On Kapteyn–Kummer Series' Integral Form

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Dedicated to Prof. RNDr. Zbyněk Nádeník to his 90th birth annyversary

***Abstract*—In this short research note we obtain double definite integral expressions for the Kapteyn type series built by Kummer's M (or confluent hypergeometric ${}_1F_1$) functions. These kind of series unify in natural way the similar fashion results for Neumann–, Schlömilch– and Kapteyn–Bessel series recently established by Pogány, Süli, Baricz and Jankov Maširević.**

***Index Terms*—Dirichlet series, Integral representation, Kampé de Fériet function, Kapteyn series, Kummer function, Neumann series, Schlömilch series.**

I. Introduction and Preliminaries

THE series of Bessel (or Struve) functions in which summation is realized with respect to the indices appearing in the order of the building term functions and/or wrapped arguments of the same input functions, can be unified in a double lacunary form:

$$\mathfrak{B}_{\ell_1,\ell_2}(z) := \sum_{n\geq 0} \alpha_n \mathscr{B}_{\ell_1(n)}(\ell_2(n)z). \tag{1}$$

Here $x \mapsto \ell_j(x) = \mu_j + a_j x,\ j \in \{1,2\},\ x \in \{0,1,\dots\}$, $z \in \mathbb{C}$ and $\mathscr{B}_\nu$ can be chosen from one of Bessel, Struve, Dini and another related special functions and/or their products, [1], [2]. This extension of the classical theory of the *so–called* Fourier–Bessel series of the first type is based on the case when $\mathscr{B}_\nu = J_\nu$ for which the thorough account was given in famous Watson's monograph [3] with extensive references list therein. However, specifying varying the coefficients of ℓ_1 and ℓ_2, we appear to three cases related not only to physical models and have physical interpretations in many branches of science, technics and technology (consult for instance the corner-stone paper by Pogány and Süli [4] and [5, Introduction]), but are also of mathematical interest, like e.g. zero function series [3]. Thus, we differ the Neumann series ($a_1 \neq 0, a_2 = 0$) [4], [6], [7], Schlömilch series ($a_1 = 0, a_2 \neq 0$) [8] and the most general Kapteyn series ($a_1 \cdot a_2 \neq 0$) introduced by Willem Kapteyn in [9], [10].

As our main goal concerns the Kapteyn series we will focus our exposition to this kind of series, pointing out that a set of problems associated with Kapteyn type series are solved in [11], [12].

The Kummer's differential equation [13, §13.2]

$$z\,\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + (b-z)\,\frac{\mathrm{d}w}{\mathrm{d}z} - aw = 0, \qquad w \equiv M(a,b,z)$$

is the limiting form of the hypergeometric differential equation with the *first standard* series solution

$$M(a,b,z) = \sum_{n\geq 0} \frac{(a)_n}{(b)_n}\,\frac{z^n}{n!}, \qquad a \in \mathbb{C},\, b \in \mathbb{C}\setminus \mathbb{Z}_0^-\,.$$

The series converges for all $z \in \mathbb{C}$. Here $(a)_n = a(a+1)\cdots(a+n-1)$ stands for the standard Pochhammer symbol. Another notations which occur for Kummer's function: $\Phi(a;b;z), {}_1F_1(a;b;z)$.

Having in mind the structure of Fourier–Bessel series (1) let us introduce the *Kapteyn–Kummer series* as

$$\begin{aligned}\mathscr{K}_\kappa(z) &=: \mathscr{K}_\kappa\Big(\begin{array}{c} a,b \\ \alpha,\beta,\zeta \end{array} ;z\Big) \\ &= \sum_{n\geq 0} \kappa_n\, M(a+\alpha n, b+\beta n, z(1+\zeta n))\,,\end{aligned} \tag{2}$$

where $\kappa_n \in \mathbb{C}$; the parameter range and the z–domain will be described in the sequel. We point out that for at least one non–zero α, β, and $\zeta = 0$, this series becomes a Neumann–, while in the case $\alpha = \beta = 0, \zeta \neq 0$ we are faced with the Schlömilch–Kummer series.

