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A SPIKY BALL

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Abstract. The illumination problem may be phrased as the problem of covering a convex body in Euclidean n-space by a minimum number of translates of its interior. By a probabilistic argument, we show that, arbitrarily close to the Euclidean ball, there is a centrally symmetric convex body of illumination number exponentially large in the dimension.

§1. *Introduction.* For two sets K and L in $\mathbb{R}^n$, let $N(K, L)$ denote the *translative covering number* of K by L, that is, the minimum number of translates of L that cover K.

Let K be a convex body (that is, a compact, convex set with non-empty interior) in $\mathbb{R}^n$. Following Hadwiger [10], we say that a point $p \in \mathbb{R}^n \setminus K$ *illuminates* a boundary point $b \in \operatorname{bd} K$ if the ray $\{p+\lambda(b-p) : \lambda > 0\}$ emanating from p and passing through b intersects the interior of K. Boltyanski [5] gave the following slightly different definition. A direction $u \in \mathbb{S}^{n-1}$ is said to *illuminate* K at a boundary point $b \in \operatorname{bd} K$ if the ray $\{b + \lambda u : \lambda > 0\}$ intersects the interior of K. It is easy to see that the minimum number of directions that illuminate each boundary point of K is equal to the minimum number of points that illuminate each boundary point of K. This number is called the *illumination number* $\mathrm{i}(K)$ of K.

We call a set of the form $\lambda K + v$ a *smaller positive homothet of* K if $0 < \lambda < 1$ and $v \in \mathbb{R}^n$. Gohberg and Markus asked how large the minimum number of smaller positive homothets of K covering K can be. It is not hard to see that this number is equal to $N(K, \operatorname{int} K)$. It is also easy to see that $\mathrm{i}(K) = N(K, \operatorname{int} K)$.

Any smooth convex body (i.e., a convex body with a unique support hyperplane at each boundary point) in $\mathbb{R}^n$ is illuminated by $n + 1$ directions. Indeed, for a smooth body, the set of directions illuminating a given boundary point is an open hemisphere of $\mathbb{S}^{n-1}$, and one can find $n + 1$ points (e.g., take the vertices of a regular simplex) in $\mathbb{S}^{n-1}$ with the property that every open hemisphere contains at least one of the points. Thus, these $n + 1$ points in $\mathbb{S}^{n-1}$ (i.e., directions) illuminate any smooth convex body in $\mathbb{R}^n$ (cf. [6] for details).

On the other hand, the illumination number of the cube is 2^n, since no two vertices of the cube share an illumination direction. Even though we do not discuss it, it would be difficult to omit mentioning the *Gohberg–Markus–Levi–Boltyanski–Hadwiger conjecture* (or illumination conjecture), according to

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which for any convex body K in $\mathbb{R}^n$, we have $\mathrm{i}(K) = 2^n$, where equality is attained only when K is an affine image of the cube.

For more background on the problem of illumination, see [1, 2, 8, 11]. In [6, Ch. VI], one can find a proof of the equivalence of the four definitions of $\mathrm{i}(K)$ given above.

The Euclidean ball is a smooth convex body and hence is of illumination number $n + 1$. Theorem 1.1 shows that, arbitrarily close to the Euclidean ball, there is a convex body of much larger illumination number.

We denote the closed Euclidean unit ball in $\mathbb{R}^n$ centered at the origin o by $\mathbf{B}^n$, and its boundary, the sphere, by $\mathbb{S}^{n-1}$.

THEOREM 1.1. *Let $1 < D < 1.116$ be given. Then, for any sufficiently large dimension n, there is an o-symmetric convex body K in $\mathbb{R}^n$, with illumination number*

$$\mathrm{i}(K) = N(K, \operatorname{int} K) \geqslant 0.05 D^n, \tag{1}$$

for which

$$\frac{1}{D}\mathbf{B}^n \subset K \subset \mathbf{B}^n. \tag{2}$$

We will use a probabilistic construction to find K. We are not aware of any lower bound for the illumination problem that was obtained by a probabilistic argument.

For a point $u \in \mathbb{S}^{n-1}$ and $0 < \varphi < \pi/2$, let $C(u, \varphi) = \{v \in \mathbb{S}^{n-1} : \sphericalangle(u, v) \leqslant \varphi\}$ denote the spherical cap centered at u of angular radius φ. We denote the normalized Lebesgue measure (that is, the Haar probability measure on $\mathbb{S}^{n-1}$) of $C(u, \varphi)$ by $\Omega_{n-1}(\varphi)$.

