Transversal Game on Hypergraphs and the $\frac{3}{4}$-Conjecture on the Total Domination Game

[1,2]Csilla Bujtás, [3]Michael A. Henning, and [1,2]Zsolt Tuza

[1]Department of Computer Science and Systems Technology
University of Pannonia
H-8200 Veszprém, Egyetem u. 10, Hungary
Email: bujtas@dcs.uni-pannon.hu, tuza@dcs.uni-pannon.hu

[2]Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
H-1053 Budapest, Reáltanoda u. 13-15, Hungary

[3]Department of Pure and Applied Mathematics
University of Johannesburg
Auckland Park, 2006 South Africa
Email: mahenning@uj.ac.za

Abstract

The $\frac{3}{4}$-Game Total Domination Conjecture posed by Henning, Klavžar and Rall [Combinatorica, to appear] states that if G is a graph on n vertices in which every component contains at least three vertices, then $\gamma_{\rm tg}(G) \le \frac{3}{4}n$, where $\gamma_{\rm tg}(G)$ denotes the game total domination number of G. Motivated by this conjecture, we raise the problem to a higher level by introducing a transversal game in hypergraphs. We define the game transversal number, $\tau_g(H)$, of a hypergraph H, and prove that if every edge of H has size at least 2, and $H \not\cong C_4$, then $\tau_g(H) \le \frac{4}{11}(n_H + m_H)$, where n_H and m_H denote the number of vertices and edges, respectively, in H. Further, we characterize the hypergraphs achieving equality in this bound. As an application of this result, we prove that if G is a graph on n vertices with minimum degree at least 2, then $\gamma_{\rm tg}(G) < \frac{8}{11}n$. As a consequence of this result, the $\frac{3}{4}$-Game Total Domination Conjecture is true over the class of graphs with minimum degree at least 2.

Keywords: Total domination game; Transversal; Game transversal; Hypergraph.
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1 Introduction

In this paper, we continue the study of the total domination game which was first investigated in [19]. A vertex *totally dominates* another vertex if they are neighbors. A *total dominating set* of a graph G is a set S of vertices such that every vertex of G is totally dominated by a vertex in S. The *total domination game* consists of two players called *Dominator* and *Staller*, who take turns choosing a vertex from G. Each vertex chosen must totally dominate at least one vertex not totally dominated by the set of vertices previously chosen. We call such a chosen vertex a *legal move* in the total domination game. The game ends when the set of vertices chosen is a total dominating set in G. Dominator wishes to minimize the number of vertices chosen, while Staller wishes to end the game with as many vertices chosen as possible.

The *game total domination number*, $\gamma_{\rm tg}(G)$, of G is the number of vertices chosen when Dominator starts the game, both players play according to the rules, and each player plays optimally to achieve his or her respective goal. The *Staller-start game total domination number*, $\gamma'_{\rm tg}(G)$, of G is the number of vertices chosen when Staller starts the game and both players play optimally.

In [19], the authors prove a Total Continuation Principle lemma from which one can readily deduce that $|\gamma_{\rm tg}(G) - \gamma'_{\rm tg}(G)| \le 1$ for every graph G with no isolated vertex. Determining the exact value of $\gamma_{\rm tg}(G)$ and $\gamma'_{\rm tg}(G)$ is a challenging problem, and is currently only known for paths and cycles [12]. A bound on the game total domination number for general graphs is established in [20] where it is shown that if G is a graph on n vertices in which every component contains at least three vertices, then $\gamma_{\rm tg}(G) \le \frac{4}{5}n$ and $\gamma'_{\rm tg}(G) \le (4n+2)/5$. Our focus in the present paper is the following conjecture posed by Henning, Klavžar and Rall [20].

$\frac{3}{4}$-Game Total Domination Conjecture ([20]) *If G is a graph on n vertices in which every component contains at least three vertices, then $\gamma_{\rm tg}(G) \le \frac{3}{4}n$.*

In this paper, we establish the $\frac{3}{4}$-Game Total Domination Conjecture over the class of graphs with minimum degree at least 2. This aim is achieved by considering the problem in a more general frame. We introduce and study a transversal game in hypergraphs, which is of interest in its own right. This demonstrates another example of the phenomenon that a general problem may turn out to be easier to handle than its particular case.

Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V(H), E(H))$ is a finite set $V(H)$ of elements, called *vertices*, together with a finite multiset $E(H)$ of nonempty subsets of $V(H)$, called *hyperedges* or simply *edges*. If the hypergraph H is clear from the context, we simply write $V = V(H)$ and $E = E(H)$. We shall use the notation $n_H = |V|$ and $m_H = |E|$ to denote the *order* and *size* of H, respectively. The hypergraph H is called *linear* if every two distinct edges of H intersect in at most one vertex. We say that two edges in H *overlap* if they intersect in at least two vertices. A linear hypergraph therefore has no overlapping edges.

A k-*edge* in H is an edge of size k. The hypergraph H is said to be k-*uniform* if every edge of H is a k-edge. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. Throughout this paper, unless stated otherwise, we assume that $|e| \geq 2$ holds for all $e \in E$; that is, we consider hypergraphs without one-element edges. We say that H is *non*-2-*uniform* if it contains at least one edge of size at least 3.

The *degree* of a vertex v in H, denoted by $d_H(v)$ or $d(v)$ if H is clear from the context, is the number of edges of H which contain v. The minimum degree among the vertices of H is denoted by $\delta(H)$ and the maximum degree by $\Delta(H)$. A vertex of degree 1 in a graph G is called a *leaf* or a *pendant vertex*, and its neighbor a *support vertex*. An *isolated vertex* is a vertex of degree 0. An *isolated edge* in a hypergraph H is an edge e that is not intersected by any other edge of H.

Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq e$. The *neighborhood* of a vertex v in H, denoted $N_H(v)$ or simply $N(v)$ if H is clear from the context, is the set of all vertices different from v that are adjacent to v, while the *closed neighborhood* of v in H, denoted $N_H[v]$ or simply $N[v]$, is the set $N_H(v) \cup \{v\}$. A vertex in $N(v)$ is a *neighbor* of v. We also use the standard notation $[k] = \{1, \ldots, k\}$.

A subset T of vertices in a hypergraph H is a *transversal* (also called *hitting set* or *vertex cover* or *blocking set* in many papers) if T has a nonempty intersection with every edge of H. A vertex *hits* or *covers* an edge if it belongs to that edge. The notion of transversal is fundamental in hypergraph theory and has been studied a great deal; a rough estimate says that it occurs in more than 25.000 papers (considering the various names listed above, and also taking 'edge cover' into account, to which it is equivalent by hypergraph duality). Two of the five chapters in the major monograph of hypergraph theory [1] deal with transversals and their fractional version (real relaxation via Linear Programming). We refer to [6, 7, 13, 21, 22, 23] for recent results and further references.

The Game Transversal Number. In this paper we introduce the study of the game transversal number in hypergraphs. The transversal game belongs to the growing family of *competitive optimization* games on graphs and hypergraphs. As remarked in [18], broadly speaking, "competitive optimization" describes a process in which multiple agents with conflicting goals collaboratively produce some special structure in an underlying host graph/hypergraph. In the transversal game, that structure is a transversal (also called hitting set), and the players' goals are completely antithetical: while Staller wants to maximize the size of a transversal constructed during the game, Edge-hitter wants to minimize it. Thus, the transversal game is a competitive optimization variant of the well-studied transversal problem on hypergraphs. One of the first and best-known competitive optimization parameters is the *game chromatic number*, which was introduced by Brams for planar graphs (cf. [15]) and independently by Bodlaender [2] for general graphs; it has seen extensive study, see the survey [28]. Recently, work has been done on competitive optimization variants of list-colouring [5] and its more studied related version called paintability as introduced in [26] (for further references see Section 8 of [28]), matching [11], domination [4], total domination [19], disjoint domination [8], Ramsey theory [9, 16, 17], and more [3].

Formally, the transversal game played on a hypergraph H consists of two players, *Edge-hitter* and *Staller*, who take turns choosing a vertex from H. Each vertex chosen must hit at least one edge not hit by the vertices previously chosen. We call such a chosen vertex a *legal move* in the transversal game. The game ends when the set of vertices chosen becomes a transversal in H. Edge-hitter wishes to end the game with a minimum number of vertices chosen, and Staller wishes to end the game with as many vertices chosen as possible. The *game transversal number* (resp. *Staller-start game transversal number*), $\tau_g(H)$ (resp. $\tau_g'(H)$), of H is the number of vertices chosen when Edge-hitter (resp. Staller) starts the game and both players play optimally according to their goals.