We are motivated by the fact that Kummer's function $M(a,b,z)$ generate diverse special functions such as [14, pp. 507-8, §13.6. Special Cases]

$$\begin{aligned} M(\nu+\tfrac12, 2\nu+1, 2\mathrm{i}z) &= \Gamma(1+\nu)\,\mathrm{e}^{\mathrm{i}z}\big(\tfrac12 z\big)^{-\nu} J_\nu(z) \\ M(-\nu+\tfrac12, -2\nu+1, 2\mathrm{i}z) &= \Gamma(1-\nu)\,\mathrm{e}^{\mathrm{i}z}\big(\tfrac12 z\big)^{\nu} \\ &\quad\times \big[\cos(\nu\pi)J_\nu(z) - \sin(\nu\pi)Y_\nu(z)\big] \\ M(\nu+\tfrac12, 2\nu+1, 2z) &= \Gamma(1+\nu)\,\mathrm{e}^{z}\big(\tfrac12 z\big)^{-\nu} I_\nu(z) \\ M(\nu+\tfrac12, 2\nu+1, 2z) &= \pi^{-\frac12}\,\mathrm{e}^{z}(2z)^{-\nu} K_\nu(z)\,, \end{aligned}$$

where $J_\nu(I_\nu), Y_\nu(K_\nu)$ stand for the Bessel (modified Bessel) functions of the first and second kind of the order ν respectively, for which their Fourier–Bessel series have been studied in [1], [2], [4], [6], [7], [8] and [12], among others. Further special cases of M listed in [14, pp. 507-8, §13.6.] are: Hankel, spherical Bessel, Coulomb wave [15], Laguerre, incomplete gamma, Poisson–Charlier, Weber, Hermite, Airy, Kelvin, error function and elementary functions like trigonometric, exponential and hyperbolic ones. These links from Kummer's M to above mentioned special functions and then *a fortiori* to their Schlömilch–, Neumann– and Kapteyn–series obviously justify the definition of the Kapteyn–Kummer $\mathscr{K}_\kappa$–series (2).

Our main aim here is to establish integral representation formula for the Kapteyn–Kummer series $\mathscr{K}_\kappa$. The main derivation tools will be the associated Dirichlet series, the famous Cahen formula [16] and the Euler–Maclaurin summation formula firstly used in similar purposes in [17] and in [4].

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II. Main results

The derivation of the integral representation formula we split into few crucial steps assuming that all auxiliary parameters a, b, α, β *mutatis mutandis* are non-negative, and ζ real. Further necessary constraints between them follow in step–by–step exposition.

1. The convergence issue. Having in mind the integral expression of Kummer's function [14, p. 505, Eq. 13.2.1]

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 \mathrm{e}^{zt}\, t^{a-1}(1-t)^{b-a-1}\, \mathrm{d}t, \quad (3)$$

valid for all $\Re(b) > \Re(a) > 0$, we transform the Kapteyn–Kummer series into

$$\mathscr{K}_\kappa(z) = \sum_{n\geq 0} \frac{\kappa_n \Gamma(b+\beta n)}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)} \times \int_0^1 \mathrm{e}^{z(1+\zeta n)t}\, t^{a+\alpha n-1}(1-t)^{b-a+(\beta-\alpha)n-1}\, \mathrm{d}t. \quad (4)$$

Hence, for all $\beta \geq \alpha \geq 0$ using (4) we yield

$$\begin{aligned}\big|\mathscr{K}_\kappa(z)\big| &\leq \sum_{n\geq 0} \frac{|\kappa_n| \Gamma(b+\beta n)}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)} \times \int_0^1 \mathrm{e}^{\Re(z)(1+\zeta n)t}\, t^{a+\alpha n-1}(1-t)^{b-a+(\beta-\alpha)n-1}\, \mathrm{d}t \\ &\leq \sum_{n\geq 0} \frac{|\kappa_n| \Gamma(b+\beta n)}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)} \times \int_0^1 \mathrm{e}^{|\Re(z)|(1+|\zeta| n)t}\, t^{a+\alpha n-1}(1-t)^{b-a+(\beta-\alpha)n-1}\, \mathrm{d}t \\ &\leq \mathrm{e}^{|\Re(z)|} \sum_{n\geq 0} \frac{|\kappa_n| \Gamma(b+\beta n)\, \mathrm{e}^{|\zeta\Re(z)|n}}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)} \times \int_0^1 t^{a+\alpha n-1}(1-t)^{b-a+(\beta-\alpha)n-1}\, \mathrm{d}t \\ &= \mathrm{e}^{|\Re(z)|} \sum_{n\geq 0} |\kappa_n|\, \mathrm{e}^{|\zeta\Re(z)|n}. \end{aligned} \quad (5)$$

