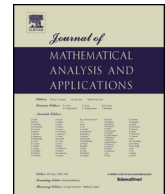




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Turán type inequalities for Struve functions [☆]

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ABSTRACT

Some Turán type inequalities for Struve functions of the first kind are deduced by using various methods developed in the case of Bessel functions of the first and second kind. New formulas, like Mittag-Leffler expansion, infinite product representation for Struve functions of the first kind, are obtained, which may be of independent interest. Moreover, some complete monotonicity results and functional inequalities are deduced for Struve functions of the second kind. These results complement naturally the known results for a particular case of Lommel functions of the first kind, and for modified Struve functions of the first and second kind.

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1. Turán type inequalities for Struve functions of the first kind

Let us start with a well-known relation between Bessel functions of the first kind J_ν and Struve functions of the first kind \mathbf{H}_ν . Namely, for all $n \in \{0, 1, \dots\}$ and $x \in \mathbb{R}$ we have [11, p. 291]

$$\mathbf{H}_{-n-\frac{1}{2}}(x) = (-1)^n J_{n+\frac{1}{2}}(x).$$

Now, let us recall the Turán type inequality for Bessel functions of the first kind, that is,

$$J_\nu^2(x) - J_{\nu-1}(x)J_{\nu+1}(x) \geq 0, \tag{1.1}$$

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where $x \in \mathbb{R}$ and $\nu > -1$. Combining the above relation with (1.1) we obtain the Turán type inequality

$$\mathbf{H}_\nu^2(x) - \mathbf{H}_{\nu-1}(x)\mathbf{H}_{\nu+1}(x) \geq 0, \tag{1.2}$$

which holds for all $x \in \mathbb{R}$ and $\nu \in \{-\frac{1}{2}, -\frac{3}{2}, \dots\}$. Moreover, since for $x \in \mathbb{R}$ and $\nu \geq 0$ the Turán type inequality (1.1) can be improved as

$$J_\nu^2(x) - J_{\nu-1}(x)J_{\nu+1}(x) \geq \frac{1}{\nu+1} J_\nu^2(x),$$

it follows that the inequality (1.2) can be improved too as

$$\mathbf{H}_\nu^2(x) - \mathbf{H}_{\nu-1}(x)\mathbf{H}_{\nu+1}(x) \geq \frac{1}{1-\nu} \mathbf{H}_\nu^2(x), \tag{1.3}$$

which holds for all $x \in \mathbb{R}$ and $\nu \in \{-\frac{1}{2}, -\frac{3}{2}, \dots\}$. For more details on the above Turán type inequalities for Bessel functions of the first kind we refer to the papers [12,14–16] and also to the survey paper [2]. Taking into account the above inequalities it is natural to ask whether the Turán type inequalities (1.2) and/or (1.3) hold true for other values of ν . In this paper we will concentrate on this problem and we present some interesting results concerning Turán type inequalities for Struve functions of the first and second kind. As we can see below the analysis of Struve functions is somewhat more complicated than that of Bessel functions, however, its nature is quite similar for some values of ν . This section is devoted to Turán type inequalities for Struve functions of the first kind, while the next section contains some results, like Turán type inequalities and complete monotonicity results on Struve functions of the second kind. Before we present the main results of this section we first show some preliminary results which will be used in the sequel and which may be of independent interest. Since Struve functions are frequently used in many places in physics and applied mathematics, we believe that our results may be useful for other scientists interested in Struve functions. We also note that the analogous results for modified Struve functions of the first and second kind were already deduced by Baricz and Pogány [3,4] by using the techniques developed in the case of modified Bessel functions of the first and second kind. Moreover, the results presented in this section complement naturally the known results for a particular case of Lommel functions of the first kind, obtained recently by Baricz and Koumandos [1].

The next result is analogous to the well-known result for Bessel functions of the first kind.

Lemma 1. *If $|\nu| \leq \frac{1}{2}$, then the Hadamard factorization of the real entire function $\mathcal{H}_\nu : \mathbb{R} \rightarrow (-\infty, 1]$, defined by $\mathcal{H}_\nu(x) = \sqrt{\pi}2^\nu x^{-\nu-1} \Gamma(\nu + \frac{3}{2}) \mathbf{H}_\nu(x)$, reads as follows*

$$\mathcal{H}_\nu(x) = \prod_{n \geq 1} \left(1 - \frac{x^2}{h_{\nu,n}^2} \right), \tag{1.4}$$

where $h_{\nu,n}$ stands for the n th positive zero of the Struve function \mathbf{H}_ν . The above infinite product is absolutely convergent and if $|\nu| \leq \frac{1}{2}$ and $x \neq h_{\nu,n}$, $n \in \{1, 2, \dots\}$, then the Mittag–Leffler expansion of the Struve function \mathbf{H}_ν is as follows

$$\frac{\mathbf{H}_{\nu-1}(x)}{\mathbf{H}_\nu(x)} = \frac{2\nu+1}{x} + \sum_{n \geq 1} \frac{2x}{x^2 - h_{\nu,n}^2}. \tag{1.5}$$

Proof. By using the power series expansion of the Struve function \mathbf{H}_ν [11, p. 288]

$$\mathbf{H}_\nu(x) = \left(\frac{x}{2}\right)^{\nu+1} \sum_{n \geq 0} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{\Gamma\left(n + \frac{3}{2}\right) \Gamma\left(n + \nu + \frac{3}{2}\right)}$$

we obtain that

$$\mathcal{H}_\nu(x) = \frac{\sqrt{\pi}}{2} \sum_{n \geq 0} \frac{(-1)^n \Gamma(\nu + \frac{3}{2}) x^{2n}}{2^{2n} \Gamma(n + \frac{3}{2}) \Gamma(n + \nu + \frac{3}{2})}.$$

Taking into consideration the well-known limits

$$\lim_{n \rightarrow \infty} \frac{\log \Gamma(n + c)}{n \log n} = 1, \quad \lim_{n \rightarrow \infty} \frac{[\Gamma(n + c)]^{1/n}}{n} = \frac{1}{e},$$

where c is a positive constant, and [9, p. 6, Theorems 2 and 3], we infer that the entire function \mathcal{H}_ν is of growth order $\rho = \frac{1}{2}$ and of exponential type $\sigma = 1$. Namely, for $\nu > -\frac{3}{2}$ we have

$$\rho = \lim_{n \rightarrow \infty} \frac{n \log n}{2n \log 2 + \log \Gamma(n + \frac{3}{2}) + \log \Gamma(n + \nu + \frac{3}{2})} = \frac{1}{2}$$

and

$$\sigma = \frac{1}{\rho e} \lim_{n \rightarrow \infty} \frac{n \sqrt[2n]{\Gamma(\nu + \frac{3}{2})}}{\sqrt[2n]{2^{2n} \Gamma(n + \frac{3}{2}) \Gamma(n + \nu + \frac{3}{2})}} = 1.$$

