Springer

Dear Author,

Here are the proofs of your article.

- You can submit your corrections **online**, via **e-mail** or by **fax**.
- For **online** submission please insert your corrections in the online correction form. Always indicate the line number to which the correction refers.
- You can also insert your corrections in the proof PDF and **email** the annotated PDF.
- For fax submission, please ensure that your corrections are clearly legible. Use a fine black pen and write the correction in the margin, not too close to the edge of the page.
- Remember to note the **journal title**, **article number**, and **your name** when sending your response via e-mail or fax.
- **Check** the metadata sheet to make sure that the header information, especially author names and the corresponding affiliations are correctly shown.
- **Check** the questions that may have arisen during copy editing and insert your answers/ corrections.
- **Check** that the text is complete and that all figures, tables and their legends are included. Also check the accuracy of special characters, equations, and electronic supplementary material if applicable. If necessary refer to the *Edited manuscript*.
- The publication of inaccurate data such as dosages and units can have serious consequences. Please take particular care that all such details are correct.
- Please **do not** make changes that involve only matters of style. We have generally introduced forms that follow the journal's style.
  Substantial changes in content, e.g., new results, corrected values, title and authorship are not allowed without the approval of the responsible editor. In such a case, please contact the Editorial Office and return his/her consent together with the proof.
- If we do not receive your corrections **within 48 hours**, we will send you a reminder.
- Your article will be published **Online First** approximately one week after receipt of your corrected proofs. This is the **official first publication** citable with the DOI. **Further changes are, therefore, not possible.**
- The **printed version** will follow in a forthcoming issue.

Please note

After online publication, subscribers (personal/institutional) to this journal will have access to the complete article via the DOI using the URL: http://dx.doi.org/[DOI].
If you would like to know when your article has been published online, take advantage of our free alert service. For registration and further information go to: http://www.link.springer.com.

Due to the electronic nature of the procedure, the manuscript and the original figures will only be returned to you on special request. When you return your corrections, please inform us if you would like to have these documents returned.

Metadata of the article that will be visualized in OnlineFirst

ArticleTitle	Proof of a Conjecture of Bárány, Katchalski and Pach
Article Sub-Title	
Article CopyRight	
Journal Name	Discrete & Computational Geometry

Corresponding Author	Family Name	**Naszódi**
	Particle	
	Given Name	**Márton**
	Suffix	
	Division	ELTE, Department of Geometry
	Organization	Lorand Eötvös University
	Address	Pázmány Péter Sétány 1/C, Budapest, 1117, Hungary
	Email	marton.naszodi@math.elte.hu

	Received	25 March 2015
Schedule	Revised	26 October 2015
	Accepted	10 November 2015

Abstract: Bárány, Katchalski and Pach (Proc Am Math Soc 86(1):109–114, 1982) (see also Bárány et al., Am Math Mon 91(6):362–365, 1984) proved the following quantitative form of Helly's theorem. If the intersection of a family of convex sets in $\mathbb{R}^d$ is of volume one, then the intersection of some subfamily of at most $2d$ members is of volume at most some constant $v(d)$. In Bárány et al. (Am Math Mon 91(6):362–365, 1984), the bound $v(d) \leq d^{2d^2}$ was proved and $v(d) \leq d^{cd}$ was conjectured. We confirm it.

Keywords (separated by '-')	Helly's theorem - Quantitative Helly theorem - Intersection of convex sets - Dvoretzky–Rogers lemma - John's ellipsoid - Volume
Mathematics Subject Classification (separated by '-')	52A35
Footnote Information	Editor in Charge: János Pach

Discrete Comput Geom
DOI 10.1007/s00454-015-9753-3

Proof of a Conjecture of Bárány, Katchalski and Pach

Márton Naszódi[1]

Received: 25 March 2015 / Revised: 26 October 2015 / Accepted: 10 November 2015

Abstract Bárány, Katchalski and Pach (Proc Am Math Soc 86(1):109–114, 1982) (see also Bárány et al., Am Math Mon 91(6):362–365, 1984) proved the following quantitative form of Helly's theorem. If the intersection of a family of convex sets in $\mathbb{R}^d$ is of volume one, then the intersection of some subfamily of at most $2d$ members is of volume at most some constant $v(d)$. In Bárány et al. (Am Math Mon 91(6):362–365, 1984), the bound $v(d) \leq d^{2d^2}$ was proved and $v(d) \leq d^{cd}$ was conjectured. We confirm it.