Here we employ the Euler Beta function's integral form and its connection to the Gamma function:

$$\mathrm{B}(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, \mathrm{d}t = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)},$$

where $\min\big(\Re(p), \Re(q)\big) > 0$. Indeed, specifying $p = a + \alpha n$, $q = b-a+(\beta-\alpha)n$ (5) immediately follows. Finally, by virtue of e.g. Cauchy's convergence test we get the convergence region of $\mathscr{K}_\kappa(z)$:

$$\mathsf{R}'_\kappa(\zeta) = \Big\{ z \in \mathbb{C} \colon |\zeta\Re(z)| < -\log \lim_{n\to\infty} \sqrt[n]{|\kappa_n|} \Big\},$$

for any fixed real ζ.

2. The associated Dirichlet series. The Dirichlet series

$$\mathscr{D}_{\boldsymbol{a}}(r) = \sum_{n\geq 1} a_n\, \mathrm{e}^{-r\lambda_n},$$

where $\Re(r) > 0$, having positive monotone increasing divergent to infinity sequence (λ_n), possesses Cahen's Laplace integral representation formula [16, p. 97]

$$\begin{aligned}\mathscr{D}_{\boldsymbol{a}}(r) &= r \int_0^\infty \mathrm{e}^{-rt} \sum_{n\colon \lambda_n \leq t} a_n\, \mathrm{d}t \\ &= r \int_0^\infty \int_0^{[\lambda^{-1}(t)]} \mathfrak{d}_u a(u)\, \mathrm{d}t\, \mathrm{d}u,\end{aligned}$$

where $\mathfrak{d}_x = 1 + \{x\}\frac{\mathrm{d}}{\mathrm{d}x}$ and $a \in \mathrm{C}^1(\mathbb{R}_+);\, (a_n) = a\big|_{\mathbb{N}}$, consult [17], [4].[1] Indeed, the so–called counting sum

$$\mathscr{A}_{\boldsymbol{a}}(t) = \sum_{n\colon \lambda_n \leq t} a_n$$

we calculate by the Euler–Maclaurin summation formula, see [17], [4]. Hence,

$$\mathscr{A}_{\boldsymbol{a}}(t) = \sum_{n=1}^{[\lambda^{-1}(t)]} a_n = \int_0^{[\lambda^{-1}(t)]} \mathfrak{d}_u a(u)\, \mathrm{d}u,$$

as $\lambda\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ is monotone, there exists unique inverse λ^{-1} for the function $\lambda\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$, being $\lambda|_{\mathbb{N}} = (\lambda_n)$.

The integral representation formula (3) of Kummer's function enables to re–formulate the series (4) into the following form

$$\begin{aligned}\mathscr{K}_\kappa(z) &= \sum_{n\geq 0} \frac{\kappa_n \Gamma(b+\beta n)}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)} \times \int_0^1 \mathrm{e}^{z(1+\zeta n)t}\, t^{a+\alpha n-1}(1-t)^{b-a+(\beta-\alpha)n-1}\, \mathrm{d}t \\ &= \int_0^1 \mathrm{e}^{zt}\, t^{a-1}(1-t)^{b-a-1}\, \mathscr{D}_\kappa(t)\, \mathrm{d}t,\end{aligned} \quad (6)$$

where the Dirichlet series

$$\mathscr{D}_\kappa(t) = \sum_{n\geq 0} \frac{\kappa_n\, \Gamma(b+\beta n)\, \mathrm{e}^{-\mathfrak{p}_t n}}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)}.$$

