Thus, Dom can keep the rules.

Matching Rule. This rule is easy to keep. Dom is forced to select an external vertex only if all neighbors of this vertex were already selected by Sepy. But then Dom can postpone the selection of all such external vertices to the last part of the game. $\Diamond$

The two lemmas above together imply that Dom can win the Bicolored Domination Game on every isolate-free graph. $\square$

7 Conclusion

In this paper we introduced two games on graphs, the Disjoint Domination Game and the Bicolored Domination Game. This new area offers many challenging open questions; below we collect some of them.

7.1 Dom-win and Sepy-win graphs

We have proved that Dom has a winning strategy in the Sepy-start Disjoint Domination Game on connected graphs. It is not very well understood, however, which properties of G ensure a winning strategy for Dom or Sepy if Dom starts the game, or if Sepy starts but the graph is disconnected.

Problem 15 *Characterize the disconnected graphs on which Dom can win the Sepy-start Disjoint Domination Game.*

Problem 16 *Characterize the graphs (connected or otherwise) on which Dom can win the Dom-start Disjoint Domination Game.*

A false intuition says that, due to the constructive goal of Dom, the possibility of passing does not increase Sepy's chances to win. But in fact this is not true, as shown by the following observation.

Proposition 17 *There exist graphs such that the possibility of passing does increase Sepy's chances to win.*

Proof. As an example, consider the graph G which is the disjoint union of a C_4 and a C_8. Note that the Disjoint Domination Game played on C_4 surely ends with two purple and two blue vertices; no monochromatic $N[v]$ can arise, independently of the strategies of the players. In addition, by Proposition 6 and Theorem 5, Sepy can win the Dom-start game and Dom can win the Sepy-start game on C_8 when passing is not allowed.

Thus, if Sepy starts the game on G and passing is not allowed, Dom has a winning strategy as he can ensure that Sepy selects the first vertex from C_8. In contrary, if Sepy may pass, first he can select a vertex from C_4 and then he passes in every turn until Dom colors a vertex from C_8. Then, Sepy can win the game. □

7.2 Biased games

If $d > 1$ or $s > 1$, weaker or stronger conditions may be imposed than those in the $(d : s)$-game. Namely, we may allow Dom to select fewer than d vertices per turn, and/or force Sepy to select exactly s vertices per turn. More generally, instead of d and s one may specify parameters $d'' \geq d' \geq 1$ and $s'' \geq s' \geq 0$, requiring that, in each turn, the number of vertices selected by Dom has to be between d' and d'' while the number of vertices selected by Sepy has to be between s' and s''. Even more generally one may specify the sets D^* and S^* of allowed numbers of selections per turn (possibly varying turn by turn), etc.

It is not our goal to analyze the similarities and differences between these variants; we leave this direction open for future research of other authors. It also remains unexplored, which kinds of substructures and legal moves should be excluded in order to make the $(d : s)$-game non-trivial on some classes of graphs. In this direction the following related question seems to be important.

Problem 18 *Determine the (sets of) restrictions that ensure the following: For every $s \geq 1$ there exists a threshold value d_s such that Dom wins both the Dom-start and Sepy-start $(d : s)$-game on every graph of minimum degree at least s.*

Minimum degrees slightly larger than s may also be of interest. Our impression is that one natural kind of conditions may be something like this: If Sepy selects a set $S = \{v_1, \dots, v_s\}$ in a move, then there must exist other s vertices $v'_1, \dots, v'_s \notin S$ which have not been dominated previously in Sepy's color and all of $v_1v'_1, \dots, v_sv'_s$ are edges in G. The strategy ONS or some variants of it may turn out to be powerful also in this context.

The following question seems to be interesting, too.

Problem 19 *Let G be a graph, and $d, s \in \mathbb{N}$. Investigate the relation of the $(d : s)$-game to the $(d + 1 : s)$-game and to the $(d : s + 1)$-game in both the Dom-start and Sepy-start versions.*

7.3 More than two colors

Some graphs contain more than two mutually disjoint dominating sets. The *domatic number* of $G = (V, E)$ is the largest integer k for which there exists a vertex partition $D_1 \cup \cdots \cup D_k = V$ into k dominating sets D_i of G.

Problem 20 *Let G be a graph with domatic number k, and let $\mathcal{C}_\ell$ be a palette of $\ell \leq k$ colors. What kind of structural properties of G imply that Dom has a winning strategy in the game where he and Sepy alternately assign colors from $\mathcal{C}_\ell$ to the vertices of G and Dom's goal is to create a domatic partition with ℓ vertex classes? In particular, characterize the graphs which admit a winning strategy for Dom in the case $k = \ell$.*

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