Keywords Helly's theorem · Quantitative Helly theorem · Intersection of convex sets · Dvoretzky–Rogers lemma · John's ellipsoid · Volume

Mathematics Subject Classification 52A35

1 Introduction and Preliminaries

Theorem 1.1 *Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$ such that the volume of its intersection is* $\mathrm{vol}\,(\cap\mathcal{F}) > 0$. *Then there is a subfamily $\mathcal{G}$ of $\mathcal{F}$ with $|\mathcal{G}| \leq 2d$ and* $\mathrm{vol}\,(\cap\mathcal{G}) \leq e^{d+1} d^{2d+\frac{1}{2}}\,\mathrm{vol}\,(\cap\mathcal{F})$.

We recall the note from [2] (see also [3]) that the number $2d$ is optimal, as shown by the $2d$ half-spaces supporting the facets of the cube.

Editor in Charge: János Pach

Márton Naszódi
marton.naszodi@math.elte.hu

[1] ELTE, Department of Geometry, Lorand Eötvös University, Pázmány Péter Sétány 1/C, Budapest 1117, Hungary

Journal: 454 Article No.: 9753 TYPESET DISK LE CP Disp.:2015/11/21 Pages: 6 Layout: Small-X

The order of magnitude d^{cd} in the Theorem (and in the conjecture in [2]) is sharp as shown in Sect. 3.

Recently, other quantitative Helly type results have been obtained by De Loera et al. [5].

We introduce notations and tools that we will use in the proof. We denote the closed unit ball centered at the origin o in the d-dimensional Euclidean space $\mathbb{R}^d$ by $\mathbf{B}$. For the scalar product of $u, v \in \mathbb{R}^d$, we use $\langle u, v\rangle$, and the length of u is $|u| = \sqrt{\langle u, u\rangle}$. The tensor product $u \otimes u$ is the rank one linear operator that maps any $x \in \mathbb{R}^d$ to the vector $(u \otimes u)x = \langle u, x\rangle u \in \mathbb{R}^d$. For a set $A \subset \mathbb{R}^d$, we denote its polar by $A^* = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1 \text{ for all } x \in A\}$. The volume of a set is denoted by $\operatorname{vol}(\cdot)$.

Definition 1.2 We say that a set of vectors $w_1, \dots, w_m \in \mathbb{R}^d$ with weights $c_1, \dots, c_m > 0$ form a *John's decomposition of the identity*, if

$$\sum_{i=1}^{m} c_i w_i = o \quad \text{and} \quad \sum_{i=1}^{m} c_i w_i \otimes w_i = I, \tag{1}$$

where I is the identity operator on $\mathbb{R}^d$.

A *convex body* is a compact convex set in $\mathbb{R}^d$ with non-empty interior. We recall John's theorem [8] (see also [1]).

Lemma 1.3 (John's theorem) *For any convex body K in $\mathbb{R}^d$, there is a unique ellipsoid of maximal volume in K. Furthermore, this ellipsoid is $\mathbf{B}$ if, and only if, there are points $w_1, \dots, w_m \in \operatorname{bd}\mathbf{B} \cap \operatorname{bd} K$ (called* contact points*) and corresponding weights $c_1, \dots, c_m > 0$ that form a John's decomposition of the identity.*

It is not difficult to see that if $w_1, \dots, w_m \in \operatorname{bd}\mathbf{B}$ and corresponding weights $c_1, \dots, c_m > 0$ form a John's decomposition of the identity, then $\{w_1, \dots, w_m\}^* \subset d\mathbf{B}$, cf. [1] or [7, Thm. 5.1]. By polarity, we also obtain that $\frac{1}{d}\mathbf{B} \subset \operatorname{conv}(\{w_1, \dots, w_m\})$.

One can verify that if Δ is a regular simplex in $\mathbb{R}^d$ such that the ball $\mathbf{B}$ is the largest volume ellipsoid in Δ, then

$$\operatorname{vol}(\Delta) = \frac{d^{d/2}(d+1)^{(d+1)/2}}{d!}. \tag{2}$$

We will use the following form of the Dvoretzky–Rogers lemma [6].