Now, recall that according to Steinig [13, p. 367] if $|\nu| < \frac{1}{2}$, then all zeros $h_{\nu,n}$ of the Struve function \mathbf{H}_ν are real and simple. Moreover, since [11, p. 291]

$$\mathbf{H}_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad \mathbf{H}_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} (1 - \cos x),$$

it is clear that all zeros of $\mathbf{H}_{-\frac{1}{2}}$ and $\mathbf{H}_{\frac{1}{2}}$ are real and simple. With this the rest of the proof of (1.4) follows by applying Hadamard’s Theorem [9, p. 26]. Now, since the infinite product in (1.4) is absolutely convergent, by taking the logarithm of both sides of (1.4) and then differentiating we obtain

$$\frac{x \mathbf{H}'_\nu(x)}{\mathbf{H}_\nu(x)} = \nu + 1 + \sum_{n \geq 1} \frac{2x^2}{x^2 - h_{\nu,n}^2}, \tag{1.6}$$

where $|\nu| \leq \frac{1}{2}$ and $x \neq h_{\nu,n}$, $n \in \{1, 2, \dots\}$. The rest of the proof of (1.5) follows from (1.6) and

$$\mathbf{H}_{\nu-1}(x) = \frac{\nu}{x} \mathbf{H}_\nu(x) + \mathbf{H}'_\nu(x) \tag{1.7}$$

which is obtained from the relations [11, p. 292]

$$\mathbf{H}_{\nu-1}(x) + \mathbf{H}_{\nu+1}(x) = \frac{2\nu}{x} \mathbf{H}_\nu(x) + \frac{(\frac{x}{2})^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{3}{2})} \tag{1.8}$$

and

$$\mathbf{H}_{\nu-1}(x) - \mathbf{H}_{\nu+1}(x) = 2\mathbf{H}'_\nu(x) - \frac{(\frac{x}{2})^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{3}{2})}. \quad \square \tag{1.9}$$

Now, we are ready to state our main result of this section.

Theorem 1. *The following assertions are valid:*

- a. *If $\nu \in [-\frac{3}{2}, -\frac{1}{2}]$ and $x \in \mathbb{R}$, then the Turán type inequality (1.2) holds true.*
- b. *If $|\nu| \leq \frac{1}{2}$ and $|x| \leq h_{\nu,1}$, then the Turán type inequality (1.2) holds true.*
- c. *If $\nu \in [-\frac{3}{2}, -\frac{1}{2}]$ and $x \in \mathbb{R}$, then*

$$\mathbf{H}_\nu^2(x) - \mathbf{H}_{\nu-1}(x)\mathbf{H}_{\nu+1}(x) \geq \frac{1}{x}\mathbf{H}_\nu(x)\mathbf{H}_{\nu+1}(x). \tag{1.10}$$

Moreover, if $\nu \in (-\frac{3}{2}, -\frac{1}{2}]$ and $|x| < h_{\nu,1}$, then the next Turán type inequality holds

$$\mathbf{H}_\nu^2(x) - \mathbf{H}_{\nu-1}(x)\mathbf{H}_{\nu+1}(x) \geq \frac{1}{2\nu+3}\mathbf{H}_\nu^2(x). \tag{1.11}$$

- d. *If $\nu \geq \frac{3}{2}$ and $|x| \leq \pi$, then the Turán type inequality (1.2) is valid.*
- e. *If $\nu > \frac{3}{2}$ and $|x| < \pi$, then the counterpart of the Turán type inequality (1.2) is as follows:*

$$\mathbf{H}_\nu^2(x) - \mathbf{H}_{\nu-1}(x)\mathbf{H}_{\nu+1}(x) \leq \frac{1}{\nu+\frac{1}{2}}\mathbf{H}_\nu^2(x). \tag{1.12}$$

Proof. a. Let us consider the notation

$$\Delta_\nu(x) = \mathbf{H}_\nu^2(x) - \mathbf{H}_{\nu-1}(x)\mathbf{H}_{\nu+1}(x).$$

By using the recurrence relation (1.7) for ν and $\nu - 1$, and the Mittag-Leffler expansion we obtain that

$$\frac{\Delta_{\nu-1}(x)}{\mathbf{H}_\nu^2(x)} = \frac{1}{x} \frac{\mathbf{H}_{\nu-1}(x)}{\mathbf{H}_\nu(x)} - \left[\frac{\mathbf{H}_{\nu-1}(x)}{\mathbf{H}_\nu(x)} \right]' = \frac{2(2\nu+1)}{x^2} + \sum_{n \geq 1} \frac{4x^2}{(x^2 - h_{\nu,n}^2)^2} \geq 0$$

for all $|\nu| \leq \frac{1}{2}$ and $x > 0$, $x \neq h_{\nu,n}$, $n \in \{1, 2, \dots\}$. Since for each $n \in \{1, 2, \dots\}$ we have

$$\Delta_{\nu-1}(h_{\nu,n}) = \mathbf{H}_{\nu-1}^2(h_{\nu,n}) > 0,$$

it follows that $\Delta_{\nu-1}(x) \geq 0$ for $x > 0$. Moreover, the expression $\Delta_{\nu-1}(x)$ is even in x , and thus the above Turán type inequality is valid for all $x \in \mathbb{R}$ and $|\nu| \leq \frac{1}{2}$. Now, changing ν to $\nu + 1$, we obtain that indeed if $\nu \in [-\frac{3}{2}, -\frac{1}{2}]$ and $x \in \mathbb{R}$, then the Turán type inequality (1.2) holds true.

b. Combining the recurrence relations (1.8) and (1.9) we obtain that

$$\mathbf{H}_{\nu+1}(x) = \frac{\nu}{x}\mathbf{H}_\nu(x) - \mathbf{H}'_\nu(x) + \frac{\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{3}{2}\right)}. \tag{1.13}$$

Now, combining (1.7) and (1.13) it follows that

$$\Delta_\nu(x) = \left(1 - \frac{\nu^2}{x^2}\right)\mathbf{H}_\nu^2(x) + [\mathbf{H}'_\nu(x)]^2 - \frac{\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{3}{2}\right)}\mathbf{H}_{\nu-1}(x).$$