In Theorem 1.2, we give an upper bound for the illumination number for bodies close to the Euclidean ball. It follows from [3] but, for the sake of completeness, we will sketch a proof.

THEOREM 1.2. *Let K be a convex body in $\mathbb{R}^n$ such that $(1/D)\mathbf{B}^n \subset K \subset \mathbf{B}^n$ for some $D > 1$. Then the illumination number of K is at most*

$$\mathrm{i}(K) \leqslant \frac{n \ln n + n \ln \ln n + 5n}{\Omega_{n-1}(\alpha)}, \tag{3}$$

where $\alpha = \arcsin(1/D)$.

By combining Theorem 1.2 with the estimate (5) on Ω_{n-1}, one can see that (1) is asymptotically sharp, that is, the base D cannot be improved.

Next, we consider an application of Theorem 1.1. Let K be an origin-symmetric convex body in $\mathbb{R}^n$, and denote its gauge function by $\|\cdot\|_K$ (that is, $\|p\|_K = \inf\{\lambda > 0 : p \in \lambda K\}$ for any $p \in \mathbb{R}^n$). We use $\operatorname{vert} P$ to denote the set of vertices of the polytope P. The *illumination parameter*, introduced by Bezdek [1], is defined as

$$\operatorname{ill}(K) = \inf\left\{ \sum_{p \in \operatorname{vert} P} \|p\|_K \;\middle|\; P \text{ a polytope such that } \operatorname{vert} P \text{ illuminates } K \right\}.$$

The *vertex index* of K, introduced by Bezdek and Litvak [4], is

$$\operatorname{vein}(K) = \inf\left\{ \sum_{p \in \operatorname{vert} P} \|p\|_K \;\middle|\; P \text{ a polytope such that } K \subseteq P \right\}.$$

Clearly, $\operatorname{ill}(K) \geqslant \operatorname{vein}(K)$ for any centrally symmetric body K, and they are equal for smooth bodies. It is shown in [4] that $\operatorname{vein}(\mathbf{B}^n)$ is of order $n^{3/2}$ (see also [9]).

By (2), for the body K constructed in Theorem 1.1 we have that $\operatorname{vein}(K)$ is of order $n^{3/2}$, while $\operatorname{ill}(K) \geqslant \mathrm{i}(K)$ is exponentially large.

Thus, as an application of Theorem 1.1, we obtain that $\operatorname{ill}(K)$ and $\operatorname{vein}(K)$ are very far from each other for some K.

§2. *Preliminaries.* We will rely heavily on the following estimates of Ω_n by Böröczky and Wintsche [7].

LEMMA 2.1 [7]. *Let* $0 < \varphi < \pi/2$.

$$\Omega_n(\varphi) > \frac{\sin^n \varphi}{\sqrt{2\pi(n+1)}}, \tag{4}$$

$$\Omega_n(\varphi) < \frac{\sin^n \varphi}{\sqrt{2\pi n}\cos\varphi} \quad \text{if } \varphi \leqslant \arccos\frac{1}{\sqrt{n+1}}, \tag{5}$$

$$\Omega_n(t\varphi) < t^n \Omega_n(\varphi) \quad \text{if } 1 < t < \frac{\pi}{2\varphi}. \tag{6}$$

The following is known as Jordan's inequality:

$$\frac{2x}{\pi} \leqslant \sin x \quad \text{for } x \in [0, \pi/2]. \tag{7}$$

§3. *Construction of a spiky ball.* We work in $\mathbb{R}^{n+1}$ instead of $\mathbb{R}^n$ to obtain slightly simpler formulas. We describe a probabilistic construction of $K \subset \mathbb{R}^{n+1}$ which is close to the Euclidean ball and has a large illumination number. We use the standard notation $[N]$ for the set $\{1, \ldots, N\}$, and $|A|$ denotes the cardinality of a set A.