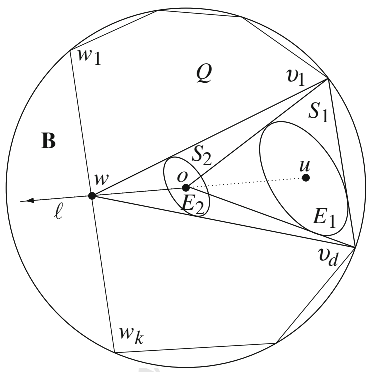
Lemma 1.4 (Dvoretzky–Rogers lemma) *Assume that $w_1, \dots, w_m \in \operatorname{bd}\mathbf{B}$ and $c_1, \dots, c_m > 0$ form a John's decomposition of the identity. Then there is an orthonormal basis $z_1, \dots, z_d$ of $\mathbb{R}^d$, and a subset $\{v_1, \dots, v_d\}$ of $\{w_1, \dots, w_m\}$ such that*

$$v_i \in \operatorname{span}\{z_1, \dots, z_i\} \quad \textit{and} \quad \sqrt{\frac{d-i+1}{d}} \leq \langle v_i, z_i\rangle \leq 1 \quad \textit{for all } i = 1, \dots, d. \tag{3}$$

This lemma is usually stated in the setting of John's theorem, that is, when the vectors are contact points of a convex body K with its maximal volume ellipsoid, which is $\mathbf{B}$.

Journal: 454 Article No.: 9753 TYPESET DISK LE CP Disp.:2015/11/21 Pages: 6 Layout: Small-X

Fig. 1 .



And often, it is assumed in the statement that K is symmetric about the origin, see for example [4]. Since we make no such assumption (in fact, we make no reference to K in the statement of Lemma 1.4), we give a proof in Sect. 4.

2 Proof of Theorem 1.1

Without loss of generality, we may assume that $\mathcal{F}$ consists of closed half-spaces, and also that $\operatorname{vol}(\cap\mathcal{F}) < \infty$, that is, $\cap\mathcal{F}$ is a convex body in $\mathbb{R}^d$. As shown in [3], by continuity, we may also assume that $\mathcal{F}$ is a finite family, that is $P = \cap\mathcal{F}$ is a d-dimensional polyhedron.

The problem is clearly affine invariant, so we may assume that $\mathbf{B} \subset P$ is the ellipsoid of maximal volume in P.

By Lemma 1.3, there are contact points $w_1, \ldots, w_m \in \operatorname{bd}\mathbf{B} \cap \operatorname{bd} P$ (and weights $c_1, \ldots, c_m > 0$) that form a John's decomposition of the identity. We denote their convex hull by $Q = \operatorname{conv}\{w_1. \ldots, w_m\}$. Lemma 1.4 yields that there is an orthonormal basis $z_1, \ldots, z_d$ of $\mathbb{R}^d$, and a subset $\{v_1, \ldots, v_d\}$ of the contact points $\{w_1, \ldots, w_m\}$ such that (3) holds.

Let $S_1 = \operatorname{conv}\{o, v_1, v_2, \ldots, v_d\}$ be the simplex spanned by these contact points, and let E_1 be the largest volume ellipsoid contained in S_1. We denote the center of E_1 by u. Let ℓ be the ray emanating from the origin in the direction of the vector $-u$. Clearly, the origin is in the interior of Q. In fact, by the remark following Lemma 1.3, $\frac{1}{d}\mathbf{B} \subset Q$. Let w be the point of intersection of the ray ℓ with $\operatorname{bd} Q$. Then $|w| \geq 1/d$. Let S_2 denote the simplex $S_2 = \operatorname{conv}\{w, v_1, v_2, \ldots, v_d\}$. See Fig. 1. 1

We apply a contraction with center w and ratio $\lambda = \frac{|w|}{|w-u|}$ on E_1 to obtain the ellipsoid E_2. Clearly, E_2 is centered at the origin and is contained in S_2. Furthermore,

$$\lambda = \frac{|w|}{|u| + |w|} \geq \frac{|w|}{1 + |w|} \geq \frac{1}{d+1}. \tag{4}$$

Since w is on $\operatorname{bd} Q$, by Caratheodory's theorem, w is in the convex hull of some set of at most d vertices of Q. By re-indexing the vertices, we may assume that $w \in \operatorname{conv}\{w_1, \ldots, w_k\}$ with $k \leq d$. Now,

Journal: 454 Article No.: 9753 TYPESET DISK LE CP Disp.:2015/11/21 Pages: 6 Layout: Small-X

$$E_2 \subset S_2 \subset \operatorname{conv}\{w_1, \ldots, w_k, v_1, \ldots, v_d\}. \tag{5}$$