On the other hand, the Struve function is the particular solution of the Struve differential equation [11, p. 288] and consequently we have

$$\mathbf{H}''_\nu(x) + \frac{1}{x}\mathbf{H}'_\nu(x) + \left(1 - \frac{\nu^2}{x^2}\right)\mathbf{H}_\nu(x) = \frac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)},$$

which implies that

$$\Delta_\nu(x) = \frac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \left[\mathbf{H}_\nu(x) - \frac{x}{2\nu + 1} \mathbf{H}_{\nu-1}(x) \right] - \frac{1}{x} \mathbf{H}_\nu^2(x) \left[\frac{x \mathbf{H}'_\nu(x)}{\mathbf{H}_\nu(x)} \right]'. \tag{1.14}$$

Differentiating both sides of (1.6) we obtain

$$\left[\frac{x \mathbf{H}'_\nu(x)}{\mathbf{H}_\nu(x)} \right]' = - \sum_{n \geq 1} \frac{4x h_{\nu,n}^2}{(x^2 - h_{\nu,n}^2)^2} < 0$$

for all $x > 0$, $x \neq h_{\nu,n}$, $n \in \{1, 2, \dots\}$ and $|\nu| \leq \frac{1}{2}$. Moreover, by using the Mittag-Leffler expansion (1.5) we obtain that

$$\frac{x \mathbf{H}_{\nu-1}(x)}{\mathbf{H}_\nu(x)} < 2\nu + 1 \tag{1.15}$$

for all $|\nu| \leq \frac{1}{2}$ and $x \in (0, h_{\nu,1})$. Finally, by using the above inequalities together with (1.14) we conclude that (1.2) is valid for $|\nu| \leq \frac{1}{2}$ and $x \in (0, h_{\nu,1})$. Since $\Delta_\nu(0) = 0$, in (1.2) we have equality when $x = 0$. Moreover, if $|\nu| \leq 1/2$, then $\Delta_\nu(h_{\nu,1}) = -\mathbf{H}_{\nu-1}(h_{\nu,1})\mathbf{H}_{\nu+1}(h_{\nu,1}) > 0$, since the smallest positive zero of $\mathbf{H}_{\nu-1}$ is nearer the origin than that of \mathbf{H}_ν , and the positive zeros of \mathbf{H}_ν and $\mathbf{H}_{\nu-1}$ separate each other, according to [13, p. 373]. Now, since the expression $\Delta_\nu(x)$ is even in x , these in turn imply that (1.2) holds true for $|x| \leq h_{\nu,1}$.

c. By using (1.7) for ν and $\nu + 1$ we obtain

$$\frac{\Delta_\nu(x)}{\mathbf{H}_\nu^2(x)} = \frac{1}{x} \frac{\mathbf{H}_{\nu+1}(x)}{\mathbf{H}_\nu(x)} + \left[\frac{\mathbf{H}_{\nu+1}(x)}{\mathbf{H}_\nu(x)} \right]'$$

On the other hand, the Mittag-Leffler expansion (1.5) for $\nu + 1$ instead of ν implies that

$$\left[\frac{\mathbf{H}_\nu(x)}{\mathbf{H}_{\nu+1}(x)} \right]' = -\frac{2\nu + 3}{x^2} + \sum_{n \geq 1} \frac{-2(x^2 + h_{\nu+1,n}^2)}{(x^2 - h_{\nu+1,n}^2)^2} \leq 0$$

for all $\nu \in [-\frac{3}{2}, -\frac{1}{2}]$, and $x \neq h_{\nu+1,n}$, $n \in \{1, 2, \dots\}$. This implies that the Turán type inequality (1.10) is valid for $x \in \mathbb{R}$ and $\nu \in [-\frac{3}{2}, -\frac{1}{2}]$. By using [13, p. 373, Proposition 6] and (1.15) for $\nu + 1$ instead of ν , we obtain that (1.11) holds for $x \in (0, h_{\nu,1})$ and $\nu \in (-\frac{3}{2}, -\frac{1}{2}]$, and hence (1.11) is valid for $|x| < h_{\nu,1}$ and $\nu \in (-\frac{3}{2}, -\frac{1}{2}]$.

d. By using (1.7), (1.9) and (1.13) we obtain

$$\Delta'_\nu(x) = \frac{2}{x} \mathbf{H}_{\nu-1}(x) \mathbf{H}_{\nu+1}(x) + \frac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \left[\frac{x}{2\nu + 1} \mathbf{H}_\nu(x) - \mathbf{H}_{\nu+1}(x) \right].$$

In view of the integral representation of \mathbf{H}_ν [11, p. 292]

$$\mathbf{H}_\nu(x) = \frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \sin(xt) dt, \tag{1.16}$$

we get that

$$\frac{x}{2\nu + 1} \mathbf{H}_\nu(x) - \mathbf{H}_{\nu+1}(x) = \frac{2\left(\frac{x}{2}\right)^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu + \frac{3}{2}\right)} \int_0^1 t^2 (1-t^2)^{\nu-\frac{1}{2}} \sin(xt) dt,$$

which is nonnegative if $x \in [0, \pi]$. On the other hand, it is known [11, p. 291] that $\mathbf{H}_\nu(x) \geq 0$ if $x > 0$ and $\nu \geq \frac{1}{2}$, and combining this with the above result, we obtain that $\Delta'_\nu(x) \geq 0$ when $x \in [0, \pi]$ and $\nu \geq \frac{3}{2}$. This implies that $\Delta_\nu(x) \geq 0$ when $x \in [0, \pi]$ and $\nu \geq \frac{3}{2}$, and using again the fact that $\Delta_\nu(x)$ is even in x , we conclude that (1.2) holds for $|x| \leq \pi$ and $\nu \geq \frac{3}{2}$.

e. We define the function $\mathbb{H}_\nu : \mathbb{R} \rightarrow \mathbb{R}$ by $\mathbb{H}_\nu(x) = 2^\nu x^{-\nu} \Gamma(\nu + \frac{1}{2}) \mathbf{H}_\nu(x)$. In view of (1.16) this function may be represented as

$$\mathbb{H}_\nu(x) = \frac{2}{\sqrt{\pi}} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \sin(xt) dt.$$

