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MODIFIED DINI FUNCTIONS: MONOTONICITY PATTERNS AND FUNCTIONAL INEQUALITIES

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Abstract. We deduce some new functional inequalities, like Turán type inequalities, Redheffer type inequalities, and a Mittag-Leffler expansion for a special combination of modified Bessel functions of the first kind, called modified Dini functions. Moreover, we show the complete monotonicity of a quotient of modified Dini functions by involving a new continuous infinitely divisible probability distribution. The key tool in our proofs is a recently developed infinite product representation for a special combination of Bessel functions of the first kind, which was very useful in determining the radius of convexity of some normalized Bessel functions of the first kind.

1. Introduction and preliminaries

Let us consider the Dini function $d_\nu\colon \Omega \subseteq \mathbb{C} \to \mathbb{C}$ defined by

$$d_\nu(z) = (1-\nu)J_\nu(z) + zJ'_\nu(z) = J_\nu(z) - zJ_{\nu+1}(z),$$

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which in view of $I_\nu(z) = \mathrm{i}^{-\nu} J_\nu(\mathrm{i}z)$ gives the modified Dini function

$$\xi_\nu \colon \Omega \subseteq \mathbb{C} \to \mathbb{C},$$

defined by

$$\xi_\nu(z) = \mathrm{i}^{-\nu} d_\nu(\mathrm{i}z) = (1-\nu) I_\nu(z) + z I'_\nu(z) = I_\nu(z) + z I_{\nu+1}(z),$$

where Ω is the whole complex plane minus an infinite slit from the origin if ν is not an integer. If ν is an integer then Ω can be taken as the whole complex plane.

In view of the Weierstrassian factorization of $d_\nu(z)$ (see [7]),

$$d_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n \geq 1} \left(1 - \frac{z^2}{\alpha_{\nu,n}^2}\right), \tag{1.1}$$

where $\nu > -1$, and the formula $\xi_\nu(z) = \mathrm{i}^{-\nu} d_\nu(\mathrm{i}z)$, we have the following Weierstrassian factorization of $\xi_\nu(z)$ for all $\nu > -1$ and $z \in \Omega$:

$$\xi_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n \geq 1} \left(1 + \frac{z^2}{\alpha_{\nu,n}^2}\right), \tag{1.2}$$

where the infinite product is uniformly convergent on each compact subset of the complex plane. Here $\alpha_{\nu,n}$ is the nth positive zero of the Dini function d_ν. The principal branches of $d_\nu(z)$ and $\xi_\nu(z)$ correspond to the principal value of $(z/2)^\nu$ and they are analytic in the z-plane cut along the negative real axis from 0 to infinity, namely, the half line $(-\infty, 0]$. Now for $\nu > -1$, define the function $\lambda_\nu \colon \mathbb{R} \to [1, \infty)$ as

$$\lambda_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} \xi_\nu(x) = \prod_{n \geq 1} \left(1 + \frac{x^2}{\alpha_{\nu,n}^2}\right). \tag{1.3}$$

Furthermore, for $\nu > -1$, let us define the function $\mathcal{D}_\nu \colon \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{D}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} d_\nu(x) = \prod_{n \geq 1} \left(1 - \frac{x^2}{\alpha_{\nu,n}^2}\right). \tag{1.4}$$

By using some ideas from [3,5,10,13], our aim is to deduce some new functional inequalities, like Turán type inequalities, Redheffer type inequalities for the above special combination of modified Bessel functions of the first kind, called modified Dini functions. Moreover, we show the complete monotonicity of a quotient of modified Dini functions by involving a new continuous infinitely divisible probability distribution. The key tool in our proofs is the above infinite product representation, which was very useful in determining the radius of convexity of some normalized Bessel functions [9].

Before we present our main results, we recall some standard definitions and basic facts. We say that a function $f\colon [a,b]\subseteqq\mathbb{R}\to\mathbb{R}$ is convex if for all $x,y\in[a,b]$ and $\alpha\in[0,1]$ we have

$$f(\alpha x+(1-\alpha)y)\leq \alpha f(x)+(1-\alpha)f(y).$$

If the above inequality is reversed then f is called a concave function. Moreover it is known that if f is differentiable, then f is convex (concave) if and only if f' is increasing (decreasing) and if f is twice differentiable, then f is convex (concave) if and only if f'' is positive (negative). A function $g\colon [a,b]\subseteqq\mathbb{R}\to(0,\infty)$ is said to be log-convex if its natural logarithm $\log g$ is convex, that is, for all $x,y\in[a,b]$ and $\alpha\in[0,1]$ we have

$$g(\alpha x+(1-\alpha)y)\leq (g(x))^{\alpha}(g(y))^{1-\alpha}.$$

If the above inequality is reversed then g is called a log-concave function. It is also known that if g is differentiable, then g is log-convex (log-concave) if and only if g'/g is increasing (decreasing). A function $h\colon [a,b]\subseteqq[0,\infty)\to[0,\infty)$ is said to be geometrically convex if it is convex with respect to the geometric mean, that is, if for all $x,y\in[a,b]$ and $\alpha\in[0,1]$ we have

$$h(x^{\alpha}y^{1-\alpha})\leq (h(x))^{\alpha}(h(y))^{1-\alpha}.$$

If the above inequality is reversed then h is called a geometrically concave function. If h is differentiable, then h is geometrically convex (concave) if and only if $x\mapsto xh'(x)/h(x)$ is increasing (decreasing). A probability distribution μ on the half line $(0,\infty)$ is infinitely divisible if for every $n\in\{1,2,\dots\}$ there exists a probability distribution μ_n on $(0,\infty)$ such that for each $n\in\{1,2,\dots\}$

$$\int_0^{\infty}e^{-xt}\,d\mu=\left(\int_0^{\infty}e^{-xt}\,d\mu_n\right)^{n}.$$

A function $f\colon (0,\infty)\to\mathbb{R}$ possessing derivatives of all order is called a completely monotonic function if $(-1)^nf^{(n)}(x)\geq 0$ for all $x>0$ and $n\in\{0,1,\dots\}$. The classes of completely monotonic and infinitely divisible distributions are related by the following result, see Feller [12, p. 425].

Lemma 1. *The function w is the Laplace transform of an infinitely divisible probability distribution if and only if $w(x)=e^{-h(x)}$, where h has a completely monotone derivative and $h(0)=0$.*

Finally, let us recall the following result (see [11,22]) which will be used in the sequel.

Lemma 2. *Consider the power series*

$$f(x)=\sum_{n\geq 0}a_nx^n \quad \text{and} \quad g(x)=\sum_{n\geq 0}b_nx^n,$$

where for all $n\geq 0$ *we have* $a_n\in\mathbb{R}$ *and* $b_n>0$, *and suppose that both series converge on* $(-r,r)$, $r>0$. *If the sequence* $\{a_n/b_n\}_{n\geq 0}$ *is increasing (decreasing), then the function* $x\mapsto f(x)/g(x)$ *is increasing (decreasing) too on* $(0,r)$. *We note that this result remains true if we have the power series*

$$f(x)=\sum_{n\geq 0}a_nx^{2n} \quad \text{and} \quad g(x)=\sum_{n\geq 0}b_nx^{2n}$$

or

$$f(x)=\sum_{n\geq 0}a_nx^{2n+1} \quad \text{and} \quad g(x)=\sum_{n\geq 0}b_nx^{2n+1}.$$

2. Monotonicity properties and inequalities for modified Dini functions

2.1. Log-convexity properties and Turán type inequalities. Our first set of results consists of some monotonicity and convexity properties of modified Dini functions as well as some Turán type inequalities.

Theorem 1. *The following assertions are valid:*

(a) *The function* $x\mapsto\lambda_\nu(x)$ *is increasing on* $(0,\infty)$ *for all* $\nu>-1$;

(b) *The function* $x\mapsto\lambda_\nu(x)$ *is strictly log-convex on* $(-\alpha_{\nu,1},\alpha_{\nu,1})$ *and strictly geometrically convex on* $(0,\infty)$ *for all* $\nu>-1$;

(c) *The functions* $\nu\mapsto\lambda_\nu(x)$ *and* $\nu\mapsto x\lambda_\nu'(x)/\lambda_\nu(x)$ *are decreasing on* $(-1,\infty)$ *for all* $x\in\mathbb{R}$;

(d) *The function* $\nu\mapsto\lambda_\nu(x)$ *is log-convex on* $(-1,\infty)$ *for all* $x\in\mathbb{R}$, *and consequently the reversed Turán type inequality*

$$\lambda_\nu^2(x)-\lambda_{\nu-1}(x)\lambda_{\nu+1}(x)\leq 0 \tag{2.1}$$

holds for all $\nu>0$ *and* $x\in\mathbb{R}$. *In addition, the Turán type inequality*

$$-\frac{1}{\nu}\lambda_\nu^2(x)\leq\lambda_\nu^2(x)-\lambda_{\nu-1}(x)\lambda_{\nu+1}(x) \tag{2.2}$$

holds for all $\nu>-1$ *and* $x>0$.