For a graph G, the *open neighborhood hypergraph*, abbreviated ONH, of G is the hypergraph H_G with vertex set $V(H_G) = V(G)$ and with edge set $E(H_G) = \{N_G(x) \mid x \in V(G)\}$ consisting of the open neighborhoods of vertices in G. Here, we assume that G contains no isolated vertex and the presence of one-element edges is allowed in H_G. We note that H_G has $n_{_G}$ vertices and $n_{_G}$ edges. The transversal number of the ONH of a graph is precisely the total domination number of the graph; that is, for a graph G, we have $\gamma_t(G) = \tau(H_G)$, where $\gamma_t(G)$ and $\tau(H_G)$ are the minimum sizes of a total dominating set in G and transversal in H_G, respectively. Further, a sequence of moves in the total domination game is legal if and only if the sequence of moves is legal in the transversal game. Thus, there is a one-to-one correspondence between the sequences of legal moves in the total domination game and the sequences of legal moves in the transversal game, implying the following observation.

Observation 1 *If G is a graph with no isolated vertex and H_G is the ONH of G, then $\gamma_{\rm tg}(G) = \tau_g(H_G)$.*

A *partially covered hypergraph* is a hypergraph together with a declaration that some edges are already covered; that is, they need not be covered in the rest of the game. Once a vertex is played in the transversal game, it has no role in the continuation and can be deleted from the hypergraph, as can all edges covered by that vertex. Further, isolated vertices also can be deleted. Therefore, during the game, we may consider this hypergraph which contains no isolated vertices and edges already covered, and we call it a *residual hypergraph*. We will also say that the original hypergraph H, before any move has been made in the game, is a residual hypergraph.

2 Main Results

In this paper we prove the $\frac{3}{4}$-Game Total Domination Conjecture over the class of graphs with minimum degree at least 2. To do this, we first establish a tight upper bound on the game transversal number of a hypergraph in terms of its order and size. The validity of the $\frac{3}{4}$-conjecture on graphs with no pendant vertices is then a simple corollary of this game transversal result. Recall that all hypergraphs considered here are assumed to not contain one-element edges, and this convention is kept for the residual hypergraphs, too. In particular, this restriction is satisfied by the ONH of a graph with minimum degree at least 2 since every edge in such an ONH has size at least 2.

Let M_1 be the hypergraph with vertex set $V(M_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and edge set

$$E(M_1) = \{\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}, \{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}\}.$$

For $i \in [3]$, we call the vertices x_i and y_i *partners* in M_1. For $k \geq 1$, let M_k consist of k vertex-disjoint copies of M_1, and let $\mathcal{H} = \{M_k : k \geq 1\}$. The hypergraph, $M_3 \in \mathcal{H}$, is illustrated in Figure 1, albeit without the vertex labels.

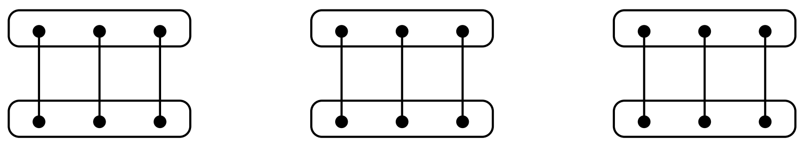

Figure 1: The hypergraph, M_3, in the family $\mathcal{H}$.

We shall establish the following upper bound on the game transversal number of a hypergraph. Its proof is given in Section 4.

Theorem 1 *If H is a hypergraph with all edges of size at least 2, and $H \not\cong C_4$, then $\tau_g(H) \leq \frac{4}{11}(n_H + m_H)$, with equality if and only if $H \in \mathcal{H}$.*

The $\frac{3}{4}$-Game Total Domination Conjecture over the class of graphs with minimum degree at least 2 now follows as an immediate consequence of our main result, Theorem 1, and the interplay between the game total domination number and the game transversal number given by Observation 1. More precisely, let G be a graph with $\delta(G) \geq 2$ and let $H = H_G$ be the ONH of G. Then, each edge of H has size at least 2. Since C_4 and any $H \in \mathcal{H}$ is not the ONH of any graph, by Observation 1 and Theorem 1 we obtain

$$\gamma_{\rm tg}(G) = \tau_g(H) < \frac{4}{11}(n_H + m_H) = \frac{4}{11}(n_G + n_G) = \frac{8}{11}n_G < \frac{3}{4}n_G.$$

We state these results formally as follows.

Corollary 1 *If G is a graph with $\delta(G) \geq 2$, then $\gamma_{\rm tg}(G) < \frac{8}{11}n_G$.*

Corollary 2 *The $\frac{3}{4}$-Game Total Domination Conjecture is true over the class of graphs with minimum degree at least 2.*

As a special case of more general results due to Tuza [27] and Chvátal and McDiarmid [10], if H is a 2-uniform hypergraph, then $\tau(H) \leq \frac{1}{3}(n_H + m_H)$. We show that this bound is almost true for the game transversal number. The proof is given in Section 5.

Theorem 2 *If H is a 2-uniform hypergraph, then $\tau_g(H) \leq \frac{1}{3}(n_H + m_H + 1)$.*

The game transversal number of a cycle is determined by Proposition 1, a proof of which is given in Section 5.

Proposition 1 *For $n \ge 3$, $\tau_g(C_n) = \left\lfloor \frac{2n+1}{3} \right\rfloor$.*

By Proposition 1, if $n \equiv 1 \pmod 3$, then $\tau_g(C_n) = \frac{1}{3}(2n+1) = \frac{1}{3}(n_G + m_G + 1)$. Thus, the bound of Theorem 2 is achieved by cycles of length congruent to 1 modulo 3.

In Section 5, we also determine the game transversal number of paths, and the corresponding values of τ'_g.

Remark 1 *While the tight bounds $\tau(G) \le (n_G + m_G)/3$ and $\tau_g(G) \le (n_G + m_G + 1)/3$ on graphs are quite similar, the behaviors of τ and τ_g become substantially different if we restrict our attention to connected graphs. Namely, the bound on τ_g remains the same, demonstrated by Theorem 2, but a much stronger inequality $\tau(G) \le \frac{2}{7}(n_G + m_G + 1)$ is valid (which is again tight), as proved in [14]. In other words, in terms of $n_G + m_G$, on connected graphs the best possible asymptotic coefficient for τ is $2/7$, whereas that for τ_g is $1/3$.*

3 Preliminary Lemmas

In this section, we prove a series of nine preliminary lemmas that we will need in order to prove our main results, namely Theorem 1 and Theorem 2. We remark that if H is a hypergraph, and H' is obtained from H by deleting all multiple edges in H (in the sense that if H has ℓ distinct edges $e_1, e_2, \ldots, e_\ell$ that are multiple edges, and so $e_1 = e_2 = \cdots = e_\ell$, then we delete $\ell - 1$ of these multiple edges), then $\tau_g(H') = \tau_g(H)$. Hence, it suffices to prove Theorem 1 and Theorem 2 in the case of hypergraphs with no multiple edges.

Throughout this section, let H be a residual hypergraph with no multiple edges, and let D be the set of played vertices, where initially $D = \emptyset$. With respect to the set D, we color the vertices and edges of H either white or red, according to the following rules. An edge is colored *white* if it is not covered by a vertex of D, and is colored *red* otherwise. Thus, if an edge e is colored white, then $e \cap D = \emptyset$, while if e is colored red, then $e \cap D \ne \emptyset$. Further, a vertex is colored *white* if it is incident to at least one white edge, and is colored *red* otherwise.

We associate a weight of 1 to each white vertex and white edge and a weight of 0 to each red vertex and red edge. We remark that as the game is played new red vertices and red edges are created and at that moment we assign them weight 0, we delete them from the residual hypergraph. We define the *weight* of the residual hypergraph H as the sum of the weights of the vertices and edges in H and denote this weight by $\mathrm{w}(H)$. Thus, $\mathrm{w}(H)$ is the number of white vertices and white edges.

If $m_H = 0$, then $\tau_g(H) = 0$ and the bounds in Theorem 1 and Theorem 2 are immediate. Hence we may assume that $m_H \ge 1$.

For a positive real number c, we say that Edge-hitter can *achieve his c-target* at a certain stage of the game if from then on he can play a sequence of moves guaranteeing that on average the weight decrease resulting from each played vertex in that part of the game is at least c. Stating this in an explicit formal way, in order to achieve his c-target, Edge-hitter must guarantee that a sequence of moves $m_1, \ldots, m_k$ are played, starting with his first move m_1, and with moves alternating between Edge-hitter and Staller such that if w_i denotes the decrease in weight after move m_i is played, then

$$\sum_{i=1}^{k} \mathrm{w}_i \geq c \cdot k\,, \tag{1}$$

where either k is odd and the game is completed after move m_k or k is any even number (in this latter case, the game may or may not be completed after move m_k). In most cases we will use $k = 1$ or 2 when proving that Edge-hitter can achieve his c-target. Thus, every move decreases the weight by at least 2, since every move results in at least one vertex and at least one edge being recolored red. In the discussion that follows, we analyse how Edge-hitter can *achieve his c-target* when $c = 3$ and when $c = \frac{11}{4}$. First, we prove a series of lemmas that establish key properties that hold in the residual hypergraph H, and thereafter, we verify Theorem 1 and Theorem 2.