Since the above integrand is log-convex in ν when $x \in (0, \pi)$ and the integral preserves the log-convexity, it follows that $\nu \mapsto \mathbb{H}_\nu(x)$ is log-convex on $(\frac{1}{2}, \infty)$ for $x \in (0, \pi)$ fixed. Here we used tacitly the inequality [11, p. 291] $\mathbf{H}_\nu(x) > 0$, which holds for $\nu > \frac{1}{2}$ and $x > 0$. Thus, for all $\nu_1, \nu_2 > \frac{1}{2}$, $\alpha \in [0, 1]$ and $x > 0$ we have

$$\mathbb{H}_{\alpha\nu_1+(1-\alpha)\nu_2}(x) \leq [\mathbb{H}_{\nu_1}(x)]^\alpha [\mathbb{H}_{\nu_2}(x)]^{1-\alpha}.$$

Choosing $\nu_1 = \nu - 1$, $\nu_2 = \nu + 1$, $\alpha = \frac{1}{2}$, the above inequality reduces to the Turán type inequality

$$\mathbb{H}_\nu^2(x) - \mathbb{H}_{\nu-1}(x)\mathbb{H}_{\nu+1}(x) < 0,$$

which is valid for $\nu > \frac{3}{2}$ and $x > 0$, and this is equivalent to the Turán type inequality (1.12). □

Concluding remarks and further results. A. We note that the right-hand side of (1.3) is negative when $\nu > 1$, and thus the Turán type inequality (1.3) is interesting only when $\nu < 1$. Now, if one looks at the right-hand sides of (1.3) and (1.11) it is natural to ask what is best constant α_ν depending on ν and not depending on x for which we have the Turán type inequality

$$\mathbf{H}_\nu^2(x) - \mathbf{H}_{\nu-1}(x)\mathbf{H}_{\nu+1}(x) \geq \alpha_\nu \mathbf{H}_\nu^2(x).$$

Since close to the origin the Struve function behaves as a simple power, that is, as $x \rightarrow 0$ we have

$$\mathbf{H}_\nu(x) \sim \frac{x^{\nu+1}}{2^\nu \sqrt{\pi} \Gamma(\nu + \frac{3}{2})},$$

it follows that as $x \rightarrow 0$ we get

$$\frac{\Delta_\nu(x)}{\mathbf{H}_\nu^2(x)} \sim \frac{1}{\nu + \frac{3}{2}}.$$

This implies that $\alpha_\nu = 1/(\nu + \frac{3}{2})$, and then (1.11) can be improved. Now, as the argument approaches infinity, the Struve function generally behaves either as a power or as a damped sinusoid, that is, for $\nu > \frac{1}{2}$ we have

$$\mathbf{H}_\nu(x) \sim \frac{x^{\nu-1}}{2^{\nu-1} \sqrt{\pi} \Gamma(\nu + \frac{1}{2})},$$

which implies that as $x \rightarrow \infty$ and $\nu > \frac{3}{2}$,

$$\frac{\Delta_\nu(x)}{\mathbf{H}_\nu^2(x)} \sim \frac{1}{\nu + \frac{1}{2}}.$$

This shows that the constant $1/(\nu + \frac{1}{2})$ in (1.12) is optimal, and cannot be improved by using any other constant depending only on ν . Thus, in this sense the Turán type inequality (1.12) is sharp.

B. It is worth also to mention that by using the formula [11, p. 292]

$$\mathbf{H}_\nu(x) = \left(\frac{x}{2\pi}\right)^{\frac{1}{2}} \sum_{n \geq 0} \frac{\left(\frac{x}{2}\right)^n}{n! \left(n + \frac{1}{2}\right)} J_{n+\nu+\frac{1}{2}}(x),$$

we obtain that

$$\Delta_\nu(x) = \frac{x}{2\pi} \sum_{n \geq 0} \sum_{m=0}^n \frac{\left(\frac{x}{2}\right)^n}{n!(n-m)! \left(m + \frac{1}{2}\right) \left(n-m + \frac{1}{2}\right)} \cdot {}_J\Delta_\nu(x),$$

where

$${}_J\Delta_\nu(x) = J_{m+\nu+\frac{1}{2}}(x)J_{n-m+\nu+\frac{1}{2}}(x) - J_{m+\nu-\frac{1}{2}}(x)J_{n-m+\nu+\frac{3}{2}}(x).$$

Now, by using Nicholson’s formula [11, p. 225]

$$J_\mu(z)J_\nu(z) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} J_{\mu+\nu}(2z \cos \theta) \cos(\mu - \nu) d\theta$$

we obtain that

$${}_J\Delta_\nu(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} J_{n+2\nu+1}(2x \cos \theta) [\cos(k\theta) - \cos((k+2)\theta)] d\theta = \frac{1}{\pi x} \int_0^{2x} J_{n+2\nu+1}(u)\Phi(u)du,$$

where

$$\Phi(u) = \Phi(2x \cos \theta) = \frac{\cos(k\theta) - \cos((k+2)\theta)}{\sin \theta}$$

and $k = n - 2m$. Thus, to verify the Turán type inequality $\Delta_\nu(x) \geq 0$ it would be enough to show the positivity of the integral $\int_0^{2x} J_{n+2\nu+1}(u)\Phi(u)du$. On the other hand, according to [5, Lemma 2.1], if the function φ is positive non-increasing and continuous for $0 < t < x$, then for $\nu > -1$ and $x > 0$ we have $\int_0^x J_\nu(t)\varphi(t)dt > 0$. Consequently, it would be enough to show that Φ is positive and non-increasing. However, we were unable to show this for each $m \in \{0, 1, \dots, n\}$ and $n \in \{0, 1, \dots\}$.

C. We would like to note also that the infinite product representation (1.4) and the Mittag-Leffler expansion (1.5) may be useful also to deduce other inequalities for the Struve function \mathbf{H}_ν . For example, we can obtain some lower and upper bounds for \mathbf{H}_ν in terms of the Bessel function of the first kind J_ν . According to Steinig [13, p. 367] for all $n \in \{1, 2, \dots\}$ and $|\nu| < \frac{1}{2}$ we have that $j_{\nu,n} < h_{\nu,n} < j_{\nu,n+1}$, where $j_{\nu,n}$ stands for the n th positive zero of the Bessel function J_ν . By using these inequalities we obtain for $|\nu| < \frac{1}{2}$ and $x \in (0, j_{\nu,1})$

$$\prod_{n \geq 1} \left(1 - \frac{x^2}{j_{\nu,n}^2}\right) < \prod_{n \geq 1} \left(1 - \frac{x^2}{h_{\nu,n}^2}\right) < \prod_{n \geq 1} \left(1 - \frac{x^2}{j_{\nu,n+1}^2}\right),$$

which in turn implies that

$$\frac{\Gamma(\nu + 1)}{\Gamma(\nu + \frac{3}{2})} x J_\nu(x) < \sqrt{\pi} \mathbf{H}_\nu(x) < \frac{\Gamma(\nu + 1)}{\Gamma(\nu + \frac{3}{2})} \frac{j_{\nu,1}^2}{j_{\nu,1}^2 - x^2} x J_\nu(x).$$