Here the parameter $\mathfrak{p}_t = \log\big(t^{-\alpha}(1-t)^{\alpha-\beta}\big) - z\zeta t$ should have positive real part. In turn, bearing in mind that for $\zeta\Re(z) < 0$ for all $t \in (0,1)$ it is

$$\Re(\mathfrak{p}_t) = -\alpha\log t - (\beta-\alpha)\log(1-t) - \zeta\Re(z)t > 0,$$

we have to take into account the following subset of $\mathsf{R}'_\kappa(\zeta)$:

$$\mathsf{R}_\kappa(\zeta) = \Big\{ z \in \mathbb{C} : \log \lim_{n\to\infty} \sqrt[n]{|\kappa_n|} < \zeta\Re(z) < 0 \Big\}.$$

Using $z \in \mathsf{R}_\kappa(\zeta)$ being ζ fixed real, applying Cahen's formula and the consequent Euler–Maclaurin summation's condensed writing developed in [17], we arrive at

Theorem 1: Let $\kappa \in \mathrm{C}^1(\mathbb{R}_+)$ be the function which restriction into $\mathbb{N}_0$ is the sequence (κ_n). For all $b > a > 0$; $\beta \geq \alpha \geq 0$; $\zeta \in \mathbb{R}$ and for all $z \in \mathsf{R}_\kappa(\zeta)$, we have

$$\mathscr{D}_\kappa(t) = \frac{\kappa_0\Gamma(b)}{\Gamma(b-a)\Gamma(a)} + \mathfrak{p}_t \int_0^\infty \mathrm{e}^{-\mathfrak{p}_t s} \mathscr{A}_\kappa(s)\, \mathrm{d}s, \quad (7)$$

[1]Here, $[x]$ and $\{x\} = x - [x]$ denote the integer and fractional part of $x \in \mathbb{R}$, respectively.

where $\mathfrak{p}_t = \log\big(t^{-\alpha}(1-t)^{\alpha-\beta}\mathrm{e}^{-z\zeta t}\big)$ *and*

$$\mathscr{A}_\kappa(s) = \int_0^{[s]} \mathfrak{d}_u\Big(\frac{\kappa(u)\,\Gamma(b+\beta u)}{\Gamma(b-a+(\beta-\alpha)u)\Gamma(a+\alpha u)}\Big)\,\mathrm{d}u\,.$$

Proof: It only remains to explain the sum–structure of (7). As to the use of Cahen formula for the Dirichlet series, which involves summation over $n \in \mathbb{N}$, we re–write

$$\mathscr{D}_\kappa(t) = \frac{\kappa_0\Gamma(b)}{\Gamma(b-a)\Gamma(a)} + \sum_{n\geq 1}\frac{\kappa_n\,\Gamma(b+\beta n)\,\mathrm{e}^{-\mathfrak{p}_t n}}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)}\,.$$

The rest is straightforward. □

Remark 1: Obviously the constituting addend constant term $\kappa_0\Gamma(b)\,(\Gamma(b-a)\Gamma(a))^{-1}$ can be avoided in the Dirichlet series' integral expression (7) by considering $\kappa_0 = 0$ without loss of any generality. ■

3. The master integral formula for $\mathscr{K}_\kappa(z)$. In this subsection of the section II we will need further special functions and auxiliary results. Firstly, we recall the double series definition of the so–called Kampé de Fériet hypergeometric function of two variables [18] in a notation given by Srivastava and Panda [19, p. 423, Eq. (26)]. For this, let (H_h) denotes the sequence of parameters $(H_1,\cdots,H_h)$ and for nonnegative integers signify the product of Pochhammer symbols $((H_h)) := (H_1)_n(H_2)_n\cdots(H_h)_n$, where when $n = 0$, the product is understood to reduce to unity. Therefore, the convenient generalization of the Kampé de Fériet function is defined as follows:

$$F^{h:a;b}_{g:c;d}\Big[\begin{array}{c}(H_h):\,(A_a)\,;\,(B_b)\\(G_g):\,(C_c)\,;\,(D_d)\end{array}\Big|\begin{array}{c}x\\y\end{array}\Big] = \sum_{m,n\geq 0}\frac{((H_h))_{m+n}((A_a))_m((B_b))_n}{((G_g))_{m+n}((C_c))_m((D_d))_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,.$$