Lemma 1 *If Edge-hitter can play as his first move a vertex which results in a decrease of at least* 4 *in the weight of the residual hypergraph H, then he can achieve his* 3*-target.*

Proof. Suppose that as his first move m_1, Edge-hitter plays a vertex in H such that $\mathrm{w}_1 \geq 4 > 3 \cdot 1$. If the game is complete after Edge-hitter's move, then Inequality (1) is satisfied with $c = 3$ and $k = 1$. Otherwise, Staller responds by playing her move m_2, which results in $\mathrm{w}_2 \geq 2$. Thus, $\mathrm{w}_1 + \mathrm{w}_2 \geq 4 + 2 = 6 = 3 \cdot 2$, and so Inequality (1) is satisfied with $c = 3$ and $k = 2$. $\Box$

Lemma 2 *If $\Delta(H) \geq 3$, then Edge-hitter can achieve his* 3*-target.*

Proof. If $\Delta(H) \geq 3$, then Edge-hitter plays as his move m_1 a vertex of maximum degree in the residual hypergraph H, which results in at least three edges and one vertex being recolored red. Thus, $\mathrm{w}_1 \geq 4$ and, by Lemma 1, Edge-hitter can achieve his 3-target. $\Box$

Lemma 3 *If $\Delta(H) = 2$ and there exist overlapping edges in H, then Edge-hitter can achieve his* 3*-target.*

Proof. If e_1 and e_2 are two overlapping edges in H, then Edge-hitter plays a vertex from $e_1 \cap e_2$ as his move m_1 in H. This results in every vertex in $e_1 \cap e_2$ being recolored red and both edges e_1 and e_2 being recolored red. Thus, since $|e_1 \cap e_2| \geq 2$, at least two vertices and two edges are recolored red. Hence, $\mathrm{w}_1 \geq 4$, and by Lemma 1 we infer that Edge-hitter can achieve his 3-target. $\Box$

From now on, we assume that every component of H is either an isolated edge or is a linear hypergraph with maximum degree 2, for otherwise by Lemma 2 and Lemma 3 Edge-hitter can achieve his 3-target. Edge-hitter henceforth applies the following rules.

(R1) He plays a degree-2 vertex which has a degree-1 neighbor (independently from the sizes of the edges containing these vertices).

(R2) If he cannot play according to (R1), he plays a vertex from an isolated edge of size at least 3.

(R3) If he cannot play according to (R1) and (R2), he plays a vertex from a 2-regular, 2-uniform component which is a cycle of length congruent to 0 or 2 modulo 3.

(R4) If he cannot play according to (R1), (R2) and (R3), he plays a vertex from a P_2-component (or, equivalently, an isolated edge of size 2).

(R5) If he cannot play according to (R1), (R2), (R3) and (R4), he plays a vertex from a 2-regular, 2-uniform component which is a cycle of length congruent to 1 modulo 3.

(R6) Otherwise, he plays a vertex v from a 2-regular, non-2-uniform component such that $|N(v)|$ is maximum.

By our assumption, H does not contain edges of size 1. Moreover, if a component is not 2-regular and not a 1-regular isolated edge, it contains a vertex of degree 2 which has a degree-1 neighbor. Therefore, the rules (R1)-(R6) together cover all possible cases.

Lemma 4 *If H is a linear hypergraph of maximum degree at most 2, and Edge-hitter can play according to rule* (R1) *or* (R2)*, then he can achieve his 3-target.*

Proof. If Edge-hitter can play as his first move, m_1, a degree-2 vertex that has a degree-1 neighbor, then such a move results in at least two vertices and two edges being recolored red, and so $\mathrm{w}_1 \geq 4$. If Edge-hitter can play as his move m_1 a vertex from an isolated edge of size at least 3, then such a move results in at least three vertices and one edge recolored red, and so $\mathrm{w}_1 \geq 4$. In both cases, by Lemma 1, Edge-hitter can achieve his 3-target. $\square$

Lemma 5 *If H is a linear hypergraph of maximum degree at most 2, and Edge-hitter can play according to rule* (R3)*, then he can achieve his 3-target.*

Proof. Suppose that Edge-hitter can play according to (R3). Since Edge-hitter cannot play according to (R1) and (R2), every component of H is therefore a 2-regular, linear hypergraph or is a P_2-component. Further, there exists a 2-regular, 2-uniform component, C say, which is a cycle of length 3ℓ or $3\ell + 2$, for some $\ell \geq 1$.

Suppose that C is a cycle of length 3ℓ. Edge-hitter now plays as his move m_1 a vertex from $V(C)$, resulting in one vertex and two edges recolored red, and so $\mathrm{w}_1 = 3$. If Staller plays as her move m_2 a non-leaf vertex in the resulting path on $3\ell - 1$ vertices or a vertex not in $V(C)$, then $\mathrm{w}_2 \geq 3$, and so Inequality (1) is satisfied with $c = 3$ and $k = 2$. Hence, we may assume that Staller plays as her move m_2 a leaf in $V(C)$. If $\ell = 1$, then Staller's move played a leaf from a P_2-component, resulting in $\mathrm{w}_2 = 3$, and so once again Inequality (1) is satisfied with $c = 3$ and $k = 2$. Thus, we may assume that $\ell \geq 2$. Edge-hitter now plays as his move m_3 a support vertex from $V(C)$ in the resulting path on $3\ell - 2$ vertices, resulting in two vertices and two edges recolored red, and so $\mathrm{w}_3 = 4$. If Staller plays as her move m_4 a non-leaf vertex from $V(C)$ or a vertex not in $V(C)$, then $\mathrm{w}_4 \geq 3$, and so Inequality (1) is satisfied with $c = 3$ and $k = 4$. Hence, we may assume that Staller plays as her move m_4 a leaf in $V(C)$. Continuing in this way, we may assume that after Edge-hitter's first move, Staller and Edge-hitter play leaves and support vertices, respectively, in $V(C)$ in subsequent moves until all vertices in $V(C)$ are colored red. In particular, Edge-hitter's move $m_{2\ell-1}$ is a support vertex from a P_4-component, while Staller's move $m_{2\ell}$ plays a leaf from a P_2-component. Thus, Staller's move $m_{2\ell}$ decreases the weight by 3, while her previous $\ell - 1$ moves each decrease the weight by 2. Edge-hitter's first move m_1 decreases the weight by 3, while his subsequent $\ell - 1$ moves each decreases the weight by 4. Hence,

$$\sum_{i=1}^{2\ell} \mathrm{w}_i = 3 \cdot 2\ell,$$

and so Inequality (1) is satisfied with $c = 3$ and $k = 2\ell$, and Edge-hitter achieves his 3-target.

Similarly, if C is a cycle of length $3\ell + 2$, then Edge-hitter plays as his first move m_1 a vertex from $V(C)$, and thereafter, we may assume that Staller and Edge-hitter play leaves and support vertices, respectively, in $V(C)$ in subsequent moves until all vertices in $V(C)$ are colored red. In this case, Staller's move $m_{2\ell}$ is a leaf from a P_4-component, while Edge-hitter's move $m_{2\ell+1}$ plays the central (support) vertex from a P_3-component. Thus, Staller's ℓ moves all decreases the weight by 2. Edge-hitter's first move m_1 decreases the weight by 3, his move $m_{2\ell+1}$ decreases the weight by 5, while his other $\ell - 1$ moves each decrease the weight by 4. Hence,

$$\sum_{i=1}^{2\ell+1} \mathrm{w}_i = 6\ell + 4 > 3(2\ell + 1).$$

If the game is complete after Edge-hitter's move $m_{2\ell+1}$, then Inequality (1) is satisfied with $c = 3$ and $k = 2\ell + 1$. Otherwise, Staller responds by playing her move $m_{2\ell+2}$, which results in $\mathrm{w}_{2\ell+2} = 3$. Thus,

$$\sum_{i=1}^{2\ell+2} \mathrm{w}_i = 6\ell + 7 > 3(2\ell + 2)$$

and Inequality (1) is satisfied with $c = 3$ and $k = 2\ell + 2$, and Edge-hitter achieves his 3-target. $\square$

Lemma 6 *If H is a linear hypergraph of maximum degree at most 2, and Edge-hitter can play according to rule* (R4)*, then he can achieve his 3-target.*

Proof. Suppose that Edge-hitter can play according to (R4). Since Edge-hitter cannot play according to (R1), (R2) and (R3), every component of H is therefore a 2-regular, linear hypergraph or is a P_2-component. Further, there exists at least one P_2-component. Edge-hitter plays as his move m_1 a vertex from a P_2-component, resulting in $\mathrm{w}_1 = 3$. If the game is complete after Edge-hitter's move, then Inequality (1) is satisfied with $c = 3$ and $k = 1$. Otherwise, Staller responds by playing a vertex from a P_2-component or a 2-regular component. This results in $\mathrm{w}_2 = 3$. Inequality (1) is now satisfied with $c = 3$ and $k = 2$. $\square$

