Depth first search in claw-free graphs

Extended abstract

Gábor Wiener*
Department of Computer Science and Information Theory
Budapest University of Technology and Economics
e-mail: wiener@cs.bme.hu

All graphs in this paper are simple, finite, and undirected; the vertex set of a graph G is denoted by $V(G)$. A graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. A graph G is *traceable* if it contains a hamiltonian path. The *minimum leaf number* $\mathrm{ml}(G)$ is the minimum number of leaves (vertices of degree 1) of the spanning trees of G. The *minimum branch number* $\mathrm{s}(G)$ is the minimum number of branches (vertices of degree at least 3) of the spanning trees of G. A tree T is a k*-tree* if all vertices have degree at most k. The minimum degree of G is denoted by $\delta(G)$ and the minimum sum of degrees of k independent vertices of G is denoted by $\delta_k(G)$. The *depth first search (DFS)* of a connected graph G (see e.g. [4]) produces a spanning tree of G, called a DFS-tree, rooted at some node r; the leaves of a DFS-tree different from r will be called *d-leaves* of the DFS-tree.

Hamiltonian properties of claw-free graphs have been examined for more than three decades; one of the early results is due to Matthews and Sumner [6] and was also found independently by Liu, Tian, and Wu [5].

Theorem 1. (Matthews and Sumner, Liu et al., 1985) *Let G be a connected claw-free graph of order n. If $\delta_3(G) \geq n-2$, then G is traceable.*

Gargano, Hammar, Hell, Stacho, and Vaccaro [2] proved a generalization of Theorem 1 concerning the minimum branch number.

Theorem 2. (Gargano et al., 2002) *Let G be a connected claw-free graph of order n and let k be a nonnegative integer. If $\delta_{k+3}(G) \geq n-k-2$, then $\mathrm{s}(G) \leq k$.*

This result was generalized further by Salamon [7].

Theorem 3. (Salamon, 2010) *Let G be a connected claw-free graph of order n and let k be a nonnegative integer. If $\delta_{k+1}(G) \geq n-k$, then $\mathrm{ml}(G) \leq k$.*

Since a branch vertex has degree at least 3, it is obvious that $\mathrm{ml}(G) \geq \mathrm{s}(G)+2$, thus Theorem 3 is a generalization of Theorem 2 indeed. Theorem 3 was rediscovered in 2012 by Kano, Kyaw, Matsuda, Ozeki, Saito, and Yamashita [3] and they also proved a stronger version.

Theorem 4. (Kano et al., 2012) *Let G be a connected claw-free graph of order n and let k be a nonnegative integer. If $\delta_{k+1}(G) \geq n-k$, then G has a spanning 3-tree with at most k leaves.*

The main result of the paper is the following theorem.

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Theorem 5. *Every connected claw-free graph G has a DFS-tree T such that no two of the d-leaves of T have a common neighbour. Moreover, if v is not a cut vertex of G, then T can be chosen such that it is rooted at v.*

Though the proof of Theorem 5 is really short, it is omitted here due to the page limit. On the other hand, we sketch how Theorem 5 implies Theorem 4. Let G be a connected claw-free graph of order n with $\delta_{k+1}(G) \geq n-k$ and let T be a DFS-tree guaranteed by Theorem 5. The set of d-leaves D of T is obviously an independent set, thus the degree sum of the vertices of D is at most $n - |D|$, since all vertices in $V(G) - D$ has at most one neighbour in D. Hence $|D| \leq k$, that is T has at most $k+1$ leaves. Notice that T, like any DFS-tree of a claw-free graph is a 3-tree. In order to find a spanning 3-tree with at most k leaves, we need a further local improvement step, which is omitted here due to lack of space.

Theorem 5 has some other connections with results concerning claw-free graphs, of which we only mention the following corollary.

Corollary 6. *Let G be a connected claw-free graph of diameter at most 2 and let v be a non-cut vertex of G. Then there exists a hamiltonian path of G starting at v.*

Proof. By Theorem 5, there exists a DFS-tree T of G rooted at v, such that no two of the d-leaves of T have a common neighbour. Since the diameter of G is at most 2, this is possible only if T has just one d-leaf, which finishes the proof. □

Corollary 6 is a stronger form of a result of Ainouche, Broersma, and Veldman [1] stating that every connected claw-free graph of diameter at most 2 is traceable (actually they also proved the more general theorem that all m-connected claw-free graphs G with $\alpha(G^2) \leq m+1$ are traceable).

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