Here we used the infinite product representation of the Bessel function of the first kind [11, p. 235]

$$J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu + 1)} \prod_{n \geq 1} \left(1 - \frac{x^2}{j_{\nu,n}^2}\right).$$

Another example is an improvement of the inequality (1.15). Namely, by using again the inequalities $h_{\nu,n} < j_{\nu,n+1}$, $n \in \{1, 2, \dots\}$, we obtain for $x \in (0, h_{\nu,1})$ and $|\nu| \leq \frac{1}{2}$ the inequality

$$\sum_{n \geq 1} \frac{2}{x^2 - h_{\nu,n}^2} < - \sum_{n \geq 1} \frac{2}{h_{\nu,n}^2} < - \sum_{n \geq 1} \frac{2}{j_{\nu,n+1}^2} = \frac{2}{j_{\nu,1}^2} - \sum_{n \geq 1} \frac{2}{j_{\nu,n}^2} = \frac{2}{j_{\nu,1}^2} - \frac{1}{2(\nu + 1)},$$

which implies that

$$\frac{x\mathbf{H}_{\nu-1}(x)}{\mathbf{H}_\nu(x)} < 2\nu + 1 + \left(\frac{2}{j_{\nu,1}^2} - \frac{1}{2(\nu + 1)}\right) x^2.$$

Note that this is indeed an improvement of (1.15) since [6, eq. 6.7] $j_{\nu,1}^2 > 4(\nu + 1)$ for $\nu > -1$.

D. We mention that the Turán type inequality (1.10) can be deduced from a Laguerre type inequality. More precisely, by using (1.4) and (1.7) we obtain that

$$\left[\frac{\mathcal{H}'_\nu(x)}{\mathcal{H}_\nu(x)}\right]' = \frac{\nu + 1}{x^2} + \left[\frac{\mathbf{H}'_\nu(x)}{\mathbf{H}_\nu(x)}\right]' = \frac{2\nu + 1}{x^2} + \left[\frac{\mathbf{H}_{\nu-1}(x)}{\mathbf{H}_\nu(x)}\right]' = - \sum_{n \geq 1} \frac{2(h_{\nu,n}^2 + x^2)}{(h_{\nu,n}^2 - x^2)^2},$$

which implies that the next Laguerre type inequality is valid

$$[\mathcal{H}'_\nu(x)]^2 - \mathcal{H}_\nu(x)\mathcal{H}''_\nu(x) \geq 0 \tag{1.17}$$

for all $|\nu| \leq \frac{1}{2}$ and $x \in \mathbb{R}$. But, this inequality is equivalent to

$$\Delta_{\nu-1}(x) \geq \frac{2\nu + 1}{x^2} \mathbf{H}_\nu^2(x) + \frac{1}{x} \mathbf{H}_\nu(x)\mathbf{H}_{\nu-1}(x),$$

which implies (1.10). Note that (1.17) is in fact a particular case of a more general inequality. For this recall that the real entire function ϕ , defined by

$$\phi(z) = \varphi(z; t) = \sum_{n \geq 0} b_n(t) \frac{z^n}{n!},$$

is said to be in the Laguerre–Pólya class, if $\phi(z)$ can be expressed in the form

$$\phi(z) = cz^d e^{-\alpha z^2 + \beta z} \prod_{n=1}^{\omega} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n}}, \quad 0 \leq \omega \leq \infty,$$

where c and β are real, z_n 's are real and nonzero for all $n \in \{1, 2, \dots, \omega\}$, $\alpha \geq 0$, d is a nonnegative integer and $\sum_{n=1}^{\omega} z_n^{-2} < \infty$. If $\omega = 0$, then, by convention, the product is defined to be 1. Now, recall the following result (for more details we refer to Skovgaard's paper [12]): if a real entire function ϕ belongs to the Laguerre–Pólya class then satisfies the Laguerre type inequalities

$$\left[\phi^{(m)}(z)\right]^2 - \phi^{(m-1)}(z)\phi^{(m+1)}(z) \geq 0,$$

for $m \in \{1, 2, \dots\}$ and all admissible values of z . The infinite product representation (1.4), and the result of Steing [13] concerning the fact that the zeros of the Struve function \mathbf{H}_ν are real when $|\nu| \leq \frac{1}{2}$, show that the function \mathcal{H}_ν belongs to the Laguerre–Pólya class, since exponential factors in the product are cancelled due to the symmetry of the zeros with respect to the origin. This in turn implies that the following Laguerre type inequality is valid for all $|\nu| \leq \frac{1}{2}$, $x \in \mathbb{R}$ and $m \in \{1, 2, \dots\}$

$$\left[\mathcal{H}_\nu^{(m)}(x)\right]^2 - \mathcal{H}_\nu^{(m-1)}(x)\mathcal{H}_\nu^{(m+1)}(x) \geq 0.$$

For $m = 1$ the above inequality reduces to (1.17).

E. Finally, we note that from Hadamard theorem [9, p. 26] actually we get that the infinite product in (1.4) is absolutely convergent on compact subsets of the complex plane. This in turn implies that if we take the purely imaginary number ix instead of x in (1.5), then for $|\nu| \leq \frac{1}{2}$ and $x \in \mathbb{R}$ we get the new Mittag–Leffler expansion

$$\frac{\mathbf{L}_{\nu-1}(x)}{\mathbf{L}_\nu(x)} = \frac{2\nu + 1}{x} + \sum_{n \geq 1} \frac{2x}{x^2 + h_{\nu,n}^2}, \tag{1.18}$$

where

$$\mathbf{L}_\nu(x) = -ie^{-\frac{1}{2}\pi i\nu} \mathbf{H}_\nu(ix) = \left(\frac{x}{2}\right)^{\nu+1} \sum_{n \geq 0} \frac{\left(\frac{x}{2}\right)^{2n}}{\Gamma\left(n + \frac{3}{2}\right) \Gamma\left(n + \nu + \frac{3}{2}\right)}$$

stands for the modified Struve function of the first kind (see [11, p. 288]). Moreover, following the lines of the proof of part **a** of Theorem 1 we obtain that