Let N be a fixed positive integer, to be given later. We pick N points, $X_1, \ldots, X_N$, independently and uniformly on the Euclidean unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. Let

$$K = \operatorname{conv}\left(\{\pm X_i : i \in [N]\} \cup \frac{1}{D}\mathbf{B}^{n+1} \right). \tag{8}$$

Clearly, K is o-symmetric and $(1/D)\mathbf{B}^{n+1} \subset K \subset \mathbf{B}^{n+1}$. We need to bound the illumination number of K from below. Let $\pi/4 < \alpha < \pi/2$ be such that $\sin\alpha = 1/D$.

We define two "bad" events, E_1 and E_2. Let E_1 be the event that there are $i \neq j \in [N]$ with $\sphericalangle(X_i, X_j) < \pi - 2\alpha$ or $\sphericalangle(-X_i, X_j) < \pi - 2\alpha$ (see Figure 1). We observe that if E_1 does not occur, then for all $i \in [N]$ we have

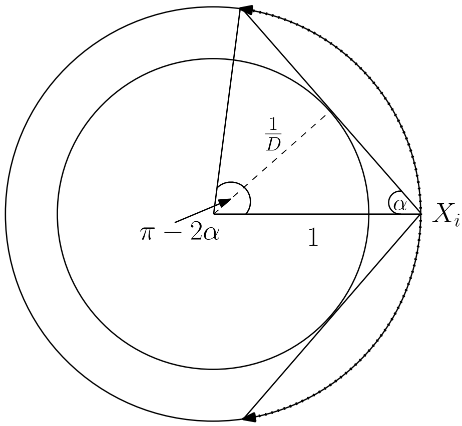

Figure 1: Event E_1: when X_j falls on the dotted cap (the arc with arrows at its end points) or on its reflection about the origin.

The set of directions (a subset of $\mathbb{S}^n$) that illuminate K at X_i is the spherical cap centered at $-X_i$ of spherical radius α. (9)

We want to prove that, with non-zero probability, no point of $\mathbb{S}^n$ belongs to too many of these caps. Thus, to illuminate K at each X_i, we will need many directions.

Let $T \in \mathbb{Z}^+$ be fixed, to be specified later. Let E_2 be the event that there is a direction $u \in \mathbb{S}^n$ with $|C(u, \alpha) \cap \{\pm X_i : i \in [N]\}| > T$.

Observe that if neither E_1 nor E_2 occurs, then $\mathrm{i}(K) \geqslant 2N/T$. However, it is difficult to bound the probability of E_2. Thus, we will replace E_2 by a “more finite” condition E_2' as follows.

We fix a $\delta > 0$. We call a set $\Lambda \subset \mathbb{S}^n$ a δ-*net* (it could also be called a metric δ-net) if $\bigcup_{v \in \Lambda} C(v, \delta) = \mathbb{S}^n$, that is, if the caps of radius δ centered at the points of Λ cover the sphere. By (4), the measure of a cap of radius δ is larger than $\sin^n(\delta)/3\sqrt{n}$. Thus, [13, Theorem 1] yields that there is a covering of the sphere by at most $n^2/\sin^n(\delta)$ caps of radius δ. That is, there is a δ-net Λ of size at most $|\Lambda| \leqslant n^2/\sin^n(\delta)$.

Let $p = 2\Omega_n(\alpha + \delta)$. Let $\Theta > 1$ be fixed, and set $T = N\Theta p$. We define the event E_2' as follows: there is a direction $v \in \Lambda$ with $|C(v, \alpha + \delta) \cap \{\pm X_i : i \in [N]\}| > N\Theta p$. Clearly, if E_2 occurs, then so does E_2'. Thus, we have

$$(\mathrm{not}(E_1) \text{ and } \mathrm{not}(E_2')) \text{ implies } \mathrm{i}(K) \geqslant 2/(\Theta p). \tag{10}$$

Now, we need to set our parameters such that the event (*not*(E_1) *and not*(E_2')) is of positive probability and $2/(\Theta p)$ is exponentially large in the dimension.