(e) *The function* $\nu\mapsto\lambda_{\nu+1}(x)/\lambda_\nu(x)$ *is increasing on* $(-1,\infty)$ *for all* $x\in\mathbb{R}$;

(f) *The function* $x \mapsto 1/\lambda_\nu(\sqrt{x})$ *is completely monotonic on* $(0,\infty)$ *for all* $\nu > -1$. *Moreover, the inequality*

$$\lambda_\nu(\sqrt{x}+\sqrt{y}) \le \lambda_\nu(\sqrt{x})\lambda_\nu(\sqrt{y}) \tag{2.3}$$

is valid for all $x, y \ge 0$ *and* $\nu > -1$.

(g) *The function* $x \mapsto \lambda_\nu(\sqrt{x})$ *is log-concave on* $(0,\infty)$ *for all* $\nu > -1$.

PROOF. (a) By taking the logarithmic derivative of (1.3) we have

$$(\log \lambda_\nu(x))' = \frac{\lambda_\nu'(x)}{\lambda_\nu(x)} = \sum_{n\ge 1} \frac{2x}{\alpha_{\nu,n}^2 + x^2}\,. \tag{2.4}$$

This implies that for $\nu > -1$ the function $x \mapsto \log \lambda_\nu(x)$ is increasing on $(0,\infty)$ and hence $x \mapsto \lambda_\nu(x)$ is increasing too on $(0,\infty)$ for $\nu > -1$.

(b) Differentiating (2.4) with respect to x we have

$$\left(\frac{\lambda_\nu'(x)}{\lambda_\nu(x)}\right)' = \sum_{n\ge 1} \frac{2(\alpha_{\nu,n}^2 - x^2)}{(\alpha_{\nu,n}^2 + x^2)^2}\,.$$

Thus, for $\nu > -1$ the function $x \mapsto \lambda_\nu'(x)/\lambda_\nu(x)$ is strictly increasing on $(-\alpha_{\nu,1}, \alpha_{\nu,1})$ and hence the function $x \mapsto \lambda_\nu(x)$ is strictly log-convex on $(-\alpha_{\nu,1}, \alpha_{\nu,1})$. Now, by using again (2.4) we obtain that

$$\left(\frac{x\lambda_\nu'(x)}{\lambda_\nu(x)}\right)' = \sum_{n\ge 1} \frac{4x\alpha_{\nu,n}^2}{(\alpha_{\nu,n}^2 + x^2)^2}\,,$$

which implies that the function $x \mapsto x\lambda_\nu'(x)/\lambda_\nu(x)$ is strictly increasing on $(0,\infty)$ for $\nu > -1$ and hence $x \mapsto \lambda_\nu(x)$ is strictly geometrically convex on $(0,\infty)$ for all $\nu > -1$.

(c) In view of the infinite product representation (1.3) we have,

$$\frac{\partial \log(\lambda_\nu(x))}{\partial \nu} = -\sum_{n\ge 1} \frac{2x^2 \frac{\partial \alpha_{\nu,n}}{\partial \nu}}{\alpha_{\nu,n}(\alpha_{\nu,n}^2 + x^2)}$$

and

$$\frac{\partial}{\partial \nu}\left(\frac{x\lambda_\nu'(x)}{\lambda_\nu(x)}\right) = -\sum_{n\ge 1} \frac{4x^2 \alpha_{\nu,n} \frac{\partial \alpha_{\nu,n}}{\partial \nu}}{(\alpha_{\nu,n}^2 + x^2)^2}\,.$$

Now in view of [19, p. 196], the expression $\partial\alpha_{\nu,n}/\partial\nu$ is positive for $\nu > -1$ and hence the functions $\nu \mapsto \lambda_\nu(x)$ and $\nu \mapsto x\lambda_\nu'(x)/\lambda_\nu(x)$ are decreasing on $(-1,\infty)$ for all $x \in \mathbb{R}$.

(d) By using (1.3) we have

$$\lambda_\nu(x) = \mathcal{I}_\nu(x) + \frac{x^2}{2(\nu+1)}\mathcal{I}_{\nu+1}(x). \tag{2.5}$$

Here for $\nu > -1$ the function $\mathcal{I}_\nu \colon \mathbb{R} \to [1,\infty)$ is defined by

$$\mathcal{I}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} I_\nu(x) = \sum_{n\geq 0} \frac{(1/4)^n}{(\nu+1)_n n!} x^{2n}, \tag{2.6}$$

where $(\nu+1)_n = (\nu+1)(\nu+2)\cdots(\nu+n) = \Gamma(\nu+n+1)/\Gamma(\nu+1)$. Using the fact that the sum of log-convex functions is log-convex and that for $x \in \mathbb{R}$ the function $\nu \mapsto \mathcal{I}_\nu(x)$ is log-convex on $(-1,\infty)$ (see [3]), to prove that $\nu \mapsto \lambda_\nu(x)$ is log-convex on $(-1,\infty)$ for $x \in \mathbb{R}$ it is enough to show that $\nu \mapsto \frac{x^2}{2(\nu+1)}\mathcal{I}_{\nu+1}(x)$ is log-convex on $(-1,\infty)$ for $x \in \mathbb{R}$. Now, the function $\nu \mapsto \frac{x^2}{2(\nu+1)}\mathcal{I}_{\nu+1}(x)$ is log-convex if and only if

$$\nu \mapsto \log(x^2/2) - \log(\nu+1) + \log(\mathcal{I}_{\nu+1}(x))$$

is convex on $(-1,\infty)$. As $\nu \mapsto -\log(\nu+1)$ and $\nu \mapsto \log(\mathcal{I}_{\nu+1}(x))$ are convex on $(-1,\infty)$ for all $x \in \mathbb{R}$, we conclude that

$$\nu \mapsto \log(x^2/2) - \log(\nu+1) + \log(\mathcal{I}_{\nu+1}(x))$$

is convex on $(-1,\infty)$ for all $x \in \mathbb{R}$ and hence $\nu \mapsto \frac{x^2}{2(\nu+1)}\mathcal{I}_{\nu+1}(x)$ is log-convex for $\nu > -1$ and $x \in \mathbb{R}$.

Alternatively, by using the idea from [3] concerning the log-convexity of $\nu \mapsto \mathcal{I}_\nu(x)$, it can be shown that $\nu \mapsto \frac{x^2}{2(\nu+1)}\mathcal{I}_{\nu+1}(x)$ is log-convex on $(-1,\infty)$ for $x \in \mathbb{R}$. Namely, consider the expression

$$\frac{x^2}{2(\nu+1)}\mathcal{I}_{\nu+1}(x) = \sum_{n\geq 0} \frac{(1/4)^n}{2(\nu+1)_{n+1} n!} x^{2n+2} = \sum_{n\geq 0} b_n(\nu) x^{2n+2},$$

where

$$b_n(\nu) = \frac{(1/4)^n}{2(\nu+1)_{n+1} n!}\,.$$

To prove the log-convexity of $\nu \mapsto \frac{x^2}{2(\nu+1)}\mathcal{I}_{\nu+1}(x)$ it is enough to show the log-convexity of each individual terms in the above sum, that is,

$$\frac{\partial^2 \log b_n(\nu)}{\partial \nu^2} = \psi'(\nu+1) - \psi'(\nu+n+2) \geq 0,$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the digamma function. But ψ is concave and hence the function $\nu\mapsto b_n(\nu)$ is log-convex on $(-1,\infty)$. Therefore

$$\nu\mapsto\frac{x^2}{2(\nu+1)}\mathcal{I}_{\nu+1}(x)$$

is log-convex on $(-1,\infty)$ for $x\in\mathbb{R}$ and consequently, the function $\nu\mapsto\lambda_\nu(x)$ is log-convex on $(-1,\infty)$ for all $x\in\mathbb{R}$.