$$\frac{\mathbf{L}_{\nu-1}^2(x) - \mathbf{L}_{\nu-2}(x)\mathbf{L}_\nu(x)}{\mathbf{L}_\nu^2(x)} = \frac{1}{x} \frac{\mathbf{L}_{\nu-1}(x)}{\mathbf{L}_\nu(x)} - \left[\frac{\mathbf{L}_{\nu-1}(x)}{\mathbf{L}_\nu(x)}\right]' = \frac{2(2\nu + 1)}{x^2} + \sum_{n \geq 1} \frac{4x^2}{(x^2 + h_{\nu,n}^2)^2} \geq 0$$

for all $|\nu| \leq \frac{1}{2}$ and $x \neq 0$. Thus, we obtained the Turán type inequality

$$\mathbf{L}_\nu^2(x) - \mathbf{L}_{\nu-1}(x)\mathbf{L}_{\nu+1}(x) \geq 0,$$

where $\nu \in \left[-\frac{3}{2}, -\frac{1}{2}\right]$ and $x \in \mathbb{R}$. For $\nu > -\frac{3}{2}$ and $x > 0$ this Turán type inequality was proved also in [4,7] by using completely different methods.

2. Turán type inequalities for Struve functions of the second kind

In this section we concentrate on the Struve function of the second kind \mathbf{K}_ν , which for $\nu > -\frac{1}{2}$ has the integral representation [11, p. 292]

$$\mathbf{K}_\nu(x) = \mathbf{H}_\nu(x) - Y_\nu(x) = \frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty (1+t^2)^{\nu-\frac{1}{2}} e^{-xt} dt. \tag{2.1}$$

Here Y_ν stands for the Bessel function of the second kind of order ν . Now, consider the function $\mathcal{K}_\nu : \mathbb{R} \rightarrow (0, \infty)$, defined by $\mathcal{K}_\nu(x) = 2^\nu x^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) \mathbf{K}_\nu(x)$. The next result is the counterpart of the similar results for modified Struve functions of the first and second kind, see [3,4].

Theorem 2. *The following assertions are true:*

- a. *The function $x \mapsto \mathcal{K}_\nu(x)$ is completely monotonic and log-convex on $(0, \infty)$ for all $\nu > -\frac{1}{2}$. Moreover, the following inequality is valid for $\nu > -\frac{1}{2}$ and $x > 0$*

$$\frac{x\mathbf{K}'_\nu(x)}{\mathbf{K}_\nu(x)} < \nu. \tag{2.2}$$

- b. *The function $\nu \mapsto \mathcal{K}_\nu(x)$ is completely monotonic and log-convex on $(-\frac{1}{2}, \infty)$ for all $x > 0$. Moreover, the following Turán type inequality is valid for all $x > 0$ and $\nu > \frac{1}{2}$*

$$\mathbf{K}_\nu^2(x) - \mathbf{K}_{\nu-1}(x)\mathbf{K}_{\nu+1}(x) \leq \frac{1}{\nu + \frac{1}{2}}\mathbf{K}_\nu^2(x). \tag{2.3}$$

- c. *The function $x \mapsto \mathbf{K}_\nu(x)$ is completely monotonic and log-convex on $(0, \infty)$ for all $\nu \in (-\frac{1}{2}, 0]$.*
- d. *The function $x \mapsto x\mathbf{K}'_\nu(x)/\mathbf{K}_\nu(x)$ is increasing on $(0, \infty)$ for all $\nu > \frac{1}{2}$. Moreover, the following Turán type inequality holds for $\nu > -\frac{1}{2}$ and $x > 0$*

$$\mathbf{K}_\nu^2(x) - \mathbf{K}_{\nu-1}(x)\mathbf{K}_{\nu+1}(x) \leq \frac{2}{x}\mathbf{K}_\nu(x)\mathbf{K}_{\nu+1}(x). \tag{2.4}$$

- e. *For all $x > 0$ and $\nu \in (-\frac{1}{2}, 0)$ we have*

$$\mathcal{K}_\nu(x) < \frac{\Gamma(-\nu)}{\Gamma(\frac{1}{2} - \nu)}. \tag{2.5}$$

- f. *For all $x, y > 0$ and $\nu \in (-\frac{1}{2}, 0)$ we have*

$$\mathcal{K}_\nu(x + y) \geq \frac{\Gamma(\frac{1}{2} - \nu)}{\Gamma(-\nu)}\mathcal{K}_\nu(x)\mathcal{K}_\nu(y). \tag{2.6}$$

- g. *For all $\nu > \frac{3}{2}$ and $x > 0$ we have*

$$\mathcal{K}_{\nu-1}(x)\mathcal{K}_{\nu+1}(x) < \mathcal{K}_{\frac{1}{2}}(x)\mathcal{K}_{2\nu-\frac{1}{2}}(x) \tag{2.7}$$

and the reverse inequality holds when $\nu \in (\frac{1}{2}, \frac{3}{2})$ and $x > 0$.

Proof. a. & b. By using (2.1) we obtain that for $\nu > -\frac{1}{2}$ the next integral representation is valid

$$\mathcal{K}_\nu(x) = 2^\nu x^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) \mathbf{K}_\nu(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty (1+t^2)^{\nu-\frac{1}{2}} e^{-xt} dt. \tag{2.8}$$

Consequently, for $n, m \in \{0, 1, 2, \dots\}$ and $\nu > -\frac{1}{2}$ we have

$$(-1)^n [\mathcal{K}_\nu(x)]^{(n)} = \frac{2}{\sqrt{\pi}} \int_0^\infty t^n (1+t^2)^{\nu-\frac{1}{2}} e^{-xt} dt,$$

and

$$(-1)^m \frac{\partial^m \mathcal{K}_\nu(x)}{\partial \nu^m} = \frac{2}{\sqrt{\pi}} \int_0^\infty \left(\log \frac{1}{1+t^2} \right)^m (1+t^2)^{\nu-\frac{1}{2}} e^{-xt} dt.$$

Therefore the functions $x \mapsto \mathcal{K}_\nu(x)$ and $\nu \mapsto \mathcal{K}_\nu(x)$ are completely monotonic and hence are log-convex, since every completely monotonic function is log convex (see [17, p. 167]). Alternatively, the log-convexity of these functions can be proved also by using the Hölder–Rogers inequality for integrals. Complete monotonicity and log-convexity of \mathcal{K}_ν can be concluded also by noticing that this function is in fact a Laplace transform.