Putting now the integral expression (7) of the Dirichlet series $\mathscr{D}_\kappa(t)$ into the integral form (6) of the Kapteyn–Kummer series $\mathscr{K}_\kappa(z)$, by (3), we deduce

$$\mathscr{K}_\kappa(z) = \kappa_0\,M(a,b,z) + \int_0^1\int_0^\infty \mathrm{e}^{zt}t^{a-1}(1-t)^{b-a-1}\mathfrak{p}_t\mathscr{A}_\kappa(s)\,\mathrm{d}t\mathrm{d}s\,. \quad (8)$$

Let us concentrate to the double integral $\mathscr{I}_\kappa(z)$ appearing above. By the legitimate change of integration order we have

$$\begin{aligned}\mathscr{I}_\kappa(z) &= -\int_0^\infty \mathscr{A}_\kappa(s)\Bigg(\int_0^1 \mathrm{e}^{z(1+\zeta s)t} \times t^{a+\alpha s-1}(1-t)^{b-a+(\beta-\alpha)s-1}\\ &\quad\times\big(\zeta zt+\alpha\log t+(\beta-\alpha)\log(1-t)\big)\,\mathrm{d}t\Bigg)\mathrm{d}s\\ &=: -\int_0^\infty \mathscr{A}_\kappa(s)\Big(\zeta z\,\mathscr{J}_\kappa(z,1)+\alpha\frac{\partial}{\partial a}\mathscr{J}_\kappa(z,0) + \beta\frac{\partial}{\partial b}\mathscr{J}_\kappa(z,0)\Big)\mathrm{d}s\,, \end{aligned} \quad (9)$$

where for $\rho \in \{0,1\}$ the following auxiliary integral occurs:

$$\mathscr{J}_\kappa(z,\rho) = \int_0^1 \mathrm{e}^{z(1+\zeta s)t}t^{a+\alpha s-1+\rho}(1-t)^{b-a+(\beta-\alpha)s-1}\mathrm{d}s.$$

In turn, by (3) it is explicitly

$$\mathscr{J}_\kappa(z,\rho) = \mathbf{\Gamma}_\rho(s)\,M\big(a+\alpha s+\rho, b+\beta s+\rho, z(1+\zeta s)\big)\,,$$

where we use the short–hand

$$\mathbf{\Gamma}_\rho(s) = \frac{\Gamma(b-a+(\beta-\alpha)s)\Gamma(a+\alpha s+\rho)}{\Gamma(b+\beta s+\rho)}\,.$$

Theorem 2: Let $\kappa \in \mathrm{C}^1(\mathbb{R}_+)$ *be the function for which* $\kappa\big|_{\mathbb{N}_0} = (\kappa_n)$. *For all* $b > a > 0$; $\beta \geq \alpha > 0$; $\zeta \in \mathbb{R}$ *and for all* $z \in \mathsf{R}_\kappa(\zeta)$, *we have*

$$\begin{aligned}\mathscr{K}_\kappa(z) &= \kappa_0\,M(a,b,z)\\ &\quad-\int_0^\infty\int_0^{[s]}\mathfrak{d}_u\Big(\frac{\kappa(u)\,\Gamma(b+\beta u)}{\Gamma(b-a+(\beta-\alpha)u)\Gamma(a+\alpha u)}\Big)\\ &\quad\times\Big(\zeta z\mathbf{\Gamma}_1(s)\,M\big(a+\alpha s+1,b+\beta s+1,z(1+\zeta s)\big)\\ &\quad+M^*\big(\beta\frac{\partial}{\partial b}\mathbf{\Gamma}_0(s)+\alpha\frac{\partial}{\partial a}\mathbf{\Gamma}_0(s)\big)\\ &\quad+\mathbf{\Gamma}_0(s)\big(\beta\frac{\partial M^*}{\partial b}+\alpha\,\frac{\partial M^*}{\partial a}\big)\Big)\mathrm{d}s\,\mathrm{d}u. \end{aligned} \quad (10)$$

where $\mathscr{A}_\kappa(s)$ *and* $\Gamma_\rho(s), \rho = 0,1$ *are described previously, while* $M^* := M\big(a+\alpha s, b+\beta s, z(1+\zeta s)\big)$. *Accordingly*