Let $X = \{w_1, \ldots, w_k, v_1, \ldots, v_d\}$ be the set of these unit vectors, and let $\mathcal{G}$ denote the family of those half-spaces which support $\mathbf{B}$ at the points of X. Clearly, $|\mathcal{G}| \le 2d$. Since the points of X are contact points of P and $\mathbf{B}$, we have that $\mathcal{G} \subseteq \mathcal{F}$. By (5),

$$\cap \mathcal{G} = X^* \subset E_2^*. \tag{6}$$

By (3),

$$\operatorname{vol}(S_1) \ge \frac{1}{d!} \cdot \frac{\sqrt{d!}}{d^{d/2}} = \frac{1}{\sqrt{d!}d^{d/2}}. \tag{7}$$

Since $\mathbf{B} \subset \cap\mathcal{F}$, by (6) and (4), (2), (7) we have

$$\begin{aligned}\frac{\operatorname{vol}(\cap\mathcal{G})}{\operatorname{vol}(\cap\mathcal{F})} &\le \frac{\operatorname{vol}\left(E_2^*\right)}{\operatorname{vol}(\mathbf{B})} = \frac{\operatorname{vol}(\mathbf{B})}{\operatorname{vol}(E_2)} \le (d+1)^d \frac{\operatorname{vol}(\mathbf{B})}{\operatorname{vol}(E_1)} = (d+1)^d \frac{\operatorname{vol}(\Delta)}{\operatorname{vol}(S_1)} \\ &= \frac{d^{d/2}(d+1)^{(3d+1)/2}}{d!\operatorname{vol}(S_1)} = \frac{d^d d^{3d/2} e^{3/2} (d+1)^{1/2}}{(d!)^{1/2}} \le e^{d+1} d^{2d+\frac{1}{2}},\end{aligned} \tag{8}$$

where Δ is as defined above (2). This completes the proof of Theorem 1.1.

Remark 2.1 In the proof, in place of the Dvoretzky–Rogers lemma, we could select the d vectors $v_1, \ldots, v_d$ from the contact points randomly: picking w_i with probability c_i/d for $i = 1, \ldots, m$, and repeating this picking independently d times. Pivovarov proved (cf. [9, Lem. 3]) that the expected volume of the random simplex S_1 obtained this way is the same as the right hand side in (7).

3 A Simple Lower Bound for $v(d)$

We outline a simple proof that one cannot hope a better bound in Theorem 1.1 than $d^{d/2}$ in place of $d^{2d+1/2}$. Indeed, consider the Euclidean ball $\mathbf{B}$, and a family $\mathcal{F}$ of (very many) supporting closed half space of $\mathbf{B}$ whose intersection is very close to $\mathbf{B}$. Suppose that $\mathcal{G}$ is a subfamily of $\mathcal{F}$ of $2d$ members. Denote by σ the Haar probability measure on the sphere $R\mathbb{S}^{d-1}$, where $R = (d/(2\ln d))^{\frac{1}{2}}$. Let $H \in \mathcal{G}$ be one of the half spaces. Then

$$\sigma(R\mathbb{S}^{d-1} \setminus H) \le \exp\left(\frac{-d}{2R^2}\right) \le 1/(4d).$$

It follows that

$$\operatorname{vol}(\cap\mathcal{G}) \ge R^d \operatorname{vol}(\mathbf{B})\, \sigma(R\mathbb{S}^{d-1} \setminus (\cup\mathcal{G})) \ge \frac{1}{2} R^d \operatorname{vol}(\mathbf{B}) \ge d^{\frac{d}{2}-\varepsilon} \operatorname{vol}(\cap\mathcal{F})$$

for any $\varepsilon > 0$ if d is large enough.

Journal: 454 Article No.: 9753 TYPESET DISK LE CP Disp.:2015/11/21 Pages: 6 Layout: Small-X

4 Proof of Lemma 1.4

We follow the proof in [4].

Claim 4.1 *Assume that* $w_1, \ldots, w_m \in \operatorname{bd}\mathbf{B}$ *and* $c_1, \ldots, c_m > 0$ *form a John's decomposition of the identity. Then for any linear map* $T : \mathbb{R}^d \to \mathbb{R}^d$ *there is an* $\ell \in \{1, \ldots, m\}$ *such that*

$$\langle w_\ell, T w_\ell \rangle \geq \frac{\operatorname{tr} T}{d}, \tag{9}$$

where $\operatorname{tr} T$ *denotes the trace of* T.

For matrices $A, B \in \Re^{d\times d}$ we use $\langle A, B\rangle = \operatorname{tr}\left(AB^T\right)$ to denote their Frobenius product.