Now, to prove the Turán type inequality (2.3), note that $\nu \mapsto \mathcal{K}_\nu(x)$ is log convex on $(-\frac{1}{2}, \infty)$ for all $x > 0$, which implies that for all $\nu_1, \nu_2 > -\frac{1}{2}$, $\alpha \in [0, 1]$ and $x > 0$ we have

$$\mathcal{K}_{\alpha\nu_1+(1-\alpha)\nu_2}(x) \leq [\mathcal{K}_{\nu_1}(x)]^\alpha [\mathcal{K}_{\nu_2}(x)]^{1-\alpha}.$$

Choosing $\nu_1 = \nu - 1$, $\nu_2 = \nu + 1$, and $\alpha = \frac{1}{2}$, the above inequality reduces to the Turán type inequality

$$\mathcal{K}_\nu^2(x) - \mathcal{K}_{\nu-1}(x)\mathcal{K}_{\nu+1}(x) \leq 0,$$

which is equivalent to the inequality (2.3).

Alternatively, (2.3) can be proved also as follows. For this let us consider the notation

$$\Theta_\nu(x) = \mathbf{K}_\nu^2(x) - \mathbf{K}_{\nu-1}(x)\mathbf{K}_{\nu+1}(x).$$

By using the integral representation (2.1) we get

$$\begin{aligned} \Theta_\nu(x) &= \frac{4}{\pi} \left(\frac{x}{2}\right)^{2\nu} \int_0^\infty \int_0^\infty e^{-x(t+s)} (1+t^2)^{\nu-\frac{1}{2}} (1+s^2)^{\nu-\frac{3}{2}} \left[\frac{1+s^2}{\Gamma^2(\nu+\frac{1}{2})} - \frac{1+t^2}{\Gamma(\nu-\frac{1}{2})\Gamma(\nu+\frac{3}{2})} \right] dt ds \\ &= \frac{4}{\pi} \left(\frac{x}{2}\right)^{2\nu} \int_0^\infty \int_0^\infty e^{-x(t+s)} (1+s^2)^{\nu-\frac{1}{2}} (1+t^2)^{\nu-\frac{3}{2}} \left[\frac{1+t^2}{\Gamma^2(\nu+\frac{1}{2})} - \frac{1+s^2}{\Gamma(\nu-\frac{1}{2})\Gamma(\nu+\frac{3}{2})} \right] dt ds \\ &= \frac{2}{\pi} \left(\frac{x}{2}\right)^{2\nu} \int_0^\infty \int_0^\infty e^{-x(t+s)} (1+t^2)^{\nu-\frac{3}{2}} (1+s^2)^{\nu-\frac{3}{2}} \cdot \mathbf{E}_\nu(t, s) dt ds, \end{aligned}$$

where

$$\begin{aligned} \mathbf{E}_\nu(t, s) &= (1+t^2) \left[\frac{1+s^2}{\Gamma^2(\nu+\frac{1}{2})} - \frac{1+t^2}{\Gamma(\nu-\frac{1}{2})\Gamma(\nu+\frac{3}{2})} \right] + (1+s^2) \left[\frac{1+t^2}{\Gamma^2(\nu+\frac{1}{2})} - \frac{1+s^2}{\Gamma(\nu-\frac{1}{2})\Gamma(\nu+\frac{3}{2})} \right] \\ &= \frac{2(1+t^2)(1+s^2)}{\Gamma^2(\nu+\frac{1}{2})} - \frac{(1+t^2)^2}{\Gamma(\nu-\frac{1}{2})\Gamma(\nu+\frac{3}{2})} - \frac{(1+s^2)^2}{\Gamma(\nu-\frac{1}{2})\Gamma(\nu+\frac{3}{2})} \\ &= \frac{1}{\Gamma(\nu+\frac{1}{2})\Gamma(\nu+\frac{3}{2})} \mathbf{d}_\nu(t, s) \end{aligned}$$

and

$$\mathbf{d}_\nu(t, s) = 2 \left(\nu + \frac{1}{2}\right) (1+t^2)(1+s^2) - \left(\nu - \frac{1}{2}\right) (1+t^2)^2 - \left(\nu - \frac{1}{2}\right) (1+s^2)^2.$$

Since

$$\mathbf{d}_\nu(t, s) \leq 2(1+t^2)(1+s^2) \iff -\left(\nu - \frac{1}{2}\right) (t^2 - s^2)^2 \leq 0$$

for all $t, s \geq 0$ and $\nu > \frac{1}{2}$, it follows that

$$\Theta_\nu(x) \leq \frac{4}{\pi} \left(\frac{x}{2}\right)^{2\nu} \frac{1}{\Gamma\left(\nu + \frac{1}{2}\right)\Gamma\left(\nu + \frac{3}{2}\right)} \int_0^\infty \int_0^\infty e^{-x(t+s)}(1+t^2)^{\nu-\frac{1}{2}}(1+s^2)^{\nu-\frac{1}{2}} dt ds,$$

which is equivalent to (2.3).

Now, we prove the inequality (2.2). Since for $\nu > -\frac{1}{2}$ the function $x \mapsto x^{-\nu}\mathbf{K}_\nu(x)$ is completely monotonic on $(0, \infty)$, in particular it is also decreasing. Consequently, the function $x \mapsto \log(x^{-\nu}\mathbf{K}_\nu(x))$ is also decreasing on $(0, \infty)$ for $\nu > -\frac{1}{2}$ and hence $(\log(x^{-\nu}\mathbf{K}_\nu(x)))' < 0$, which in turn implies (2.2).

c. By definition of $\mathcal{K}_\nu(x)$ we have

$$\mathbf{K}_\nu(x) = \frac{x^\nu \mathcal{K}_\nu(x)}{2^\nu \Gamma\left(\nu + \frac{1}{2}\right)}.$$

Note that $x \mapsto x^\nu$ is completely monotonic on $(0, \infty)$ for all $\nu \leq 0$. Thus by part a of this theorem the function $x \mapsto \mathbf{K}_\nu(x)$, as a product of two completely monotonic functions, is completely monotonic and hence log-convex on $(0, \infty)$ for all $\nu \in \left(-\frac{1}{2}, 0\right]$.

d. By using $\mathbf{K}_\nu(x) = \mathbf{H}_\nu(x) - Y_\nu(x)$ and the corresponding recurrence relations for \mathbf{H}_ν and Y_ν we can see that the Struve function of the second kind \mathbf{K}_ν satisfies the same recurrence relations like the Struve function of the first kind \mathbf{H}_ν , that is, if we replace \mathbf{H}_ν with \mathbf{K}_ν in (1.7), (1.8), (1.9) and (1.13), then these recurrence relations remain true. The analogous of (1.7) is the following

$$\mathbf{K}_{\nu-1}(x) = \frac{\nu}{x}\mathbf{K}_\nu(x) + \mathbf{K}'_\nu(x). \tag{2.9}$$