Now, to prove the inequality (2.1), we first observe that the function $\nu\mapsto\lambda_\nu(x)$ is log-convex on $(-1,\infty)$ for all $x\in\mathbb{R}$ and hence for all $\nu_1,\nu_2>-1$, $x\in\mathbb{R}$ and $\alpha\in[0,1]$ we have

$$\lambda_{\alpha\nu_1+(1-\alpha)\nu_2}(x)\le\left(\lambda_{\nu_1}(x)\right)^{\alpha}\left(\lambda_{\nu_2}(x)\right)^{1-\alpha}.$$

Taking $\nu_1=\nu-1$, $\nu_2=\nu+1$ and $\alpha=\frac{1}{2}$ we get the Turán type inequality (2.1) for $\nu>0$ and $x\in\mathbb{R}$.

To prove the inequality (2.2) let us consider the Turánian

$$\Delta_\nu(x)=\xi_\nu^2(x)-\xi_{\nu-1}(x)\xi_{\nu+1}(x),$$

which can be rewritten as

$$\Delta_\nu(x)=\left[I_\nu^2(x)-I_{\nu-1}(x)I_{\nu+1}(x)\right]$$
$$+x^2\left[I_{\nu+1}^2(x)-I_\nu(x)I_{\nu+2}(x)\right]+x\left[I_\nu(x)I_{\nu+1}(x)-I_{\nu-1}(x)I_{\nu+2}(x)\right].$$

Observe that the inequality (2.2) is equivalent to $\Delta_\nu(x)\ge0$. Using the Turán inequality for modified Bessel functions (see for example [4]),

$$I_\nu^2(x)-I_{\nu-1}(x)I_{\nu+1}(x)\ge0, \tag{2.7}$$

which holds for $\nu>-1$ and $x\in\mathbb{R}$, and by changing the parameter ν to $\nu+1$ in it, we get

$$I_{\nu+1}^2(x)-I_\nu(x)I_{\nu+2}(x)\ge0.$$

We also note that (see [6]) for $\nu>-1$ and $x>0$ we have

$$I_\nu(x)I_{\nu+1}(x)-I_{\nu-1}(x)I_{\nu+2}(x)\ge0, \tag{2.8}$$

and therefore $\Delta_\nu(x)\ge0$ for $\nu>-1$ and $x>0$. This completes the proof of the Turán type inequality (2.2).

(e) From part (d) the function $\nu\mapsto\log\lambda_\nu(x)$ is convex and hence

$$\nu\mapsto\log\left[\lambda_{\nu+\varepsilon}(x)\right]-\log\left[\lambda_\nu(x)\right]$$

is increasing for all $\varepsilon>0$. In particular, by taking $\varepsilon=1$ we get that the function $\nu\mapsto\lambda_{\nu+1}(x)/\lambda_\nu(x)$ is increasing on $(-1,\infty)$ for all $x\in\mathbb{R}$.

(f) The infinite product representation (1.3) implies that

$$\left(-\log\frac{1}{\lambda_\nu(\sqrt{x})}\right)'=\sum_{n\geq1}\frac{1}{\alpha_{\nu,n}^2+x},$$

which is completely monotonic on $(0,\infty)$ for each fixed $\nu>-1$ as an infinite series of completely monotonic functions. Since $h\colon(0,\infty)\to(0,\infty)$ is completely monotonic whenever $(-\log h)'$ is completely monotonic (see [1]), we conclude that the function $x\mapsto 1/\lambda_\nu(\sqrt{x})$ is completely monotonic on $(0,\infty)$ for all $\nu>-1$. The result of Kimberling [18] asserts that if f is a completely monotonic function from $[0,\infty)$ into (0,1] then

$$f(x+y)\geq f(x)f(y)$$

for all $x,y\geq0$. Applying this result to the function $x\mapsto 1/\lambda_\nu(\sqrt{x})$, the inequality (2.3) follows.

(g) From part (f) of this theorem it follows that $x\mapsto 1/\lambda_\nu(\sqrt{x})$ is log-convex on $(0,\infty)$ for all $\nu>-1$, since every completely monotonic function is log convex, see [24, p. 167]. Consequently, $x\mapsto\lambda_\nu(\sqrt{x})$ is log-concave on $(0,\infty)$ for all $\nu>-1$. Note that another proof of the log-concavity of $x\mapsto\lambda_\nu(\sqrt{x})$ can be given by using the infinite product representation (1.3). Namely, from (1.3) we have

$$\log(\lambda_\nu(\sqrt{x}))=\sum_{n\geq1}\log\left(1+\frac{x}{\alpha_{\nu,n}^2}\right).$$

Since $x\mapsto\log(1+x/\alpha_{\nu,n}^2)$ is concave on $(0,\infty)$ for all $\nu>-1$ and for all $n\geq1$ it follows that $x\mapsto\log(\lambda_\nu(\sqrt{x}))$ is concave as an infinite sum of concave functions. Hence $x\mapsto\lambda_\nu(\sqrt{x})$ is log-concave on $(0,\infty)$ for all $\nu>-1$. □

2.2. Monotonicity of some quotients. Now, we are going to prove some other monotonicity properties of the modified Dini functions by using Lemma 2. Moreover, we present some simple bounds for these functions in terms of hyperbolic functions. The idea of this result comes from the survey paper [5], where a similar result has been proved for modified Bessel functions of the first kind.

THEOREM 2. *Let $\mu,\nu>-1$ and k be a non-negative integer. Then the following assertions are valid*:

(a) *If $\mu>\nu$ $(\mu<\nu)$, then the function $x\mapsto\lambda_\nu(x)/\lambda_\mu(x)$ is strictly increasing* (*decreasing*) *on $(0,\infty)$*;

(b) *If $-1<\nu<\frac{1}{2}$ $(\nu>\frac{1}{2})$, then $x\mapsto\lambda_\nu^{(2k)}(x)/\cosh x$ is strictly increasing* (*decreasing*) *on $(0,\infty)$*;

(c) *If* $-1 < \nu < \frac{1}{2}$ $(\nu > \frac{1}{2})$, *then* $x \mapsto \lambda_\nu^{(2k+1)}(x)/\sinh x$ *is strictly increasing* (*decreasing*) *on* $(0,\infty)$;

(d) *If* $-1 < \nu < \frac{1}{2}$ $(\nu > \frac{1}{2})$, *then* $x \mapsto \lambda_\nu(x)/\cosh x$ *is strictly increasing* (*decreasing*) *on* $(0,\infty)$;

(e) *If* $-1 < \nu < -\frac{1}{2}$ $(\nu > -\frac{1}{2})$, *then* $x \mapsto \lambda_\nu(x)/(\cosh x + x\sinh x)$ *is strictly increasing* (*decreasing*) *on* $(0,\infty)$;

(f) *The following inequalities are valid for all* $x > 0$

$$\lambda_\nu(x) > \cosh x, \quad \textit{for} \quad \nu \in (-1, \tfrac{1}{2}), \tag{2.9}$$

$$\lambda_\nu(x) < \cosh x, \quad \textit{for} \quad \nu > \tfrac{1}{2}, \tag{2.10}$$

$$\lambda_\nu(x) > \cosh x + x\sinh x, \quad \textit{for} \quad \nu \in (-1, -\tfrac{1}{2}), \tag{2.11}$$

and

$$\lambda_\nu(x) < \cosh x + x\sinh x, \quad \textit{for} \quad \nu > -\tfrac{1}{2}. \tag{2.12}$$

Moreover, all the above inequalities become equalities when $x = 0$.

Proof. (a) Using (2.5) and (2.6) we have the power series

$$\lambda_\nu(x) = \sum_{n\geq 0} \frac{(2n+1)x^{2n}}{4^n n!(\nu+1)_n}. \tag{2.13}$$

In view of Lemma 2 and the power series representations of $\lambda_\nu(x)$ and $\lambda_\mu(x)$, it is enough to study the monotonicity of the sequence $\{\alpha_n\}_{n\geq 0} = (\mu+1)_n/(\nu+1)_n$. Now, it can be checked that for all $n \in \{0,1,\dots\}$ we have $\alpha_{n+1}/\alpha_n = (\mu+n+1)/(\nu+n+1) > 1$ if and only if $\mu > \nu$, and hence the conclusion follows.