Lemma 7 *If each component of H is a cycle of length congruent to* 1 *modulo* 3*, and there are at least two components, then Edge-hitter can achieve his* 3*-target. If $H \cong C_{3\ell+1}$ for some $\ell \geq 1$, then either Edge-hitter can achieve his* 3*-target, or the game ends after move $m_{2\ell+1}$ and*

$$\sum_{i=1}^{2\ell+1} \mathrm{w}_i = 3(2\ell+1) - 1. \tag{2}$$

Proof. Let C be an arbitrary component of H, and so C is a cycle of length $3\ell + 1$, for some $\ell \geq 1$. Edge-hitter now plays as his move m_1 a vertex from $V(C)$, resulting in $\mathrm{w}_1 = 3$. If Staller plays as her move m_2 a non-leaf vertex from the resulting path on 3ℓ vertices or a vertex not in $V(C)$, then $\mathrm{w}_2 \geq 3$, and so Inequality (1) is satisfied with $c = 3$ and $k = 2$. Hence, we may assume that Staller plays as her move m_2 a leaf in $V(C)$, and so $\mathrm{w}_2 = 2$. If $\ell > 1$, Edge-hitter plays as his move m_3 a support vertex from $V(C)$ in the resulting path on $3\ell - 1$ vertices, resulting in $\mathrm{w}_3 = 4$. If Staller plays as her move m_4 a non-leaf vertex from $V(C)$ or a vertex not in $V(C)$, then $\mathrm{w}_4 = 3$, and so Inequality (1) is satisfied with $c = 3$ and $k = 4$. Hence, we may assume that Staller plays as her move m_4 a leaf in $V(C)$. Continuing in this way, we may assume that after Edge-hitter's first move, Staller and Edge-hitter play leaves and support vertices, respectively, in $V(C)$ in subsequent moves in the residual hypergraph until all vertices in $V(C)$ are colored red. In particular, Edge-hitter's move $m_{2\ell+1}$ is a vertex in $V(C)$ from a P_2-component. Thus, Staller's first ℓ moves all decrease the weight by 2. Edge-hitter's first move m_1 and his move $m_{2\ell+1}$ both decrease the weight by 3, while his other $\ell - 1$ moves each decrease the weight by 4. This exactly yields Equation (2).

If the residual hypergraph H contained only this component $C \cong C_{3\ell+1}$, the game is complete after Edge-hitter's move $m_{2\ell+1}$ and Equality (2) is satisfied. Otherwise, the game is not complete after Edge-hitter's move $m_{2\ell+1}$. Staller responds by playing as her move $m_{2\ell+2}$ a vertex not in $V(C)$, and she therefore opens up a cycle, C' say, of length $3r + 1$, for some $r \geq 1$, that is different from C, by playing the first vertex in C'. Thus, $\mathrm{w}_{2\ell+2} = 3$. Using analogous arguments as with the component C, we may assume that after Staller opens the cycle C', Edge-hitter and Staller play support vertices and leaves, respectively, in $V(C')$ in subsequent moves in the residual graph until all vertices in $V(C')$ are colored red. In particular, both Staller and Edge-hitter play r moves in $V(C')$. Further, Edge-hitter's last move in $V(C')$, namely his move $m_{2\ell+2r+1}$, is the central vertex in $V(C')$ from a P_3-component in the residual graph. Thus, Staller's first move in $V(C')$ decreases the weight by 3, while her subsequent $r - 1$ moves in $V(C')$ all decrease the weight by 2. Edge-hitter's first $r-1$ moves in $V(C')$ all decrease the weight by 4, while his rth move in $V(C')$ decreases the weight by 5. Thus,

$$\sum_{i=1}^{2\ell+2r+1} \mathrm{w}_i = \sum_{i=1}^{2\ell+1} \mathrm{w}_i + \sum_{i=2\ell+2}^{2\ell+2r+1} \mathrm{w}_i = (3(2\ell+1)-1) + (6r+2) = 3(2\ell+2r+1)+1.$$

If the game is complete after Edge-hitter's move $m_{2\ell+2r+1}$, then Inequality (1) is satisfied with $c = 3$ and $k = 2\ell + 2r + 1$. Otherwise, Staller plays as her move $m_{2\ell+2r+2}$ a vertex that opens up a cycle, different from C and C', implying that $\mathrm{w}_{2\ell+2r+2} = 3$. Thus, in this case,

$$\sum_{i=1}^{2\ell+2r+2} \mathrm{w}_i = 3(2\ell+2r+2)+1,$$

and Inequality (1) is satisfied with $c = 3$ and $k = 2\ell + 2r + 2$. $\Box$

Lemma 8 *If H is a linear hypergraph of maximum degree at most 2 and $H \not\cong C_4$, and Edge-hitter can play according to rule* (R5)*, then he can achieve his $\frac{11}{4}$-target.*

Proof. Suppose that Edge-hitter can play according to (R5). Since Edge-hitter cannot play according to (R1), (R2), (R3) and (R4), every component of H is therefore a 2-regular, linear hypergraph. Further, no 2-regular, 2-uniform, component is a cycle of length 3ℓ or $3\ell + 2$. By assumption, there exists a 2-regular, 2-uniform component, C say, which is a cycle of length $3\ell + 1$, for some $\ell \ge 1$. Edge-hitter now plays as his first move, m_1, a vertex from $V(C)$, resulting in $\mathrm{w}_1 = 3$. If Staller plays as her move m_2 a vertex that is not a leaf in the resulting path on 3ℓ vertices or is a vertex not in $V(C)$, then $\mathrm{w}_2 \ge 3$, and so Inequality (1) is satisfied with $c = 3$ and $k = 2$ (and hence, also with $c = \frac{11}{4}$). Therefore, we may assume that Staller plays as her move m_2 a leaf in $V(C)$, and so $\mathrm{w}_2 = 2$.

Suppose that $\ell = 1$. In this case, $C \cong C_4$ and Edge-hitter plays as his move m_3 a vertex in $V(C)$ from the resulting P_2-component, implying that $\mathrm{w}_3 = 3$. Since $H \not\cong C_4$, the game is not complete after Edge-hitter's move. Staller plays as her move m_4 a vertex not in $V(C)$, and therefore she plays a degree-2 vertex in a 2-regular, linear component, C' say. Thus, $\mathrm{w}_4 = 3$. Then, $\sum_{i=1}^{4} \mathrm{w}_i \ge 3+2+3+3 = 11 = 4 \cdot \frac{11}{4}$ and Edge-hitter achieves his $\frac{11}{4}$-target.

If $\ell \ge 2$, an analogous proof as the proof of Lemma 7 shows that we may assume that after Edge-hitter's first move, Staller and Edge-hitter play leaves and support vertices, respectively, in $V(C)$ in subsequent moves in the residual hypergraph until all vertices in $V(C)$ are colored red (otherwise, Edge-hitter achieves his 3-target). In particular, Edge-hitter's move $m_{2\ell+1}$ is a vertex in $V(C)$ from a P_2-component. Thus, Staller's first ℓ moves all decrease the weight by 2. Edge-hitter's first move m_1 and his move $m_{2\ell+1}$ both decrease the weight by 3, while his other $\ell - 1$ moves all decrease the weight by 4. Hence, as $\ell \ge 2$,

$$\sum_{i=1}^{2\ell+1} \mathrm{w}_i = 3(2\ell+1) - 1 \ge \frac{11}{4}(2\ell+1) + \frac{1}{4}. \tag{3}$$

If the game is not over, for the next move $m_{2\ell+2}$ of Staller, $\mathrm{w}_{2\ell+2} \geq 3$ holds, implying that

$$\sum_{i=1}^{2\ell+2} \mathrm{w}_i = 3(2\ell+2) - 1 \geq \frac{11}{4}(2\ell+2) + \frac{1}{2}. \tag{4}$$

In both cases, Edge-hitter can achieve his $\frac{11}{4}$-target. □

Lemma 9 *If H is a linear hypergraph of maximum degree 2 and Edge-hitter can play according to rule* (R6)*, then he can achieve his $\frac{11}{4}$-target.*

Proof. Since Edge-hitter cannot play according to (R1)–(R5), every component of H is therefore a 2-regular, non-2-uniform, linear component. Let v be a vertex of H such that $|N(v)|$ is maximum, and let C be the component of H containing the vertex v. We note that $|N(v)| \geq 3$. Edge-hitter now plays as his move m_1 the vertex v, resulting in $\mathrm{w}_1 = 3$. After move m_1, the vertices in $N(v)$ have degree 1 in the residual hypergraph, while the remaining vertices in $V(C)$ have degree 2. If Staller plays as her move m_2 a vertex of degree 2, or a vertex of degree 1 which has a degree-1 neighbor, then $\mathrm{w}_2 \geq 3$, and so Inequality (1) is satisfied with $c = 3$ and $k = 2$ (and also with $c = \frac{11}{4}$). Hence, we may assume that Staller plays as her move m_2 a vertex of degree 1 that has only neighbors of degree 2, resulting in $\mathrm{w}_2 = 2$. After Staller's move m_2, we have at least three vertices of degree 1 in the residual hypergraph.