Clearly,

$$\mathbb{P}(E_1) \leqslant N^2 \Omega_n(\pi - 2\alpha). \tag{11}$$

Consider a fixed $v \in \Lambda$. When X_i is picked randomly, the probability that v is contained in $C(X_i, \alpha+\delta)$ or in $C(-X_i, \alpha+\delta)$ is p (recall that $p = 2\Omega_n(\alpha+\delta)$). Thus, the probability that v is contained in more than $N\Theta p$ caps of the form $C(\pm X_i, \alpha + \delta)$ is $\mathbb{P}(\xi > N\Theta p)$, where ξ is a binomial random variable of distribution $\mathtt{Binom}(N, p)$. Thus,

$$\mathbb{P}(E_2') \leqslant \frac{n^2}{\sin^n(\delta)}\mathbb{P}(\xi > N\Theta p) \quad \text{with } \xi \sim \mathtt{Binom}(N, p). \tag{12}$$

By a Chernoff-type inequality (cf. [**12**, p. 64]),

$$\mathbb{P}(\xi > N\Theta p) < 2^{-N\Theta p} \quad \text{for any } \Theta \geqslant 6. \tag{13}$$

Consider the following three inequalities:

$$N \leqslant \left(\frac{1}{4\Omega_n(\pi - 2\alpha)}\right)^{1/2}, \tag{14}$$

$$\frac{n^2}{\sin^n \delta} 2^{-\Theta N p} \leqslant \frac{1}{4}, \tag{15}$$

$$6 \leqslant \Theta. \tag{16}$$

Combining (10)–(13), we obtain the following. If there are $N \in \mathbb{Z}^+$, $\delta > 0$ and $\Theta \geqslant 0$ (all depending on n) such that the three inequalities (14)–(16) hold, then there is a $K \subset \mathbb{R}^{n+1}$ o-symmetric convex body with $\mathrm{i}(K) \geqslant 2/(\Theta p)$, where $p = 2\Omega_n(\alpha + \delta)$. In fact, in this case, our construction yields such a K with probability at least $1/2$.

Now, (15) holds if $\Theta N p > 2n \log_2 (1/\sin\delta)$. Thus, an integer N satisfying (14) and (15) exists if

$$4n \log_2 \frac{1}{\sin \delta} \leqslant \Theta p\left(\frac{1}{4\Omega_n(\pi - 2\alpha)}\right)^{1/2},$$

which we rewrite as

$$\frac{1}{\Theta p} \leqslant \frac{1}{8n(\Omega_n(\pi - 2\alpha))^{1/2} \log_2 (1/\sin \delta)}.$$

By (7), we can replace it by the following stronger inequality:

$$\frac{1}{\Theta p} \leqslant \frac{1}{24n(\Omega_n(\pi - 2\alpha))^{1/2} \log_2(1/\delta)}. \tag{17}$$

On the other hand, by substituting the value of p, we see that (16) is equivalent to

$$\frac{1}{\Theta p} \leqslant \frac{1}{12\Omega_n(\alpha + \delta)}. \tag{18}$$

Finally, let $\delta = \alpha/n$.

Since $1 < D = 1/\sin\alpha < 1.116$, we have that $1.11 < \alpha < \pi/2$ and thus $\sin^2(\alpha+\delta) > \sin(\pi - 2\alpha)$. Now, by Lemma 2.1, (18) is a stronger inequality than (17). Thus, so far we have that if we can satisfy (18), then the proof is complete.

By (6), we have that (18) holds if

$$\frac{1}{\Theta p} \leqslant \frac{1}{36\Omega_n(\alpha)}. \tag{19}$$

By (5), it holds for $1/\Theta p = \frac{1}{36}D^n$. Since $\mathrm{i}(K) \geqslant 2/(\Theta p)$, this finishes the proof of Theorem 1.1.

Remark 3.1. The body K is not a polytope. However, the construction can easily be modified to obtain a polytope. One simply replaces the ball of radius $1/D$ by a sufficiently dense finite subset A of this ball in the definition of K as follows: $K = \mathrm{conv}(\{\pm X_i : i \in [N]\} \cup A)$.

Proof of Theorem 1.2. Since $(1/D)\mathbf{B}^n \subset K \subset \mathbf{B}^n$, it follows that for any boundary point b of K, the set of directions (as a subset of $\mathbb{S}^{n-1}$) that illuminate K at b contains an open spherical cap of radius $\alpha = \arcsin(1/D)$. Thus, any subset A of $\mathbb{S}^{n-1}$ that pierces each such cap illuminates K. However, finding such A is equivalent to finding a covering of $\mathbb{S}^{n-1}$ by caps of radius α. Such a covering of the desired size exists by [**13**] (see also [**7**]). □

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