(b) By using (2.13) we obtain that

$$\lambda_\nu^{(2k)}(x) = \sum_{n\geq 0} \frac{(2n+2k+1)!}{(2n)!4^{n+k}(n+k)!(\nu+1)_{n+k}} x^{2n}$$

and

$$\lambda_\nu^{(2k+1)}(x) = \sum_{n\geq 0} \frac{(2n+2k+3)!}{(2n+1)!4^{n+k+1}(n+k+1)!(\nu+1)_{n+k+1}} x^{2n+1}.$$

We also note that

$$\mathcal{I}_{-1/2}(x) = \cosh x, \quad \mathcal{I}_{1/2}(x) = \frac{\sinh x}{x} \quad \text{and} \quad \mathcal{I}_{3/2}(x) = -3\Big(\frac{\sinh x}{x^3} - \frac{\cosh x}{x^2}\Big).$$

Hence using (2.5) we have

$$\lambda_{1/2}(x) = \cosh x = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!}, \tag{2.14}$$

$$\lambda_{-1/2}(x) = \cosh x + x\sinh x = \sum_{n\geq 0} \frac{(2n+1)x^{2n}}{4^n n!(1/2)_n}, \tag{2.15}$$

and

$$\frac{\lambda_{-1/2}(x) - \lambda_{1/2}(x)}{x} = \sinh x = \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!}. \tag{2.16}$$

By using Lemma 2 and (2.14), it is enough to verify the monotonicity of the sequence $\{\alpha_n\}_{n\geq 0}$ where

$$\alpha_n = \frac{(2n+2k+1)!}{4^{n+k}(n+k)!(\nu+1)_{n+k}}.$$

But $\alpha_{n+1}/\alpha_n = (2n+2k+3)/(2\nu+2n+2k+2) > 1$ if and only if $\nu < \frac{1}{2}$ and the conclusion follows.

(c) Again using Lemma 2 and (2.16) the result follows as the sequence $\{\beta_n\}_{n\geq 0}$ where

$$\beta_n = \frac{(2n+2k+3)!}{4^{n+k+1}(n+k+1)!(\nu+1)_{n+k+1}},$$

strictly increases for $-1 < \nu < \frac{1}{2}$ and decreases for $\nu > \frac{1}{2}$.

(d) This follows from part (a) by taking $\mu = \frac{1}{2}$ and observing that $\lambda_{1/2}(x) = \cosh x$. Alternatively, this can be proved from part (b) by choosing $k = 0$.

(e) This part again follows from part (a) by taking $\mu = -\frac{1}{2}$ and noticing $\lambda_{-1/2}(x) = \cosh x + x\sinh x$.

(f) The inequalities (2.9) and (2.10) follow from part (d) while (2.11) and (2.12) follow from part (e).

We note that for $-1 < \nu < -1/2$ the inequality (2.11) improves the inequality (2.9) and for $\nu > 1/2$ the inequality (2.10) improves the inequality (2.12). □

2.3. Some remarks and further results. We first note that the Turán type inequality (2.7) and the fact that $I_\nu(x) > 0$ for all $\nu > -1$ and $x > 0$ actually imply the Turán type inequality (2.8) for $\nu > 0$ and $x > 0$.

Moreover, by using the power series representation of the product of modified Bessel functions [21, p. 252]

$$I_\nu(x)I_\mu(x)=\sum_{k\geq 0}\frac{(\nu+\mu+k+1)_k(\frac{x}{2})^{2k+\nu+\mu}}{k!\Gamma(\nu+k+1)\Gamma(\mu+k+1)}$$

we can also prove the inequality (2.8) for $\nu>-1$ and $x>0$.

We also note that by using the power series representation (2.13) and [17, Theorem 3], we can get another proof for the log-convexity of the function $\nu\mapsto\lambda_\nu(x)$ on $(-1,\infty)$ for all $x\in\mathbb{R}$. Moreover, if we consider the expression $f(\mu,x)=\lambda_\nu(x)/\Gamma(\nu+1)$, where $\mu=\nu+1>0$ and $x\in\mathbb{R}$, then by using [16, Theorem 3.1] we can conclude that the function $\nu\mapsto\lambda_\nu(x)/\Gamma(\nu+1)$ is log-concave on $(-1,\infty)$ for each fixed $x\in\mathbb{R}$ which in turn implies the Turán type inequality (2.2) for $\nu>0$. Now, using [16, equations (17) and (19)], we have the bounds

$$\frac{1}{(\nu+1)\Gamma^2(\nu+1)}<\frac{\lambda_\nu^2(x)}{\Gamma^2(\nu+1)}-\frac{\lambda_{\nu-1}(x)\lambda_{\nu+1}(x)}{\Gamma(\nu)\Gamma(\nu+2)}<\frac{1}{\nu+1}\frac{\lambda_\nu^2(x)}{\Gamma^2(\nu+1)}$$

for the Turánian of $\lambda_\nu(x)/\Gamma(\nu+1)$ for $\nu>0$ and $x\in\mathbb{R}$. We note that the right-hand side of the above inequality is equivalent to (2.1) for $\nu>0$, while the left-hand side gives the Turán type inequality

$$\frac{1}{\nu}-\frac{1}{\nu}\lambda_\nu^2(x)\leq\lambda_\nu^2(x)-\lambda_{\nu-1}(x)\lambda_{\nu+1}(x),$$

where $\nu>0$ and $x\in\mathbb{R}$. This improves (2.2).

We mention that part (a) of Theorem 2 can also be proved using part (c) of Theorem 1 for all $x>0$. Namely, for $\mu>\nu>-1$ and $x>0$ we have

$$\frac{\lambda_\nu'(x)}{\lambda_\nu(x)}>\frac{\lambda_\mu'(x)}{\lambda_\mu(x)},$$

which implies that

$$\left(\frac{\lambda_\nu(x)}{\lambda_\mu(x)}\right)'=\frac{\lambda_\mu(x)\lambda_\nu'(x)-\lambda_\mu'(x)\lambda_\nu(x)}{\lambda_\mu^2(x)}>0.$$

Finally, we mention that by using the Weierstrassian decomposition (1.3) it is possible to deduce a Mittag-Leffler type expansion for the function λ_ν. Namely, by using the infinite product representation (1.3) we have the Mittag-Leffler expansion

$$\frac{\lambda_{\nu+1}(x)}{\lambda_\nu(x)}=-\frac{4(\nu+1)}{x^2-1+2\nu}+\frac{4(\nu+1)(x^2+1+2\nu)}{x^2-1+2\nu}\sum_{n\geq 1}\frac{1}{\alpha_{\nu,n}^2+x^2}\tag{2.17}$$

for all $\nu > -1$ and $x \in \mathbb{R}$. To prove the above expression, we note that

$$\lambda_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} \big[I_\nu(x) + x I_{\nu+1}(x) \big]$$
$$\lambda_\nu'(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} \big[x I_\nu(x) + (1-2\nu) I_{\nu+1}(x) \big],$$

and

$$\lambda_{\nu+1}(x) = 2^{\nu+1} \Gamma(\nu+2) x^{-\nu-1} \big[x I_\nu(x) - (2\nu+1) I_{\nu+1}(x) \big].$$

Here we have used the formula [21, p. 252]

$$(x^{-\nu} I_\nu(x))' = x^{-\nu} I_{\nu+1}(x)$$

and the recurrence relations [21, p. 251]

$$x I_\nu'(x) + \nu I_\nu(x) = x I_{\nu-1}(x) \quad \text{and} \quad x I_{\nu-1}(x) - x I_{\nu+1} = 2\nu I_\nu(x).$$

Combining the above equations on $\lambda_\nu(x)$, $\lambda_\nu'(x)$ and $\lambda_{\nu+1}(x)$ we obtain that

$$\lambda_{\nu+1}(x) = \frac{2(\nu+1)}{x(x^2-1+2\nu)} \big[-2x\lambda_\nu(x) + (x^2+1+2\nu)\lambda_\nu'(x) \big],$$

which in view of (2.4) gives (2.17).

2.4. Monotonicity properties of the Dini functions. Our third set of main results presents some monotonicity properties for the Dini function itself, which is a special combination of Bessel functions of the first kind. Observe that by (1.4) and the definition of $d_\nu(x)$ we have

$$\mathcal{D}_\nu(x) = \mathcal{J}_\nu(x) - \frac{x^2}{2(\nu+1)} \mathcal{J}_{\nu+1}(x), \tag{2.18}$$

where $\nu > -1$ and the function $\mathcal{J}_\nu \colon \mathbb{R} \to (-\infty, 1]$ is defined by

$$\mathcal{J}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} J_\nu(x) = \sum_{n \geq 0} \frac{(-1/4)^n}{(\nu+1)_n n!} x^{2n}.$$

By using (2.18) we have the power series

$$\mathcal{D}_\nu(x) = \sum_{n \geq 0} \frac{(-1)^n (2n+1) x^{2n}}{4^n n! (\nu+1)_n}.$$

Using this we get

$$\mathcal{D}_\nu'(x) = -\frac{x}{2(\nu+1)} \mathcal{D}_{\nu+1}(x) - \frac{x}{\nu+1} \mathcal{J}_{\nu+1}(x). \tag{2.19}$$

We also note that using the power series (2.13), we get

$$\lambda_\nu'(x) = \frac{x}{2(\nu+1)}\lambda_{\nu+1}(x) + \frac{x}{\nu+1}\mathcal{I}_{\nu+1}(x). \tag{2.20}$$

The next results may be proved by using some ideas from [3, Theorem 3]. We note that in the sequel by the log-concavity on disconnected set we mean actually the log-concavity on each connected component, which is of course different from log-concavity on the entire set.