$$\frac{\partial M^*}{\partial a} = \frac{z(1+\zeta s)}{b+\beta s} \times F^{1:1;2}_{2:0;1}\Big[\begin{array}{c}a+\alpha s+1:1;1,a+\alpha s\\2,b+\beta s+1:-;a+\alpha s+1\end{array}\Big|\begin{array}{c}z(1+\zeta s)\\z(1+\zeta s)\end{array}\Big]$$

$$\frac{\partial M^*}{\partial b} = -\frac{(a+\alpha s)z(1+\zeta s)}{(b+\beta s)^2} \times F^{1:1;2}_{2:0;1}\Big[\begin{array}{c}a+\alpha s+1:1;1,b+\beta s\\2,b+\beta s+1:-;b+\beta s+1\end{array}\Big|\begin{array}{c}z(1+\zeta s)\\z(1+\zeta s)\end{array}\Big]\,.$$

Proof: Collecting all these expressions, that is (8) and (9), we finish the proof. So, from

$$\begin{aligned}\mathscr{K}_\kappa(z) &= \kappa_0\,M(a,b,z)\\ &\quad-\int_0^\infty\int_0^{[s]}\mathfrak{d}_u\Big(\frac{\kappa(u)\,\Gamma(b+\beta u)}{\Gamma(b-a+(\beta-\alpha)u)\Gamma(a+\alpha u)}\Big)\\ &\quad\times\Big(\zeta z\mathbf{\Gamma}_1(s)\,M\big(a+\alpha s+1,b+\beta s+1,z(1+\zeta s)\big)\\ &\quad+\beta\frac{\partial}{\partial b}\mathbf{\Gamma}_0(s)\,M\big(a+\alpha s,b+\beta s,z(1+\zeta s)\big)\\ &\quad+\alpha\,\frac{\partial}{\partial a}\mathbf{\Gamma}_0(s)M\big(a+\alpha s,b+\beta s,z(1+\zeta s)\big)\Big)\mathrm{d}s\,\mathrm{d}u, \end{aligned}$$

with some algebra the double integral will take the form

$$\begin{aligned}&\int_0^\infty\int_0^{[s]}\mathfrak{d}_u\Big(\frac{\kappa(u)\,\Gamma(b+\beta u)}{\Gamma(b-a+(\beta-\alpha)u)\Gamma(a+\alpha u)}\Big)\\ &\quad\times\Big(\zeta z\mathbf{\Gamma}_1(s)\,M\big(a+\alpha s+1,b+\beta s+1,z(1+\zeta s)\big)\\ &\quad+M^*\Big(\beta\frac{\partial}{\partial b}\mathbf{\Gamma}_0(s)+\alpha\frac{\partial}{\partial a}\mathbf{\Gamma}_0(s)\Big)\\ &\quad+\mathbf{\Gamma}_0(s)\Big(\beta\frac{\partial M^*}{\partial b}+\alpha\,\frac{\partial M^*}{\partial a}\Big)\Big)\mathrm{d}s\,\mathrm{d}u. \end{aligned}$$

Applying the formulae [20], [21]

$$\frac{\partial}{\partial a}M(a,b,z) = \frac{z}{b}F^{1:1;2}_{2:0;1}\Big[\begin{array}{c}a+1:1;1,a\\2,b+1:-;a+1\end{array}\Big|\begin{array}{c}z\\z\end{array}\Big]$$

$$\frac{\partial}{\partial b}M(a,b,z) = -\frac{az}{b^2}F^{1:1;2}_{2:0;1}\Big[\begin{array}{c} a+1:1;1,b \\ 2,b+1:-;b+1 \end{array}\Big|\begin{array}{c} z \\ z \end{array}\Big]$$

getting the partial derivatives of M^*, in which should be specified $a \to a+\alpha s$, $b \to b+\beta s$ and $z \to z(1+\zeta s)$, we arrive at the assertion of the Theorem 2. □

III. Toward to Neumann–Kummer and Schlömilch–Kummer series

As we have mentioned earlier in limiting case A. $\alpha \to 0$ we get a *two–parameter Kapteyn–Kummer series*; when either B. $\zeta \to 0$ or C. $\alpha, \zeta \to 0$, this imply a *Neumann–Kummer series*.