To prove the claim, we observe that

$$\frac{\operatorname{tr} T}{d} = \frac{1}{d}\langle T, I\rangle = \frac{1}{d}\sum_{i=1}^{m} c_i \langle T, w_i \otimes w_i\rangle = \frac{1}{d}\sum_{i=1}^{m} c_i \langle T w_i, w_i\rangle .$$

Since $\sum_{i=1}^m c_i = d$, the right hand side is a weighted average of the values $\langle T w_i, w_i\rangle$. Clearly, some value is at least the average, yielding Claim 4.1.

We define z_i and v_i inductively. First, let $z_1 = v_1 = w_1$. Assume that, for some $k < d$, we have found z_i and v_i for all $i = 1, \ldots, k$. Let $F = \operatorname{span}\{z_1, \ldots, z_k\}$, and let T be the orthogonal projection onto the orthogonal complement $F^\perp$ of F. Clearly, $\operatorname{tr} T = \dim F^\perp = d - k$. By Claim 4.1, for some $\ell \in \{1, \ldots, m\}$ we have

$$|T w_\ell|^2 = \langle T w_\ell, w_\ell\rangle \geq \frac{d-k}{d}.$$

Let $v_{k+1} = w_\ell$ and $z_{k+1} = \frac{T w_\ell}{|T w_\ell|}$. Clearly, $v_{k+1} \in \operatorname{span}\{z_1, \ldots, z_{k+1}\}$. Moreover,

$$\langle v_{k+1}, z_{k+1}\rangle = \frac{\langle T w_\ell, w_\ell\rangle}{|T w_\ell|} = \frac{|T w_\ell|^2}{|T w_\ell|} = |T w_\ell| \geq \sqrt{\frac{d-k}{d}},$$

finishing the proof of Lemma 1.4.

Note that in this proof, we did not use the fact that, in a John's decomposition of the identity, the vectors are balanced, that is $\sum_{i=1}^m c_i w_i = o$.

Acknowledgments I am grateful for János Pach for the many conversations that we had on the subject and for the inspiring atmosphere that he creates in his DCG group at EPFL. I also thank the referee for helping to make the presentation more clear. The support of the János Bolyai Research Scholarship of the Hungarian Academy of Sciences, and the Hung. Nat. Sci. Found. (OTKA) Grant PD104744 is acknowledged.

References

1. Ball, K.: An elementary introduction to modern convex geometry. Flavors of Geometry, pp. 1–58. Cambridge University Press, Cambridge (1997)

Journal: 454 Article No.: 9753 TYPESET DISK LE CP Disp.:2015/11/21 Pages: 6 Layout: Small-X

2. Bárány, I., Katchalski, M., Pach, J.: Quantitative Helly-type theorems. Proc. Am. Math. Soc. **86**(1), 109–114 (1982)
3. Bárány, I., Katchalski, M., Pach, J.: Helly's theorem with volumes. Am. Math. Mon. **91**(6), 362–365 (1984)
4. Brazitikos, S., Giannopoulos, A., Valettas, P., Vritsiou, B.-H.: Geometry of Isotropic Convex Bodies, Mathematical Surveys and Monographs, vol. 196. American Mathematical Society, Providence, RI (2014)
5. De Loera, J.A., La Haye, R.N., Rolnick, D., Soberón, P.: Quantitative Tverberg, Helly, & Carathéodory Theorems. http://arxiv.org/abs/1503.06116 [math] (March 2015)
6. Dvoretzky, A., Rogers, C.A.: Absolute and unconditional convergence in normed linear spaces. Proc. Natl Acad. Sci. USA **36**, 192–197 (1950)
7. Gordon, Y., Litvak, A.E., Meyer, M., Pajor, A.: John's decomposition in the general case and applications. J. Differ. Geom. **68**(1), 99–119 (2004)
8. John, F.: Extremum problems with inequalities as subsidiary conditions. Studies and Essays Presented to R. Courant on his 60th Birthday, pp. 187–204, 8 Jan 1948
9. Pivovarov, P.: On determinants and the volume of random polytopes in isotropic convex bodies. Geom. Dedicata **149**, 45–58 (2010)

Journal: 454
Article: 9753

Author Query Form

Please ensure you fill out your response to the queries raised below and return this form along with your corrections

Dear Author

During the process of typesetting your article, the following queries have arisen. Please check your typeset proof carefully against the queries listed below and mark the necessary changes either directly on the proof/online grid or in the 'Author's response' area provided below

Query	Details required	Author's response
1.	Please provide caption for Figure 1.	