Suppose that one of the vertices of degree 1 has a neighbor of degree 2. In this case, Edge-hitter plays as his move m_3 such a vertex of degree 2, resulting in $\mathrm{w}_3 \geq 4$. If the game is complete after Edge-hitter's move m_3, then Inequality (1) is satisfied with $c = 3$ and $k = 3$. Otherwise, Staller responds by playing her move m_4, which results in $\mathrm{w}_4 \geq 2$. Thus, $\sum_{i=1}^{4} \mathrm{w}_i \geq 3 + 2 + 4 + 2 = 11 = 4 \cdot \frac{11}{4}$, and Inequality (1) is satisfied with $c = \frac{11}{4}$ and $k = 4$. Hence, we may assume that after Staller's move m_2, every component in the residual hypergraph that contains a vertex of degree 1 is an isolated edge.

If there is an isolated edge of size at least 3, then Edge-hitter plays as his move m_3 a vertex from such an isolated edge, resulting in $\mathrm{w}_3 \geq 4$, and, analogously as before, Inequality (1) is satisfied with $c = 3$ and $k = 3$ or with $c = \frac{11}{4}$ and $k = 4$. Hence, we may assume that every isolated edge has size 2. Since there are at least three vertices of degree 1 (after Staller's move m_2), we note that there are at least two isolated edges (each of size 2). Further, every component that is not a P_2-component is a 2-regular, non-2-uniform, linear component. Edge-hitter now plays as his move m_3 a vertex from an isolated edge, resulting in $\mathrm{w}_3 = 3$. Staller plays as her move m_4 a vertex of degree 2, or a vertex from an isolated edge. In both cases, her move results in $\mathrm{w}_4 = 3$. Thus, $\sum_{i=1}^{4} \mathrm{w}_i = 3 + 2 + 3 + 3 = 11 = 4 \cdot \frac{11}{4}$, and Inequality (1) is satisfied with $c = \frac{11}{4}$ and $k = 4$. □

4 Proof of Theorem 1

In this section, we present a proof of Theorem 1. First, we prove the following proposition.

Proposition 2 *If $H \in \mathcal{H}$, then $\tau_g(H) = \frac{4}{11}(n_{_H} + m_{_H})$.*

Proof. Let $H = M_k \in \mathcal{H}$, for some $k \geq 1$. Staller's strategy is as follows: whenever Edge-hitter opens a component of H by playing the first vertex in that copy of M_1, Staller responds by playing the partner of Edge-hitter's move in that component, thereby reducing that component in the residual hypergraph to two isolated edges of size 2. We call these two isolated edges *twin edges*. If Edge-hitter plays a vertex in one of these isolated edges, then Staller plays on a vertex in its twin. In this way, Staller can force four vertices to be played in each copy of M_1 in H, implying that $\tau_g(H) \geq 4k$. However, at most four vertices can be played in each copy of M_1 in H, and so $\tau_g(H) \leq 4k$. Consequently, $\tau_g(H) = 4k$. We note that $n_{_H} = 6k$ and $m_{_H} = 5k$, and so $\tau_g(H) = 4k = \frac{4}{11}(6k + 5k) = \frac{4}{11}(n_{_H} + m_{_H})$. $\Box$

We are now in a position to present a proof of Theorem 1. Recall its statement.

Theorem 1. *If H is a hypergraph with all edges of size at least 2, and $H \ncong C_4$, then $\tau_g(H) \leq \frac{4}{11}(n_{_H} + m_{_H})$, with equality if and only if $H \in \mathcal{H}$.*

Proof. Lemmas 2, 3, 4, 5, 6, 8, and 9 together cover all possible cases and prove that Edge-hitter can achieve a $\frac{11}{4}$-target. Thus, Edge-hitter can make sure that the average decrease in the weight of the residual graph resulting from each played vertex in the game is at least $\frac{11}{4}$. Thus, in the residual hypergraph H, where $H \ncong C_4$,

$$\tau_g(H) \leq \frac{\mathrm{w}(H)}{\frac{11}{4}} = \frac{4}{11}(n_{_H} + m_{_H}).$$

Now, assume that Edge-hitter and Staller are playing on a hypergraph H, which satisfies the equality $\tau_g(H) = \frac{4}{11}(n_{_H} + m_{_H})$. We observe that if $H \cong C_4$, then $\tau_g(H) = 3 > \frac{4}{11}(4+4) = \frac{4}{11}(n_{_H} + m_{_H})$, and hence, $H \ncong C_4$.

Consider the residual hypergraphs $H = H_1, H_2, \ldots, H_s, H_{s+1}$, where H_{s+1} is the empty hypergraph and moreover, for each $i \in [s]$, H_{i+1} is obtained from H_i by Edge-hitter achieving a 3-target or $\frac{11}{4}$-target as described in Lemmas 2, 3, 4, 5, 6, 8, and 9. For notational simplicity, if the first move played by Edge-hitter in H_i is according to rule (Rj) for some $j \in [6]$, then we simply say that "H_{i+1} is obtained from H_i by playing according to rule (Rj)". For $i \in [s]$, let k_i denote the number of moves played to obtain H_{i+1} from H_i. By Lemmas 2, 3, 4, 5, and 6, for $i \in [s]$ if the residual hypergraph H_i has maximum degree at least 3, or contains overlapping edges, or contains a vertex of degree 2 with a degree-1 neighbor, or contains an isolated edge, or contains a cycle of length congruent to 0 or 2 modulo 3, then

$$\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}) \geq 3k_i > \frac{11}{4}k_i$$

holds. Further, if $i < s$, then $k_i \geq 2$ implies that

$$\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}) \geq \frac{11}{4}k_i + \frac{1}{2}\,. \tag{5}$$

By Lemmas 8 and 9, for $i \in [s]$ if Edge-hitter can play according to rules (R5) and (R6), then

$$\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}) \geq \frac{11}{4}k_i, \tag{6}$$

except when $i = s$ and $H_s \cong C_4$.

We show firstly that $H_s \not\cong C_4$. Suppose, to the contrary, that $H_s \cong C_4$. Thus, $k_s = 3$ and $\mathrm{w}(H_s) - \mathrm{w}(H_{s+1}) = 8 = \frac{11}{4}k_s - \frac{1}{4}$. Since $H \not\cong C_4$, we note that $s \geq 2$. Further, we note that $H_s \cong C_4$ cannot be obtained from H_{s-1} by an application of rule (R6). It is also evident from the proof of Lemma 8 that $H_s \cong C_4$ cannot be obtained from H_{s-1} by an application of rule (R5). Thus, either H_{s-1} has maximum degree at least 3 or contains overlapping edges or H_s is obtained from H_{s-1} by an application of one of the rules (R1)-(R4). Thus, Inequality (5) is satisfied by $i = s-1$, and so $\mathrm{w}(H_{s-1}) - \mathrm{w}(H_s) \geq \frac{11}{4}k_s + \frac{1}{2}$. Therefore,

$$(\mathrm{w}(H_{s-1}) - \mathrm{w}(H_s)) + (\mathrm{w}(H_s) - \mathrm{w}(H_{s+1})) \geq \frac{11}{4}(k_{s-1} + k_s) + \frac{1}{4}.$$

If $s \geq 3$, then by Inequality (5) and Inequality (6), for all $i \in [s-2]$, we have $\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}) \geq \frac{11}{4}k_i$; or, equivalently, $k_i \leq \frac{4}{11}(\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}))$. Hence,

$$\tau_g(H) = \sum_{i=1}^{s} k_i < \sum_{i=1}^{s} \frac{4}{11}(\mathrm{w}(H_i) - \mathrm{w}(H_{i+1})) = \frac{4}{11}\mathrm{w}(H) = \frac{4}{11}(n_{_H} + m_{_H}),$$

a contradiction. Therefore, $H_s \not\cong C_4$. Thus, by Inequality (5) and Inequality (6), for all $i \in [s]$, we have $\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}) \geq \frac{11}{4}k_i$; or, equivalently, $k_i \leq \frac{4}{11}(\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}))$. If $k_i < \frac{4}{11}(\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}))$ for some $i \in [s]$, then $\tau_g(H) < \frac{4}{11}(n_{_H} + m_{_H})$, a contradiction. Hence for all $i \in [s]$,

$$k_i = \frac{4}{11}(\mathrm{w}(H_i) - \mathrm{w}(H_{i+1})), \tag{7}$$

implying that H_{i+1} is obtained from H_i by playing according to rule (R5) or (R6). We show that, in fact, Edge-hitter can never play according to rule (R5).