THEOREM 3. *Let* $\nu > -1$, *and define* $\triangle = \triangle_1 \cup \triangle_2$, *where*

$$\triangle_1 = \bigcup_{n\geq 1}[-\alpha_{\nu,2n}, -\alpha_{\nu,2n-1}] \quad \text{and} \quad \triangle_2 = \bigcup_{n\geq 1}[\alpha_{\nu,2n-1}, \alpha_{\nu,2n}].$$

Then the following are valid.

(a) *The function* $x \mapsto \mathcal{D}_\nu(x)$ *is negative on* $\triangle$ *and strictly positive on* $\mathbb{R} \setminus \triangle$;

(b) *The function* $x \mapsto \mathcal{D}_\nu(x)$ *is increasing on* $(-\alpha_{\nu,1}, 0]$ *and decreasing on* $[0, \alpha_{\nu,1})$;

(c) *The function* $x \mapsto \mathcal{D}_\nu(x)$ *is strictly log-concave on* $\mathbb{R} \setminus \triangle$;

(d) *The function* $x \mapsto d_\nu(x)$ *is strictly log-concave on* $(0,\infty) \setminus \triangle_2$, *provided* $\nu > 0$;

(e) *The function* $\nu \mapsto \mathcal{D}_\nu(x)$ *is increasing on* $(-1,\infty)$ *for all*

$$x \in (-\alpha_{\nu,1}, \alpha_{\nu,1}).$$

PROOF. (a) From the infinite product representation (1.4) and the fact that

$$0 < \alpha_{\nu,1} < \alpha_{\nu,2} < \cdots < \alpha_{\nu,n} < \cdots ,$$

we see that if $x \in [\alpha_{\nu,2n-1}, \alpha_{\nu,2n}]$ or $x \in [-\alpha_{\nu,2n}, -\alpha_{\nu,2n-1}]$ then the first $(2n-1)$ terms of the product (1.4) are negative and the remaining terms are strictly positive. Therefore $\mathcal{D}_\nu(x)$ becomes negative on $\triangle$. Now if $x \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$ then each term of the product (1.4) are strictly positive and if $x \in (\alpha_{\nu,2n}, \alpha_{\nu,2n+1})$ or $x \in (-\alpha_{\nu,2n+1}, -\alpha_{\nu,2n})$, then the first $2n$ terms are strictly negative while the rest is strictly positive. Therefore $\mathcal{D}_\nu(x) > 0$ on $\mathbb{R} \setminus \triangle$.

(b) From part (a) we have $\mathcal{D}_\nu(x) > 0$ on $(-\alpha_{\nu,1}, \alpha_{\nu,1})$. Therefore by taking the logarithmic derivative of both sides of (1.4), we have

$$\big(\log \mathcal{D}_\nu(x)\big)' = \frac{\mathcal{D}_\nu'(x)}{\mathcal{D}_\nu(x)} = -\sum_{n\geq 1}\frac{2x}{\alpha_{\nu,n}^2 - x^2}\,.$$

From this we conclude that the function $x \mapsto \mathcal{D}_\nu(x)$ is increasing on $(-\alpha_{\nu,1}, 0]$ and decreasing on $[0, \alpha_{\nu,1})$.

(c) Differentiating both sides of the above relation with respect to x, we have

$$(\log \mathcal{D}_\nu(x))'' = -\sum_{n\geq 1} \frac{2(\alpha_{\nu,n}^2 + x^2)}{(\alpha_{\nu,n}^2 - x^2)^2}\,.$$

Thus we conclude that the function $x \mapsto \mathcal{D}_\nu(x)$ is strictly log-concave on $\mathbb{R} \setminus \triangle$. Note that this part has been proved also in [7, Theorem 4] but only for $x \in (0,\infty) \setminus \triangle_2$.

(d) From (1.4) we have

$$d_\nu(x) = \frac{x^\nu \mathcal{D}_\nu(x)}{2^\nu \Gamma(\nu+1)}\,.$$

Now from part (c) and using the fact that product of a log-concave function is log-concave, the conclusion follows, as $x \mapsto x^\nu$ is log-concave on $(0,\infty)$ for all $\nu \geq 0$. Another proof can be seen in [7, Theorem 3].

(e) We note that this part has been proved in [7, Theorem 6] for $\nu \in (-1,\infty)$ and $x \in (0, \alpha_{\nu,1})$ but because of the expression

$$\frac{\partial}{\partial\nu}\big(\log(\mathcal{D}_\nu(x))\big) = \sum_{n\geq 1} \frac{2x^2 \frac{\partial \alpha_{\nu,n}}{\partial \nu}}{\alpha_{\nu,n}(\alpha_{\nu,n}^2 - x^2)}\,,$$

the result is true for $\nu \in (-1,\infty)$ and $x \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$. □

Now, we show another result on Dini functions and modified Dini functions.

THEOREM 4. *Let $\nu > -1$. Then the function*

$$x \mapsto \lambda_\nu(x)/\mathcal{D}_\nu(x) = \xi_\nu(x)/d_\nu(x)$$

is strictly log-convex on $(-\alpha_{\nu,1}, \alpha_{\nu,1})$. Moreover,

$$\frac{\lambda_\nu^2(\frac{x+y}{2})}{\mathcal{D}_\nu^2(\frac{x+y}{2})} \leq \frac{\lambda_\nu(x)\lambda_\nu(y)}{\mathcal{D}_\nu(x)\mathcal{D}_\nu(y)}\,,$$

for all $x, y \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$, and in particular,

$$\frac{\big(\cosh(\frac{x+y}{2}) + (\frac{x+y}{2})\sinh(\frac{x+y}{2})\big)^2}{(\cosh x + x\sinh x)(\cosh y + y\sinh y)} \leq \frac{\big(\cos(\frac{x+y}{2}) - (\frac{x+y}{2})\sin(\frac{x+y}{2})\big)^2}{(\cos x - x\sin x)(\cos y - y\sin y)}\,,$$

for all $x, y \in (-\alpha_{-1/2,1}, \alpha_{1/2,1})$, where $\alpha_{-1/2,1} \simeq 0.8603335890\ldots$ is the first positive root of the equation $\cos x = x\sin x$.

Proof. By using part (b) of Theorem 1, the function $x \mapsto \lambda_\nu(x)$ is strictly log-convex on $(-\alpha_{\nu,1}, \alpha_{\nu,1})$ and by part (c) of Theorem 3 the function $x \mapsto 1/\mathcal{D}_\nu(x)$ is strictly log-convex on $(-\alpha_{\nu,1}, \alpha_{\nu,1})$. Therefore, the function $x \mapsto \lambda_\nu(x)/\mathcal{D}_\nu(x) = \xi_\nu(x)/d_\nu(x)$ is strictly log-convex on $(-\alpha_{\nu,1}, \alpha_{\nu,1})$, as a product of two strictly log-convex functions. The first inequality follows by the definition of log-convexity and the other inequality follows from (2.15) and observing that

$$\mathcal{D}_{-1/2}(x) = \cos x - x \sin x,$$

which can be derived by using

$$\mathcal{J}_{-1/2}(x) = \cos x, \quad \mathcal{J}_{1/2}(x) = \frac{\sin x}{x}. \quad \square$$

2.5. An infinitely divisible probability distribution involving Dini functions. The next result is motivated by [13, Theorem 1.9]. The next distributions are very natural companions to the distributions considered by Ismail and Kelker [13]. The main motivation to represent these functions as the Laplace transforms of continuous infinitely divisible probability distributions lies in the fact that similar results for modified Bessel functions of the first and second kind and Tricomi hypergeometric functions were used in the proof of the infinite divisibility of Student, gamma and Fisher–Snedecor distributions. For more details we refer to [13, p. 886].