Now, by using (2.8) and (2.9) we obtain

$$\begin{aligned} \left[\frac{x\mathbf{K}'_\nu(x)}{\mathbf{K}_\nu(x)}\right]' &= \left[\frac{x\mathbf{K}_{\nu-1}(x)}{\mathbf{K}_\nu(x)}\right]' = (2\nu - 1) \left[\frac{\int_0^\infty (1+t^2)^{\nu-\frac{3}{2}} e^{-xt} dt}{\int_0^\infty (1+t^2)^{\nu-\frac{1}{2}} e^{-xt} dt} \right]' \\ &= (2\nu - 1) \frac{\int_0^\infty \int_0^\infty (1+t^2)^{\nu-\frac{3}{2}}(1+s^2)^{\nu-\frac{1}{2}} e^{-x(t+s)}(s-t) dt ds}{\left(\int_0^\infty (1+t^2)^{\nu-\frac{1}{2}} e^{-xt} dt\right)^2} \\ &= (2\nu - 1) \frac{\int_0^\infty \int_0^\infty (1+s^2)^{\nu-\frac{3}{2}}(1+t^2)^{\nu-\frac{1}{2}} e^{-x(t+s)}(t-s) dt ds}{\left(\int_0^\infty (1+t^2)^{\nu-\frac{1}{2}} e^{-xt} dt\right)^2} \\ &= \left(\nu - \frac{1}{2}\right) \frac{\int_0^\infty \int_0^\infty ((1+t^2)(1+s^2))^{\nu-\frac{3}{2}} e^{-x(t+s)}(t-s)^2(t+s) dt ds}{\left(\int_0^\infty (1+t^2)^{\nu-\frac{1}{2}} e^{-xt} dt\right)^2}. \end{aligned}$$

Thus, indeed the function $x \mapsto x\mathbf{K}'_\nu(x)/\mathbf{K}_\nu(x)$ is increasing on $(0, \infty)$ for all $\nu > \frac{1}{2}$. Now, appealing to the above result and to the recurrence relation (2.9) we obtain for $x > 0$ and $\nu > \frac{1}{2}$

$$0 \leq \left[\frac{x\mathbf{K}_{\nu-1}(x)}{\mathbf{K}_\nu(x)}\right]' = 2\frac{\mathbf{K}_{\nu-1}(x)}{\mathbf{K}_\nu(x)} - \frac{x\Theta_{\nu-1}(x)}{\mathbf{K}_{\nu-1}(x)}.$$

Changing ν with $\nu + 1$ we get (2.4). Alternatively, (2.4) can be proved by using

$$0 \geq \left[\frac{\mathbf{K}_{\nu+1}(x)}{x\mathbf{K}_\nu(x)}\right]' = \frac{\Theta_\nu(x)}{x\mathbf{K}_\nu^2(x)} - \frac{2}{x^2} \frac{\mathbf{K}_{\nu+1}(x)}{\mathbf{K}_\nu(x)},$$

where $x > 0$ and $\nu > -\frac{1}{2}$.

e. By part a of this theorem, the function \mathcal{K}_ν is decreasing on $(0, \infty)$ for all $\nu > -\frac{1}{2}$, and hence we get

$$\mathcal{K}_\nu(x) < \frac{2}{\sqrt{\pi}} \int_0^\infty (1+t^2)^{\nu-\frac{1}{2}} dt.$$

Now, by using [11, p. 142]

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^\infty \frac{t^{a-1} dt}{(1+t)^{a+b}} = \int_0^\infty \frac{2u^{2a-1} du}{(1+u^2)^{a+b}}$$

it can be shown that for $\nu < 0$ we have

$$\frac{2}{\sqrt{\pi}} \int_0^\infty (1+t^2)^{\nu-\frac{1}{2}} dt = \frac{\Gamma(-\nu)}{\Gamma(\frac{1}{2}-\nu)}.$$

Consequently, for all $x > 0$ and $\nu \in (-\frac{1}{2}, 0)$ we obtain the inequality (2.5).

f. From inequality (2.5) and part a of this theorem we have that the function

$$x \mapsto \frac{\Gamma(\frac{1}{2}-\nu)}{\Gamma(-\nu)} \mathcal{K}_\nu(x)$$

maps $(0, \infty)$ into $(0, 1)$ and it is completely monotonic on $(0, \infty)$ for all $\nu \in (-\frac{1}{2}, 0)$. Now, recall the result of Kimberling [8], which says that if a function f , defined on $(0, \infty)$, is continuous and completely monotonic and maps $(0, \infty)$ into $(0, 1)$, then $\log f$ is super-additive, that is, for all $x, y > 0$ we have

$$\log f(x+y) \geq \log f(x) + \log f(y)$$

or

$$f(x+y) \geq f(x)f(y).$$

Applying this result the inequality (2.6) follows.

g. To prove the inequality (2.7) we use the Chebyshev integral inequality [10, p. 40]: Let f and g be functions which are integrable and monotone in the same sense (i.e. either both increasing or both decreasing) on (a, b) and let p be a positive and integrable function on (a, b) . Then

$$\int_a^b p(t)f(t)dt \int_a^b p(t)g(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)f(t)g(t)dt. \tag{2.10}$$

If f and g are monotone in opposite sense (i.e. one is decreasing and the other is increasing), then the inequality (2.10) is reversed.

Now let p, f and g be functions defined on $(0, \infty)$ such that

$$p(t) = e^{-xt}, \quad f(t) = \frac{2}{\sqrt{\pi}}(1+t^2)^{\nu-\frac{3}{2}} \quad \text{and} \quad g(t) = \frac{2}{\sqrt{\pi}}(1+t^2)^{\nu+\frac{1}{2}}.$$

Note that f is increasing if $\nu > \frac{3}{2}$, and g is increasing if $\nu > -\frac{1}{2}$. Now substituting p, f and g in (2.10) and using the following

$$\mathbf{H}_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}(1 - \cos x), \quad Y_{\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}} \cos x$$

and the definition of $\mathcal{K}_\nu(x)$ we have the desired inequality (2.7) valid for all $\nu > \frac{3}{2}$ and $x > 0$. As f is decreasing for $\nu < \frac{3}{2}$, the inequality (2.7) is reversed when $\nu \in (\frac{1}{2}, \frac{3}{2})$. \square

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