Claim A. *For all $i \in [s]$, H_{i+1} is obtained from H_i by playing according to rule (R6).*

Proof. Suppose, to the contrary, that H_{i+1} is obtained from H_i by playing according to rule (R5) for some $i \in [s]$. Thus, the residual graph H_i contains a 2-regular, 2-uniform component, C say, which is a cycle of length $3\ell + 1$, for some $\ell \geq 1$, and we may assume that Edge-hitter plays as his first move, m_1, a vertex from $V(C)$, resulting in $\mathrm{w}_1 = 3$.

If $i = s$, then $H_s = C$, and so $H_s \cong C_{3\ell+1}$. Since $H_s \not\cong C_4$, we note that $\ell \geq 2$. However, by Inequality (3) in the proof of Lemma 8, we note that in this case $\mathrm{w}(H_s) - \mathrm{w}(H_{s+1}) > \frac{11}{4}k_s$, contradicting Equation (7). Hence, $i \in [s-1]$.

If $C \not\cong C_4$, then $\ell \geq 2$ and, once again, by Inequality (3) in the proof of Lemma 8, $\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}) > \frac{11}{4}k_i$, a contradiction. Hence, $C \cong C_4$, and so $\ell = 1$. According to rule (R5), after Edge-hitter plays his first move, m_1, which is a vertex from $V(C)$, Staller and

Edge-hitter subsequently play a leaf and a support vertex, respectively, from the resulting P_3-component as their moves m_2 and m_3, resulting in $\mathrm{w}_2 = 2$ and $\mathrm{w}_3 = 3$. Staller plays as her move m_4 a vertex not in $V(C)$, and therefore she plays a vertex (of degree 2) from a 2-regular, linear component, C' say. Hence, $\mathrm{w}_4 = 3$. According to rule (R5), the resulting residual hypergraph is the hypergraph H_{i+1}. Further, $\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}) = 11$ and $k_i = 4 = \frac{4}{11}(\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}))$.

We note that $C' \not\cong C_3$, for otherwise H_i contains a C_3-component and Edge-hitter could have applied rule (R3) when played on H_i, a contradiction. After Staller's move m_4, which is played on $C' \not\cong C_3$, either there is a vertex v of degree 2 in H_{i+1} having a degree-1 neighbor, or there are at least two isolated edges. Thus, in the residual hypergraph H_{i+1}, Edge-hitter can play according to one of the rules (R1)-(R4), which contradicts our earlier observation that, by Equation (7), H_{i+2} is obtained from H_{i+1} by playing according to rule (R5) or (R6). $\square$

By Claim A, H_{i+1} is obtained from H_i by playing according to rule (R6) for all $i \in [s]$. In particular, each component of H_i is a 2-regular, non-2-uniform, linear component, for all $i \in [s]$. To characterize these residual hypergraphs H_i that achieve equality in Equation (7), we proceed as in the proof of Lemma 9. Thus, Edge-hitter plays a vertex v from a component, C, of H_i such that $|N(v)|$ is maximum, as his move m_1, resulting in $\mathrm{w}_1 = 3$. Since H_i is not 2-uniform, we note that $|N(v)| \ge 3$ and that after move m_1, the vertices in $N(v)$ have degree 1 in the residual hypergraph, while the remaining vertices in $V(H_i)$ have degree 2.

If Staller plays as her move m_2 a vertex of degree 2, or a vertex of degree 1 which has a degree-1 neighbor, then $\mathrm{w}_2 \ge 3$. Thus, $\mathrm{w}_1 + \mathrm{w}_2 \ge 3 \cdot 2$ and Edge-hitter can achieve his 3-target in H_i, contradicting Equation (7). Thus, Staller plays as her move m_2 a vertex, u say, of degree 1 that has only neighbors of degree 2, resulting in $\mathrm{w}_2 = 2$. In the continuation of the game, Edge-hitter always plays a vertex from this component while there is at least one white vertex in $V(C)$. Note that the last move in $V(C)$, taken by any player, results in a decrease of at least 3 in the weight of the residual hypergraph. After Staller's move m_2, we have at least three vertices of degree 1 in the residual hypergraph. We proceed further with one more claim.

Claim B. *There are no vertices of degree* 2 *in* $V(C)$ *after the move* m_2.

Proof. Suppose, to the contrary, that one of vertices of degree 1 after Staller's move m_2 has a neighbor of degree 2. Among all such vertices of degree 2, Edge-hitter plays a vertex, say z, with the maximum possible number of degree-1 neighbors as his move m_3. If z has at least two degree-1 neighbors, then $\mathrm{w}_3 \ge 5$ and either the game is complete after Edge-hitter plays his move z, in which case Inequality (1) is satisfied with $c = 3$ and $k = 3$, or Staller responds and achieves $\mathrm{w}_4 \ge 2$, in which case Inequality (1) is satisfied with $c = 3$ and $k = 4$. In both cases, Edge-hitter can achieve his 3-target in H_i, contradicting Equation (7). Thus, z has exactly one degree-1 neighbor, resulting in $\mathrm{w}_3 = 4$.

Since at least one vertex of degree 1 remains after Edge-hitter plays his move m_3, the game cannot be finished at this point. Hence, Staller plays a vertex as her move m_4. If her move results in $w_4 \geq 3$, Edge-hitter can achieve his 3-target in H_i, contradicting Equation (7). Hence, $w_4 = 2$, and at least one vertex from $V(C)$ remains white after her move m_4. Thus, Edge-hitter can play a vertex from $V(C)$ as his move m_5, and either $w_5 \geq 4$ or $w_5 = 3$. We note that the case $w_5 = 3$ can occur only if every component induced by $V(C)$ in the residual hypergraph after Edge-hitter's move m_5 is a P_2-component. If the game is complete after Edge-hitter plays his move m_5, then $w(H_i) - w(H_{i+1}) \geq 3+2+4+2+3 = 14$ and $k_i = 5 < \frac{4}{11}(w(H_i) - w(H_{i+1}))$, a contradiction. Hence, Staller plays a vertex as her move m_6, which results in $w_6 \geq 2$. If $w_5 \geq 4$, then

$$\sum_{i=1}^{6} w_i \geq 3+2+4+2+4+2 = 17 > 6 \cdot \frac{11}{4},$$

contradicting Equation (7). Hence, $w_5 = 3$. Thus, only isolated edges and 2-regular components remain in the residual hypergraph after Edge-hitter's move m_4. Therefore, Staller can play either a vertex in an isolated edge or a vertex of degree 2. In both cases, her move results in $w_6 \geq 3$, once again contradicting Equation (7). $\Box$

By Claim B, there are no vertices of degree 2 in $V(C)$ after the move m_2. Hence, after Staller's move m_2, every component induced by $V(C)$ in the residual hypergraph is an isolated edge, while every other component, if any, in the residual hypergraph is a 2-regular, non-2-uniform, linear component.

If there is an isolated edge of size at least 3, then Edge-hitter plays as his move m_3 a vertex from such an isolated edge, resulting in $w_3 \geq 4$. If the game is complete after Edge-hitter plays his move m_3, then $w(H_i) - w(H_{i+1}) \geq 3+2+4 = 9$ and $k_i = 3 < \frac{4}{11}(w(H_i) - w(H_{i+1}))$, a contradiction. Hence, Staller plays a vertex as her move m_4, resulting in $w_4 \geq 3$. Thus, $\sum_{i=1}^{4} w_i \geq 3+2+4+3 = 12 > 4 \cdot \frac{11}{4}$, contradicting Equation (7). Therefore, after Staller's move m_2, every component induced by $V(C)$ in the residual hypergraph is an isolated edge of size 2.

Recall that after Staller's move m_2, we have at least three vertices of degree 1 in $V(C)$. Hence, there exist at least two isolated edges (of size 2) in $V(C)$. Edge-hitter plays as his move m_3 a vertex from an isolated edge, resulting in $w_3 = 3$.

Staller responds by playing as her move m_4 either a vertex from an isolated edge or a vertex of degree 2 from a 2-regular, non-2-uniform, linear component. In both cases, $w_4 = 3$. If Staller's move m_4 is a vertex of degree 2 from a 2-regular, non-2-uniform, linear component, then Edge-hitter can respond by playing as his move m_5 a vertex from such a component that either belongs to an isolated edge of size at least 3 or is a vertex of degree 2 with a degree 1-neighbor. Thus, Edge-hitter's move m_5 would result in $w_5 \geq 4$. If the game is complete after Edge-hitter plays his move m_5, then $w(H_i) - w(H_{i+1}) \geq 3+2+3+3+4 = 15$ and $k_i = 5 < \frac{4}{11}(w(H_i) - w(H_{i+1}))$, a contradiction. Hence, Staller plays a vertex as her move m_6, which results in $w_6 \geq 2$, and so $\sum_{i=1}^{6} w_i \geq 3+2+3+3+4+2 = 17 > 6 \cdot \frac{11}{4}$, contradicting Equation (7). Therefore, Staller's move m_4 is a vertex from an isolated edge (of size 2).