Theorem 5. *Let $\mu > \nu > -1$. Then $x \mapsto \lambda_\mu(\sqrt{x})/\lambda_\nu(\sqrt{x})$ is a completely monotonic function with*

$$\frac{\lambda_\mu(\sqrt{x})}{\lambda_\nu(\sqrt{x})} = \int_0^\infty e^{-xt}\rho(t,\nu,\mu)\,dt,$$

where $\rho(t,\nu,\mu)$ is a probability density function on $(0,\infty)$ of an infinitely divisible distribution.

An immediate consequence of Theorem 5 (taking $\mu \to \infty$) is the following corollary.

Corollary 6. *Let $\nu > -1$. Then we have*

$$\frac{1}{\lambda_\nu(\sqrt{x})} = \int_0^\infty e^{-xt}\rho(t,\nu,\infty)\,dt,$$

where $\rho(t,\nu,\infty)$ is an infinitely divisible probability density.

We remark that part (f) of Theorem 1 may be obtained as a consequence of this corollary.

Proof. Let us consider $h(x)=-\log\big(\lambda_\mu(\sqrt{x})/\lambda_\nu(\sqrt{x})\big)$. Hence using (1.3) we have

$$h'(x)=\big(\log\lambda_\nu(\sqrt{x})\big)'-\big(\log\lambda_\mu(\sqrt{x})\big)'$$
$$=\sum_{n\geq1}\frac{1}{\alpha_{\nu,n}^2+x}-\sum_{n\geq1}\frac{1}{\alpha_{\mu,n}^2+x}=\sum_{n\geq1}\frac{\alpha_{\mu,n}^2-\alpha_{\nu,n}^2}{(\alpha_{\nu,n}^2+x)(\alpha_{\mu,n}^2+x)}.$$

Since $\alpha_{\mu,n}>\alpha_{\nu,n}$ for $\mu>\nu$ and for each $n\in\{1,2,\dots\}$ [19, p. 196], therefore each term in above series is positive and completely monotonic which implies that $x\mapsto h'(x)$ is completely monotonic as a sum of completely monotonic functions, consequently in view of [1, Lemma 2.4] the function $x\mapsto\lambda_\mu(\sqrt{x})/\lambda_\nu(\sqrt{x})$ is completely monotonic on $(0,\infty)$, as required. Now as $x\mapsto h'(x)$ is completely monotonic and from (1.3), $h(0)=0$ and hence by Lemma 1, $x\mapsto\lambda_\mu(\sqrt{x})/\lambda_\nu(\sqrt{x})$ is the Laplace transform of an infinitely divisible probability distribution. □

2.6. Redheffer-type inequalities for modified Dini functions. In this subsection we prove some Redheffer-type inequalities for modified Dini functions. Similar investigations were carried out in [10] for Bessel functions and modified Bessel functions. Here, we will also study the monotonicity of the product of Dini function and modified Dini function.

Theorem 7. *Let $\nu>-1$ and $x\in(-\alpha_{\nu,1},\alpha_{\nu,1})$. Then the modified Dini function satisfies the sharp exponential Redheffer-type inequality*

$$\left(\frac{\alpha_{\nu,1}^2+x^2}{\alpha_{\nu,1}^2-x^2}\right)^{a_\nu}\leq\lambda_\nu(x)\leq\left(\frac{\alpha_{\nu,1}^2+x^2}{\alpha_{\nu,1}^2-x^2}\right)^{b_\nu},\tag{2.21}$$

where $a_\nu=0$ and $b_\nu=\frac{3\alpha_{\nu,1}^2}{8(\nu+1)}$ are the best possible constants. In particular, the following exponential Redheffer type inequality is also valid

$$\left(\frac{\alpha_{-1/2,1}^2+x^2}{\alpha_{-1/2,1}^2-x^2}\right)^{a_{-1/2}}\leq\cosh(x)+x\sinh(x)\leq\left(\frac{\alpha_{-1/2,1}^2+x^2}{\alpha_{-1/2,1}^2-x^2}\right)^{b_{-1/2}},$$

where $a_{-1/2}=0$ and $b_{-1/2}=\frac{3}{4}\alpha_{-1/2,1}^2\simeq0.5551304132\dots$ are the best possible constants, and $\alpha_{-1/2,1}\simeq0.8603335890\dots$ is the first positive root of the equation $\cos x=x\sin x$.

Proof. Since all the functions in (2.21) are even in x it is enough to prove the result for $x\in(0,\alpha_{\nu,1})$. From part (a) of Theorem 1 the function $x\mapsto\lambda_\nu(x)$ is increasing on $(0,\alpha_{\nu,1})$ for all $\nu>-1$ and hence $\lambda_\nu(x)\geq1$ which gives the left-hand side of (2.21). To prove the right-hand side of (2.21), we consider the function $f_\nu\colon (0,\alpha_{\nu,1})\to\mathbb{R}$, defined by

$$f_\nu(x)=\frac{3\alpha_{\nu,1}^2}{8(\nu+1)}\log\left(\frac{\alpha_{\nu,1}^2+x^2}{\alpha_{\nu,1}^2-x^2}\right)-\log\lambda_\nu(x),$$

which in view of (2.4) and the formula [7]

$$\sum_{n\geq1}\frac{1}{\alpha_{\nu,n}^2}=\frac{3}{4(\nu+1)}$$

gives

$$f_\nu'(x)=\frac{3\alpha_{\nu,1}^2}{8(\nu+1)}\cdot\frac{4x\alpha_{\nu,1}^2}{\alpha_{\nu,1}^4-x^4}-\sum_{n\geq1}\frac{2x}{\alpha_{\nu,n}^2+x^2}$$

$$=\frac{2x\alpha_{\nu,1}^4}{\alpha_{\nu,1}^4-x^4}\sum_{n\geq1}\frac{1}{\alpha_{\nu,n}^2}-\sum_{n\geq1}\frac{2x}{\alpha_{\nu,n}^2+x^2}=2x^3\sum_{n\geq1}\frac{\alpha_{\nu,1}^4+\alpha_{\nu,n}^2x^2}{\alpha_{\nu,n}^2(\alpha_{\nu,1}^4-x^4)(\alpha_{\nu,n}^2+x^2)}.$$

Therefore the function f_ν is increasing on $[0,\alpha_{\nu,1})$ for all $\nu>-1$ and hence $f_\nu(x)\geq f_\nu(0)=0$ which implies the right-hand side of (2.21). Now, to prove that $a_\nu=0$ and $b_\nu=\frac{3\alpha_{\nu,1}^2}{8(\nu+1)}$ are the best possible constants, we consider the function $g_\nu\colon(0,\alpha_{\nu,1})\to\mathbb{R}$ defined as

$$g_\nu(x)=\frac{\log\lambda_\nu(x)}{\log(\frac{\alpha_{\nu,1}^2+x^2}{\alpha_{\nu,1}^2-x^2})}.$$

We note that $\lim\limits_{x\to\alpha_{\nu,1}}g_\nu(x)=0=a_\nu$ and using the l'Hospital rule we have

$$\lim_{x\to0}g_\nu(x)=\lim_{x\to0}\frac{\lambda_\nu'(x)}{\lambda_\nu(x)}\cdot\frac{\alpha_{\nu,1}^4-x^4}{4x\alpha_{\nu,1}^2}=\lim_{x\to0}\sum_{n\geq1}\frac{2x}{\alpha_{\nu,n}^2+x^2}\cdot\frac{\alpha_{\nu,1}^4-x^4}{4x\alpha_{\nu,1}^2}=b_\nu.$$

Therefore $a_\nu=0$ and $b_\nu=\frac{3\alpha_{\nu,1}^2}{8(\nu+1)}$ are indeed the best possible constants. Alternatively, inequality (2.21) can be proved using the monotone form of l'Hospital's rule [2, Lemma 2.2]. Namely, it is enough to observe that

$$x\mapsto\frac{\frac{d}{dx}\log\lambda_\nu(x)}{\frac{d}{dx}\log\left(\frac{\alpha_{\nu,1}^2+x^2}{\alpha_{\nu,1}^2-x^2}\right)}=\frac{1}{2\alpha_{\nu,1}^2}\sum_{n\geq1}\frac{\alpha_{\nu,1}^4-x^4}{\alpha_{\nu,n}^2+x^2}$$

is decreasing on $(0, \alpha_{\nu,1})$ as each term in the above series is decreasing. Therefore g_ν is decreasing too on $(0, \alpha_{\nu,1})$ and hence

$$a_\nu = \lim_{x\to\alpha_{\nu,1}} g_\nu(x) < g_\nu(x) < \lim_{x\to 0} g_\nu(x) = b_\nu,$$

which gives the inequality (2.21). □

We continue with another result on Dini and modified Dini functions.