In the last possible common–sense case D. $\beta \to 0$ we earn a *Schlömilch–Kummer series* – all from $\mathscr{K}_\kappa(z)$ under the conditions of Theorem 2.

We point out that for the sake of simplicity in this section we take vanishing κ_0.

A. $\alpha \to 0$. Since $\alpha \to 0$ independently of β, in this case we have a Kapteyn–Kummer series:

$$\begin{aligned}\mathscr{K}_\kappa\Big(\begin{array}{c} a,b \\ 0,\beta,\zeta \end{array};z\Big) &= \int_0^\infty\int_0^{[s]}\mathfrak{d}_u\Big(\frac{-\kappa(u)\,\Gamma(b+\beta u)}{\Gamma(b-a+\beta u)}\Big) \\ &\times\Bigg(\zeta za\boldsymbol{\Gamma}_1(s)\,M\big(a+1,b+\beta s+1,z(1+\zeta s)\big) \\ &\quad+\beta\Big(M^*\big|_{\alpha=0}\frac{\partial}{\partial b}\boldsymbol{\Gamma}_0(s)+\boldsymbol{\Gamma}_0(s)\frac{\partial M^*|_{\alpha=0}}{\partial b}\Big)\Bigg)\mathrm{d}s\,\mathrm{d}u.\end{aligned}$$

B. $\zeta \to 0$. This case results in a two–parameter Neumann–Kummer series

$$\begin{aligned}\mathscr{K}_\kappa\Big(\begin{array}{c} a,b \\ \alpha,\beta,0 \end{array};z\Big) &= \int_0^\infty\int_0^{[s]}\mathfrak{d}_u\Big(\frac{-\kappa(u)\Gamma(b+\beta u)/\Gamma(a+\alpha u)}{\Gamma(b-a+(\beta-\alpha)u)}\Big) \\ &\times\Bigg(M^*|_{\zeta=0}\Big(\beta\frac{\partial}{\partial b}\boldsymbol{\Gamma}_0(s)+\alpha\frac{\partial}{\partial a}\boldsymbol{\Gamma}_0(s)\Big) \\ &\quad+\boldsymbol{\Gamma}_0(s)\Big(\beta\,\frac{\partial M^*|_{\zeta=0}}{\partial b}+\alpha\,\frac{\partial M^*|_{\zeta=0}}{\partial a}\Big)\Bigg)\mathrm{d}s\,\mathrm{d}u.\end{aligned}$$

C. $\alpha, \zeta \to 0$. Further simplification of the previous integral gives one–parameter Neumann–Kummer series, reads as follows:

$$\begin{aligned}\mathscr{K}_\kappa\Big(\begin{array}{c} a,b \\ 0,\beta,0 \end{array};z\Big) &= -\frac{\beta}{\Gamma(a)}\int_0^\infty\int_0^{[s]}\mathfrak{d}_u\Big(\frac{\kappa(u)\Gamma(b+\beta u)}{\Gamma(b-a+\beta u)}\Big) \\ &\times\Bigg(M^*|_{\alpha,\zeta=0}\frac{\partial}{\partial b}\boldsymbol{\Gamma}_0(s)+\boldsymbol{\Gamma}_0(s)\frac{\partial M^*|_{\alpha,\zeta=0}}{\partial b}\Bigg)\mathrm{d}s\,\mathrm{d}u.\end{aligned}$$

D. $\beta \to 0$. We end this overview of special cases of Master Theorem 2 with the Schlömilch–Kummer series integral representation formula

$$\begin{aligned}\mathscr{K}_\kappa\Big(\begin{array}{c} a,b \\ 0,0,\zeta \end{array};z\Big) &= -\frac{a\zeta z}{b}\int_0^\infty\int_0^{[s]}\mathfrak{d}_u\kappa(u) \\ &\times M\big(a+1,b+1,z(1+\zeta s)\big)\,\mathrm{d}s\,\mathrm{d}u.\end{aligned}$$

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