Suppose there remains an isolated edge (of size 2) in $V(C)$ in the resulting residual hypergraph after Staller's move m_4. In this case, Edge-hitter plays as his move m_5 a vertex from such an isolated edge, resulting in $\mathrm{w}_5 = 3$. If the game is complete after Edge-hitter plays his move m_5, then $k_i = 5 < \frac{4}{11} \cdot 14 \leq \frac{4}{11}(\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}))$, a contradiction. Hence, Staller plays a vertex as her move m_6, which results in $\mathrm{w}_6 \geq 3$, and so $k_i = 6 < \frac{4}{11} \cdot 17 \leq \frac{4}{11}(\mathrm{w}(H_i) - \mathrm{w}(H_{i+1}))$, a contradiction. Therefore, after Staller's move m_4, no isolated edge remains.

Consequently, the component C is a 2-regular, non-2-uniform, linear component in which Edge-hitter plays a vertex v of degree 2 with $|N(v)| \geq 3$ as his move m_1, resulting in $\mathrm{w}_1 = 3$. Staller plays as her move m_2 a vertex u of degree 1 that has only neighbors of degree 2, resulting in $\mathrm{w}_2 = 2$. After Staller's move m_2, $V(C)$ induces two components both of which are isolated edges of size 2. Edge-hitter plays as his m_3 a vertex from one of these isolated edges, and Staller plays as her move m_4 a vertex from the remaining isolated edge. After the moves m_1 and m_2 are played, two vertices and three edges were recolored red, and four vertices and two edges in C remain white. Hence, before Edge-hitter plays his move m_1, the component C contained six vertices and five edges. Further, $N[v]$ does not contain all vertices from $V(C)$, since u has a degree-2 neighbor after move m_1. This implies that C contains no edge of size 5 or greater. Let $V(C) = \{v_1, v_2, \ldots, v_6\}$.

Suppose that C contains an edge of size 4, say $e = \{v_1, v_2, v_3, v_4\}$. The linearity and 2-regularity conditions on C are achieved only if the remaining four edges are all 2-edges. Further, renaming vertices if necessary, we may assume that these edges are v_1v_5, v_2v_5, v_3v_6, and v_4v_6. However, then Edge-hitter could play as his move m_1 the vertex v_5. After Edge-hitter plays this move, there is no vertex u of degree 1 that has only neighbors of degree 2, contradicting our earlier supposition. Hence, every edge in C has size at most 3. Since C is a 2-regular hypergraph on six vertices with five edges, this implies that there C contains exactly two 3-edges and three 2-edges.

Suppose that C contains two 3-edges that share a common vertex. Renaming vertices, if necessary, we may assume that these two edges are $e_1 = \{v_1, v_2, v_3\}$ and $e_2 = \{v_1, v_4, v_5\}$. The linearity and 2-regularity conditions on C are achieved only if the remaining three edges are all 2-edges. Further, renaming vertices, if necessary, we may assume that these edges are v_2v_4, v_3v_6, v_5v_6. However, then Edge-hitter could play as his move m_1 the vertex v_6. Whatever move Staller plays as her move m_2 in response to this move by Edge-hitter, he can complete the game on his next move m_3. Thus, three moves are needed, and $\sum_{i=1}^{3} \mathrm{w}_i = 11 > 3 \cdot \frac{11}{4}$, contradicting Equation (7). Hence, C has exactly two 3-edges, and these 3-edges do not intersect.

Renaming vertices, if necessary, we may assume that the two 3-edges in C are $e_1 = \{v_1, v_2, v_3\}$ and $e_2 = \{v_4, v_5, v_6\}$. Since C is 2-regular, the three 2-edges of C form a matching. Renaming vertices, if necessary, we may assume that these edges are v_1v_4, v_2v_5, and v_3v_6. Thus, the component C of H belongs to the family $\mathcal{H}$. Since C is an arbitrary component in H, this implies that $H \in \mathcal{H}$. This, together with Proposition 2, completes the proof of Theorem 1.

5 Proof of Theorem 2

In this section, we first present a proof of Theorem 2. Recall its statement.

Theorem 2. *If H is a 2-uniform hypergraph, then $\tau_g(H) \le \frac{1}{3}(n_H + m_H + 1)$.*

Proof. Suppose that H is a 2-uniform hypergraph. Thus, H is a graph. Lemmas 2, 3, 4, 5, 6, and 7 together cover all possibilities. The only case when Edge-hitter cannot achieve his 3-target might be when at a point of the game, the current residual graph is a cycle of length $3\ell+1$, for some $\ell \ge 1$, and it is his move. By Lemma 7, this may happen only once, at the end of the game. Thus, if $r \ge 0$ moves are played before such a residual graph is reached that is a cycle of length $3\ell+1$, then, by Equation (2), we have

$$\mathrm{w}(H) = \sum_{i=1}^{r+2\ell+1} \mathrm{w}_i \ge 3(r+2\ell+1) - 1.$$

Hence, $\tau_g(H) \le r + 2\ell + 1 \le \frac{1}{3}(n_H + m_H + 1)$. $\square$

Next, we determine the game transversal number of a cycle, thereby showing that the upper bound of Theorem 2 is tight. Recall the statement of Proposition 1.

Proposition 1. *For $n \ge 3$, $\tau_g(C_n) = \lfloor \frac{2n+1}{3} \rfloor$.*

Proof. For $n \ge 3$, let $G = C_n$. By Theorem 2, $\tau_g(C_n) \le \lfloor \frac{2n+1}{3} \rfloor$. Hence it suffices for us to show that $\tau_g(C_n) \ge \lfloor \frac{2n+1}{3} \rfloor$. For this purpose, we define the function $f(G) = n_G + m_G - c_G$, where c_G denotes the number of path components in the residual graph G. Hence, we start with $f(G) = 2n$. Let $m_1, \ldots, m_k$ be a sequence of moves played starting with Edge-hitter's first move, m_1, and with moves alternating between Edge-hitter and Staller. Initially, we let $G_0 = G$, and we let G_i be the residual graph G after move m_i is played for $i \in [k]$. Further, we let $f_i = f(G_{i-1}) - f(G_i)$ for $i \in [k]$. Thus, f_i is the decrease in the function value $f(G)$ of the residual graph after move m_i is played.

Since $\Delta(G) \le 2$ for every residual graph G, after every move played m_G decreases by at most 2. We show that $n_G - c_G$ decreases by at most 2 after every move played. If c_G increases by 1, then n_G decreases by 1, and so $n_G - c_G$ remains unchanged. If c_G does not change, then n_G decreases by at most 2, and so $n_G - c_G$ decreases by at most 2. If c_G decreases by 1, then n_G decreases by at most 3, and so $n_G - c_G$ decreases by at most 2. In all cases, $n_G - c_G$ decreases by at most 2. Thus, after every move of Edge-hitter, $n_G + m_G - c_G$ decreases by at most 4; that is, $f_{2i-1} \le 4$ for all $i \ge 1$. If the game is not complete after Edge-hitter's move m_{2i-1}, then Staller can play as her move m_{2i} a leaf, which decreases $n_G + m_G - c_G$ by 2 (even if the leaf played by Staller is from a P_2-component); that is, $f_{2i} = 2$. Thus, $f_{2i-1} + f_{2i} \le 6$. If the last move m_k is played by Staller, then k is even and

$$2n = \sum_{i=1}^{k} f_i = \sum_{i=1}^{\frac{k}{2}} (f_{2i-1} + f_{2i}) \le \frac{1}{2}k \cdot 6 = 3k,$$

and so $k \ge \frac{2n}{3}$. If the last move m_k is played by Edge-hitter, then we may have that $f_k = 4$.

Thus in this case, k is odd and

$$2n = \sum_{i=1}^{k} f_i = \sum_{i=1}^{\frac{k-1}{2}} (f_{2i-1} + f_{2i}) + f_k \le \frac{1}{2}(k-1) \cdot 6 + 4 = 3k + 1,$$

and so $k \ge \frac{2n-1}{3}$. Therefore, $\tau_g(C_n) \ge k \ge \lceil \frac{2n-1}{3} \rceil = \lfloor \frac{2n+1}{3} \rfloor$. Consequently, $\tau_g(C_n) = \lfloor \frac{2n+1}{3} \rfloor$. $\Box$

By Proposition 1, if $n \equiv 1 \pmod 3$, then $\tau_g(C_n) = \frac{1}{3}(2n+1) = \frac{1}{3}(n_G + m_G + 1)$. We remark that equality in the bound of Theorem 2 is also achieved by connected graphs that are not cycles. For instance, consider the graph H' on the vertex set $V(H') = \{v_1, v_2, v_3, v_4, u_1, u_2\}$ and with the edge set

$$E(H') = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, u_1v_1, u_1v_3, u_2v_2, u_2v_4\}.$$

Consider the transversal game played in H'. For $i \in \{1, 2\}$, if Edge-hitter plays u_i as his first move, then Staller plays u_{2-i}. If Edge-hitter plays v_1 or v_3, then Staller plays u_1. If Edge-hitter plays v_2 or v_4, then Staller plays u_2. In all cases, the edges not hit by these two moves form a 4-cycle. Edge-hitter must play as his second move a vertex on the resulting 4-cycle. The remaining two non-covered edges induce a path P_3. Staller responds by playing as her second move a leaf from this path. Thus, Staller has a strategy to guarantee that at least five moves are played. Hence, $\tau_g(H') \ge 5 = \frac{1}{3}(n_{H'} + m_{H'} + 1)$. Theorem 2 implies that in fact the equality holds.