THEOREM 8. *Let* $\nu > -1$. *The following assertions are valid*:
(a) *The function* $x \mapsto \mathcal{D}_\nu(x)\lambda_\nu(x)$ *is increasing on* $(-\alpha_{\nu,1}, 0]$ *and decreasing on* $[0, \alpha_{\nu,1})$;
(b) *The function* $\nu \mapsto \mathcal{D}_\nu(x)\lambda_\nu(x)$ *is increasing on* $(-1, \infty)$ *for all fixed* $x \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$;
(c) $0 < \mathcal{D}_\nu(x)\lambda_\nu(x) < \mathcal{D}_{\nu+1}(x)\lambda_{\nu+1}(x) < 1$, *for all* $x \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$ *and* $\nu > -1$.

PROOF. (a) Using (1.3) and (1.4) we get,

$$\big[\log(\mathcal{D}_\nu(x)\lambda_\nu(x))\big]' = \frac{\mathcal{D}'_\nu(x)}{\mathcal{D}_\nu(x)} + \frac{\lambda'_\nu(x)}{\lambda_\nu(x)}$$
$$= -\sum_{n\geq 1} \frac{2x}{\alpha^2_{\nu,n} - x^2} + \sum_{n\geq 1} \frac{2x}{\alpha^2_{\nu,n} + x^2} = -\sum_{n\geq 1} \frac{4x^3}{(\alpha^2_{\nu,n} - x^2)(\alpha^2_{\nu,n} + x^2)}$$

where $\nu > -1$ and $x \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$. Therefore the conclusion follows.

Alternatively, this part can be proved as follows. Using (2.19) and (2.20) we have

$$\begin{aligned}\big(\mathcal{D}_\nu(x)\lambda_\nu(x)\big)' &= \mathcal{D}_\nu(x)\lambda'_\nu(x) + \mathcal{D}'_\nu(x)\lambda_\nu(x)\\
&= \mathcal{D}_\nu(x)\Big[\frac{x}{2(\nu+1)}\lambda_{\nu+1}(x) + \frac{x}{\nu+1}\mathcal{I}_{\nu+1}(x)\Big]\\
&+ \Big[-\frac{x}{2(\nu+1)}\mathcal{D}_{\nu+1}(x) - \frac{x}{\nu+1}\mathcal{J}_{\nu+1}(x)\Big]\lambda_\nu(x)\\
&= \frac{x}{2(\nu+1)}\big[\mathcal{D}_\nu(x)\lambda_{\nu+1}(x) - \mathcal{D}_{\nu+1}(x)\lambda_\nu(x)\big]\\
&\quad + \frac{x}{\nu+1}\big[\mathcal{D}_\nu(x)\mathcal{I}_{\nu+1}(x) - \mathcal{J}_{\nu+1}(x)\lambda_\nu(x)\big].\end{aligned}$$

Hence it is enough to prove that

$$\mathcal{D}_\nu(x)\lambda_{\nu+1}(x) - \mathcal{D}_{\nu+1}(x)\lambda_\nu(x) < 0 \text{ and } \mathcal{D}_\nu(x)\mathcal{I}_{\nu+1}(x) - \mathcal{J}_{\nu+1}(x)\lambda_\nu(x) < 0 \tag{2.22}$$

for all $\nu > -1$ and $x \in [0, \alpha_{\nu,1})$. Now as the function $\nu \mapsto \mathcal{D}_\nu(x)$ is increasing on $(-1, \infty)$ for each fixed $x \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$ and the function $\nu \mapsto \lambda_\nu(x)$ is decreasing on $(-1, \infty)$ for all $x \in \mathbb{R}$ fixed, we have

$$\frac{\lambda_{\nu+1}(x)}{\lambda_\nu(x)} \leq 1 \leq \frac{\mathcal{D}_{\nu+1}(x)}{\mathcal{D}_\nu(x)},$$

and hence the first inequality in (2.22) is valid for all $\nu > -1$ and $x \in [0, \alpha_{\nu,1})$. Alternatively, in view of the infinite product representation (1.3) and (1.4), it is enough to show the inequality

$$\Big(1 - \frac{x^2}{\alpha_{\nu,n}^2}\Big)\Big(1 + \frac{x^2}{\alpha_{\nu+1,n}^2}\Big) \leq \Big(1 - \frac{x^2}{\alpha_{\nu+1,n}^2}\Big)\Big(1 + \frac{x^2}{\alpha_{\nu,n}^2}\Big)$$

for all $\nu > -1$, $n \in \{1, 2, \ldots\}$ and $x \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$ which is indeed true. Here we used the fact [19, p. 196], $\alpha_{\nu,n} < \alpha_{\nu+1,n}$ holds for all $\nu > -1$ and $n \in \{1, 2, \ldots\}$. Now to prove the second inequality in (2.22), recall the infinite product representation of $\mathcal{J}_\nu(z)$ and $\mathcal{I}_\nu(z)$ [23], namely

$$\mathcal{J}_\nu(z) = \prod_{n\geq 1}\Big(1 - \frac{z^2}{j_{\nu,n}^2}\Big), \quad \mathcal{I}_\nu(z) = \prod_{n\geq 1}\Big(1 + \frac{z^2}{j_{\nu,n}^2}\Big),$$

where $j_{\nu,n}$ is the nth positive zero of the Bessel function J_ν. In view of the above infinite product representations (1.3) and (1.4), it is enough to show the inequality

$$\Big(1 - \frac{x^2}{\alpha_{\nu,n}^2}\Big)\Big(1 + \frac{x^2}{j_{\nu+1,n}^2}\Big) \leq \Big(1 - \frac{x^2}{j_{\nu+1,n}^2}\Big)\Big(1 + \frac{x^2}{\alpha_{\nu,n}^2}\Big)$$

for all $\nu > -1$, $n \in \{1, 2, \ldots\}$ and $x \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$, that is, $\alpha_{\nu,n}^2 < j_{\nu+1,n}^2$, which is indeed true because

$$\alpha_{\nu,n} < \alpha_{\nu+1,n} < j_{\nu+1,n}. \tag{2.23}$$

The first inequality in (2.23) follows from the monotonicity of $\nu \mapsto \alpha_{\nu,n}$ [19, p. 196], and the second inequality follows from Dixon's theorem [23, p. 480], which says that when $\nu > -1$ and a, b, c, d are constants such that $ad \neq bc$, then the positive zeros of $x \mapsto aJ_\nu(x) + bxJ_\nu'(x)$ are interlaced with those of $x \mapsto cJ_\nu(x) + dxJ_\nu'(x)$. Therefore if we choose $a = 1 - \nu$, $b = c = 1$ and $d = 0$ then for $\nu > -1$ we have, $j_{\nu,n-1} < \alpha_{\nu,n} < j_{\nu,n}$, $n \geq 2$, and for $n = 1$, $0 < \alpha_{\nu,1} < j_{\nu,1}$.

(b) Since $\nu \mapsto \alpha_{\nu,n}$ is increasing on $(-1, \infty)$ for each $n \in \{1, 2, \ldots\}$, it follows that the function $\nu \mapsto \log(1 - x^4/\alpha_{\nu,n}^4)$ is increasing on $(-1, \infty)$ for each

$n \in \{1, 2, \dots\}$ and $x \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$ fixed. Again using the infinite products (1.3) and (1.4), the function

$$\nu \mapsto \log \left[\mathcal{D}_\nu(x) \lambda_\nu(x) \right] = \sum_{n \geq 1} \log \left(1 - \frac{x^4}{\alpha_{\nu,n}^4} \right)$$

is increasing on $(-1, \infty)$ for each $x \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$ fixed and hence the conclusion follows.