For the Staller-start game, we have the following consequence of Theorem 2.

Corollary 3 *If H is a 2-uniform hypergraph, then $\tau'_g(H) \le \frac{1}{3}(n_H + m_H + 2)$.*

Proof. Let H be a residual 2-uniform hypergraph. The first move of Staller decreases $n_H + m_H$ by at least 2, since at least one vertex and one edge are deleted by her move. Let H' denotes the resulting residual hypergraph, and so $n_{H'} + m_{H'} \le n_H + m_H - 2$. By Theorem 2,

$$\begin{array}{rcl} \tau'_g(H) & = & 1 + \tau_g(H') \\ & \le & 1 + \frac{1}{3}(n_{H'} + m_{H'} + 1) \\ & \le & 1 + \frac{1}{3}(n_H + m_H - 1) \\ & = & \frac{1}{3}(n_H + m_H + 2). \end{array} \qquad \Box$$

We close this section with the following two results.

Proposition 3 *For $n \ge 2$, $\tau'_g(P_n) = \lfloor \frac{2n}{3} \rfloor$ and $\tau_g(P_n) = \lfloor \frac{2n-1}{3} \rfloor$.*

Proof. We remark that the Staller-start game played on a path P_n can be considered as the second move on the Edge-hitter start game played on a cycle C_{n+1}. Hence, $\tau'_g(P_n) = \tau_g(C_{n+1}) - 1 = \lfloor \frac{2n+3}{3} \rfloor - 1 = \lfloor \frac{2n}{3} \rfloor$. We show next that $\tau_g(P_n) = \lfloor \frac{2n-1}{3} \rfloor$. If $n \in \{2, 3\}$, then $\tau_g(P_n) = 1 = \lfloor \frac{2n-1}{3} \rfloor$. Hence, we may assume that $n \geq 4$. Edge-hitter plays as his first move a support vertex on the path, thereby ensuring that the game is completed in at most $\tau'_g(P_{n-2})$ further moves. Therefore, $\tau_g(P_n) \leq \tau'_g(P_{n-2}) + 1 = \lfloor \frac{2n-4}{3} \rfloor + 1 = \lfloor \frac{2n-1}{3} \rfloor$.

Next, we prove that Staller can ensure that the length of the Edge-hitter start game on P_n is at least $\lfloor \frac{2n-1}{3} \rfloor$. We adopt the notation used in the proof of Proposition 1. In particular, $f(G) = n_G + m_G - c_G$, where c_G denotes the number of path components in the residual graph G. Hence, we start with $f(G) = 2n - 2$. If the last move m_k is played by Staller, then k is even and $2n - 2 = \sum_{i=1}^{k} f_i \leq 3k$, and so $k \geq \frac{2n-2}{3}$. If the last move m_k is played by Edge-hitter, then we may have that $f_k = 4$. Thus in this case, k is odd and $2n - 2 = \sum_{i=1}^{k} f_i \leq 3k + 1$, and so $k \geq \frac{2n-3}{3}$. Therefore, $\tau_g(P_n) \geq k \geq \lceil \frac{2n-3}{3} \rceil = \lfloor \frac{2n-1}{3} \rfloor$. Consequently, $\tau_g(P_n) = \lfloor \frac{2n-1}{3} \rfloor$. $\square$

Proposition 4 *For $n \geq 3$, $\tau'_g(C_n) = \lfloor \frac{2n}{3} \rfloor$.*

Proof. Suppose that Staller makes the first move on C_n, where $n \geq 3$. Thereafter, the game can be considered as the Edge-hitter start game played on a path P_{n-1}. Hence, $\tau'_g(C_n) = \tau_g(P_{n-1}) + 1 = \lfloor \frac{2n-3}{3} \rfloor + 1 = \lfloor \frac{2n}{3} \rfloor$. $\square$

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References

[1] C. Berge, Hypergraphs – Combinatorics of Finite Sets. North-Holland, 1989.

[2] H. L. Bodlaender, On the complexity of some coloring games. *Internat. J. Found. Comp. Sci.* **2** (1991), 133–147.

[3] A. Bonato, W. B. Kinnersley, and P. Prałat, Game toppling number for complete and random graphs. *Discrete Math. Theor. Comp. Sci.* **16** (2014), 229–252.

[4] B. Brešar, S. Klavžar, and D. Rall, Domination game and an imagination strategy. *SIAM J. Discrete Math.* **24** (2010), 979–991.

[5] M. Borowiecki, E. Sidorowicz, and Zs. Tuza, Game list colouring of graphs. *Electron. J. Combin.* **14** (2007), #R26, 11 pp.

[6] Cs. Bujtás, M. A. Henning and Zs. Tuza, Transversals and domination in uniform hypergraphs. *European J. Combin.* **33** (2012), 62–71.

[7] Cs. Bujtás, M. A. Henning, Zs. Tuza and A. Yeo, Total transversals and total domination in uniform hypergraphs. *Electronic J. Combin.* **21**(2) (2014), #P2.24.

[8] Cs. Bujtás and Zs. Tuza, The disjoint domination game. *Discrete Math.* **339**(7) (2016), 1985–1992.

[9] J. Butterfield, T. Grauman, W. B. Kinnersley, K. G. Milans, C. Stocker, and D. B. West, On-line Ramsey theory for bounded-degree graphs. *Electron. J. Combin.* **18** (2011), #P136.

[10] V. Chvátal and C. McDiarmid, Small transversals in hypergraphs. *Combinatorica* **12** (1992), 19–26.

[11] D. W. Cranston, W. B. Kinnersley, S. O, and D. B. West, Game matching number of graphs. *Discrete Applied Math.* **161** (2013), 1828–1836.

[12] P. Dorbec and M. A. Henning, Game total domination for cycles and paths. *Discrete Applied Math.* **208** (2016), 7–18.

[13] M. Dorfling and M. A. Henning, Linear hypergraphs with large transversal number and maximum degree two. *European J. Combin.* **36** (2014), 231–236.

[14] P. Erdős and Zs. Tuza, Vertex coverings of the edge set in a connected graph. In Graph Theory, Combinatorics, and Applications (Y. Alavi and A. Schwenk, eds.), John Wiley & Sons, 1995, 1179–1187.

[15] M. Gardner, Mathematical games. *Scientific American* **244** (1981), 18–26.

[16] J. A. Grytczuk, M. Hałuszczak, and H. A. Kierstead, On-line Ramsey theory. *Electron. J. Combin.* **11** (2004), #R57, 10 pp.

[17] J. A. Grytczuk, H. A. Kierstead, and P. Prałat, On-line Ramsey numbers for paths and stars. *Discrete Math. Theor. Comp. Sci.* **10** (2008), 63–74.

[18] M. A. Henning and W. B. Kinnersley, Domination Game: A proof of the 3/5-Conjecture for graphs with minimum degree at least two. *SIAM J. Discrete Math.* **30** (2016), 20–35.

[19] M. A. Henning, S. Klavžar, and D. F. Rall, Total version of the domination game. *Graphs Combin.*, **31**(5) (2015), 1453–1462.

[20] M. A. Henning, S. Klavžar, and D. F. Rall, The 4/5 upper bound on the game total domination number. *Combinatorica*, to appear. DOI 10.1007/s00493-015-3316-3.

[21] M. A. Henning and C. Löwenstein, Hypergraphs with large transversal number and with edge sizes at least four. *Central European J. Math.* **10**(3) (2012), 1133–1140.

[22] M. A. Henning and C. Löwenstein, A characterization of the hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem. *Discrete Math.* **323** (2014), 69–75.

[23] M. A. Henning and A. Yeo, Hypergraphs with large transversal number. *Discrete Math.* **313** (2013), 959–966.

[24] M. A. Henning and A. Yeo, Transversals and matchings in 3-uniform hypergraphs. *European J. Combin.* **34** (2013), 217–228.

[25] Z. Lonc and K. Warno, Minimum size transversals in uniform hypergraphs. *Discrete Math.* **313**(23) (2013), 2798–2815.

[26] U. Schauz, Mr. Paint and Mrs. Correct. *Electron. J. Combin.* **16** (2009), #R77, 18 pp.

[27] Zs. Tuza, Covering all cliques of a graph. *Discrete Math.* **86** (1990), 117–126.

[28] Zs. Tuza and X. Zhu, Colouring games. Chapter 13 in: Topics in Chromatic Graph Theory (L. W. Beineke and R. J. Wilson, eds.), Encyclopedia of Mathematics and Its Applications 156, Cambridge University Press, 304–326, 2014.