(c) This is an immediate consequence of parts (a) and (b) of this theorem. $\square$

2.7. Bounds for Bessel and modified Bessel functions. It is important to mention here that by using a similar approach as in [8, Remark C] we can find bounds for Dini and modified Dini functions in terms of Bessel and modified Bessel functions, which in turn give bounds for ratios of Bessel and modified Bessel functions. Namely, by Dixon's theorem [23, p. 480] for all $n \geq 2$ and $\nu > -1$ we have $j_{\nu,n-1} < \alpha_{\nu,n} < j_{\nu,n}$, where $j_{\nu,n}$ is the nth positive zero of the Bessel function J_ν. Therefore by these inequalities for $\nu > -1$ and $x \in \mathbb{R}$ we have,

$$\prod_{n \geq 2} \left(1 + \frac{x^2}{j_{\nu,n}^2} \right) < \prod_{n \geq 2} \left(1 + \frac{x^2}{\alpha_{\nu,n}^2} \right) < \prod_{n \geq 2} \left(1 + \frac{x^2}{j_{\nu,n-1}^2} \right),$$

which in view of (1.2) and the infinite product representation of the modified Bessel function I_ν implies

$$\frac{j_{\nu,1}^2}{\alpha_{\nu,1}^2} \cdot \frac{\alpha_{\nu,1}^2 + x^2}{j_{\nu,1}^2 + x^2} I_\nu(x) < \xi_\nu(x) < \frac{\alpha_{\nu,1}^2 + x^2}{\alpha_{\nu,1}^2} I_\nu(x).$$

Now, using the definition of $\xi_\nu(x)$ and the fact that $\alpha_{\nu,1} < j_{\nu,1}$ (see [14]) the above inequality gives

$$0 < \frac{x}{\alpha_{\nu,1}^2} \cdot \frac{j_{\nu,1}^2 - \alpha_{\nu,1}^2}{j_{\nu,1}^2 + x^2} < \frac{I_{\nu+1}(x)}{I_\nu(x)} < \frac{x}{\alpha_{\nu,1}^2}$$

for all $\nu > -1$ and $x > 0$. Using the formula

$$\mathcal{I}_\nu'(x) = 2^\nu \, \Gamma(\nu + 1)(x^{-\nu} I_\nu(x))' = 2^\nu \, \Gamma(\nu + 1) x^{-\nu} I_{\nu+1}(x),$$

the above inequality is equivalent to

$$\frac{t}{\alpha_{\nu,1}^2} \cdot \frac{j_{\nu,1}^2 - \alpha_{\nu,1}^2}{j_{\nu,1}^2 + t^2} < \frac{\mathcal{I}_\nu'(t)}{\mathcal{I}_\nu(t)} < \frac{t}{\alpha_{\nu,1}^2}, \tag{2.24}$$

where $t > 0$ and $\nu > -1$. Integrating (2.24) we obtain

$$\int_0^x \frac{j_{\nu,1}^2 - \alpha_{\nu,1}^2}{2\alpha_{\nu,1}^2} \big(\log(j_{\nu,1}^2 + t^2)\big)' \, dt < \int_0^x \big(\log \mathcal{I}_\nu(t)\big)' \, dt < \int_0^x \Big(\frac{t^2}{2\alpha_{\nu,1}^2}\Big)' \, dt,$$

which implies that

$$\Big(1 + \frac{x^2}{j_{\nu,1}^2}\Big)^{\frac{j_{\nu,1}^2 - \alpha_{\nu,1}^2}{2\alpha_{\nu,1}^2}} < \mathcal{I}_\nu(x) < e^{\frac{x^2}{2\alpha_{\nu,1}^2}}. \tag{2.25}$$

Here we have used the fact that $\mathcal{I}_\nu(0) = 1$. We also note that when $x = 0$, the above inequality (2.25) is sharp. The left-hand side of the inequality (2.25) is stronger than the inequality $\mathcal{I}_\nu(x) > 1$ for all $x > 0$ and $\nu > -1/2$, given by Luke [20], while using the inequality $\alpha_{\nu,1}^2 < 2(\nu + 1)$ (see [14]), we conclude that the right-hand side of (2.25) is weaker than the existing inequality

$$I_\nu(x) < \frac{x^\nu}{2^\nu \Gamma(\nu + 1)} e^{\frac{x^2}{4(\nu+1)}}$$

for all $x > 0$ and $\nu > -1$, given in [5].

Again integrating (2.24) over $0 < x < y$, we have

$$\int_x^y \frac{j_{\nu,1}^2 - \alpha_{\nu,1}^2}{2\alpha_{\nu,1}^2} \big(\log(j_{\nu,1}^2 + t^2)\big)' \, dt < \int_x^y \big(\log \mathcal{I}_\nu(t)\big)' \, dt < \int_x^y \Big(\frac{t^2}{2\alpha_{\nu,1}^2}\Big)' \, dt$$

and

$$\Big(\frac{x}{y}\Big)^\nu e^{\frac{x^2 - y^2}{2\alpha_{\nu,1}^2}} < \frac{I_\nu(x)}{I_\nu(y)} < \Big(\frac{x}{y}\Big)^\nu \Big(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 + y^2}\Big)^{\frac{j_{\nu,1}^2 - \alpha_{\nu,1}^2}{2\alpha_{\nu,1}^2}}. \tag{2.26}$$

In view of the inequality $\alpha_{\nu,1}^2 < 2(\nu + 1)$ (see [14]) the left-hand side of (2.26) improves the inequality

$$\Big(\frac{x}{y}\Big)^\nu e^{\frac{x^2 - y^2}{4(\nu+1)}} < \frac{I_\nu(x)}{I_\nu(y)}$$

given by Joshi and Bissu [15], where $\nu > -1$ and $0 < x < y$.

Next we find a bound for Dini functions in terms of Bessel functions, which in turn gives a bound for the ratio $J_{\nu+1}(x)/J_\nu(x)$. For $\nu > -1$, using the inequalities $j_{\nu,n-1} < \alpha_{\nu,n} < j_{\nu,n}$, $n \geq 2$ and $0 < \alpha_{\nu,1} < j_{\nu,1}$, we have for all $x \in (-\alpha_{\nu,1}, \alpha_{\nu,1})$,

$$\prod_{n \geq 2} \Big(1 - \frac{x^2}{j_{\nu,n-1}^2}\Big) < \prod_{n \geq 2} \Big(1 - \frac{x^2}{\alpha_{\nu,n}^2}\Big) < \prod_{n \geq 2} \Big(1 - \frac{x^2}{j_{\nu,n}^2}\Big),$$

which in view of (1.1) and the infinite product representation of the Bessel functions of the first kind implies

$$\frac{\alpha_{\nu,1}^2 - x^2}{\alpha_{\nu,1}^2} J_\nu(x) < d_\nu(x) < \frac{j_{\nu,1}^2}{\alpha_{\nu,1}^2} \cdot \frac{\alpha_{\nu,1}^2 - x^2}{j_{\nu,1}^2 - x^2} J_\nu(x) \tag{2.27}$$

Now, using the definition of $d_\nu(x)$ and the fact that $0 < \alpha_{\nu,1} < j_{\nu,1}$, (2.27) gives

$$0 < \frac{x}{\alpha_{\nu,1}^2} \cdot \frac{j_{\nu,1}^2 - \alpha_{\nu,1}^2}{j_{\nu,1}^2 - x^2} < \frac{J_{\nu+1}(x)}{J_\nu(x)} < \frac{x}{\alpha_{\nu,1}^2} \tag{2.28}$$

for all $\nu > -1$ and $x \in (0, \alpha_{\nu,1})$. In view of the formula

$$\mathcal{J}_\nu'(x) = 2^\nu \Gamma(\nu+1)(x^{-\nu} J_\nu(x))' = -2^\nu \Gamma(\nu+1) x^{-\nu} J_{\nu+1}(x),$$

(2.28) implies

$$\frac{t}{\alpha_{\nu,1}^2} . \frac{j_{\nu,1}^2 - \alpha_{\nu,1}^2}{j_{\nu,1}^2 - t^2} < -\frac{\mathcal{J}_\nu'(t)}{\mathcal{J}_\nu(t)} < \frac{t}{\alpha_{\nu,1}^2} . \tag{2.29}$$

Integrating (2.29) as above, we have for all $\nu > -1$ and $0 < x < y < \alpha_{\nu,1}$

$$e^{-\frac{x^2}{2\alpha_{\nu,1}^2}} < \mathcal{J}_\nu(x) < \left(1 - \frac{x^2}{j_{\nu,1}^2}\right)^{\frac{j_{\nu,1}^2 - \alpha_{\nu,1}^2}{2\alpha_{\nu,1}^2}} .$$

and

$$\left(\frac{x}{y}\right)^\nu \left(\frac{j_{\nu,1}^2 - x^2}{j_{\nu,1}^2 - y^2}\right)^{\frac{j_{\nu,1}^2 - \alpha_{\nu,1}^2}{2\alpha_{\nu,1}^2}} < \frac{J_\nu(x)}{J_\nu(y)} < \left(\frac{x}{y}\right)^\nu e^{-\frac{x^2 - y^2}{2\alpha_{\nu,1}^2}} .$$

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