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Finiteness results for Diophantine triples with repdigit values

by

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1. Introduction. A classical *Diophantine m-tuple* is a set of m positive integers, $\{a_1, \dots, a_m\}$, such that $a_i a_j + 1$ is a square for all $1 \leq i < j \leq m$. Dujella [6] proved that there is no Diophantine sextuple and that there are only finitely many Diophantine quintuples. A folklore conjecture is that there are no Diophantine quintuples. Various variants of the notion of Diophantine tuples have been considered in which the set of squares has been replaced by some other arithmetically interesting subsets of the positive integers. For instance, the case of kth powers was considered in [3], while the case of the members of a fixed binary recurrence was considered in [7, 13, 14]. In [10], it is proved that there is no triple $\{a, b, c\}$ of positive integers such that all of $ab + 1$, $ac + 1$, $bc + 1$ belong to the sequence $\{u_n\}_{n \geq 0}$ of the recurrence $u_n = Au_{n-1} - u_{n-2}$ for $n \geq 2$ with initial values $u_0 = 0$ and $u_1 = 1$. For related results, see [1, 8, 9]. Diophantine tuples with values in the set of S-units for a fixed finite set S of primes were considered in [16, 19]. For a survey on this topic, we recommend the Diophantine m-tuples page maintained by A. Dujella [5].

Here we take an integer $g \geq 2$ and recall that a *repdigit* N in base g is a positive integer all of whose base g digits are the same. That is,

$$N = d\frac{g^k - 1}{g - 1} \quad \text{for some } d \in \{1, \dots, g-1\}. \tag{1.1}$$

These numbers fascinated both mathematicians and amateurs. Questions concerning Diophantine equations involving repdigits have been considered by Keith [11], Marques and Togbé [17] and Kovács et al. [12], to name just

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a few. In this paper, we combine the Diophantine tuples with repdigits and thus consider Diophantine triples having products increased by 1 in the set of repdigits in a fixed base g.

To avoid trivialities, we only look at repdigits with at least two digits. That is, the parameter k appearing in (1.1) satisfies $k \geq 2$. We denote by $\mathcal{R}_g$ the set of all positive integers that are repdigits in base g. In this paper, we are interested in triples $(a, b, c) \in \mathbb{N}^3$ with $c < b < a$ such that $ab+1$, $ac+1$ and $bc+1$ are all elements of $\mathcal{R}_g$. Let us denote by $\mathcal{D}_g$ the set of all such triples. The reason why we exclude the one-digit numbers from our analysis is, that in some sense, these are degenerate examples. Furthermore, if we allow $ab+1$, $ac+1$ and $bc+1$ to be one-digit numbers in a large base g, we will have many small examples, which however are of no interest.

Our main result is the following.

Theorem 1.1. *Assume that* $(a, b, c) \in \mathcal{D}_g$. *Then*

$$a \leq \frac{g^{186}-2}{2} \qquad \text{for all integers } g \geq 2, \text{ and}$$
$$a \leq \frac{g^{124}-2}{2} \qquad \text{for all integers } g \geq 10^6.$$

Moreover,

$$\#\mathcal{D}_g \leq \frac{(185g-185)(185g-186)(185g-187)}{6}$$

for all bases g *and*

$$\#\mathcal{D}_g \ll g^{1+o(1)} \qquad \text{as } g \to \infty. \tag{1.2}$$

In the next section, we estimate the greatest common divisor of two numbers of a special shape, which is an important step in the proof of Theorem 1.1. In Section 3, we prove Theorem 1.1 except for the asymptotic bound (1.2), which is proved later in Section 4.

We want to emphasize that our proof of Theorem 1.1 yields a rather efficient algorithm to compute $\mathcal{D}_g$ for a given g. In particular, we have computed all sets $\mathcal{D}_g$ for $2 \leq g \leq 200$, and we give the details and the results of this computation in the last section.

2. Estimates for the GCD of some numbers of special shape.

The main result of this section is:

Lemma 2.1. *Let* $g \geq 2$, $k_1, k_2 \geq 1$, t_1, w_1, t_2, w_2 *be non-zero integers, and set* $C := \max\{g, |t_1|, |w_1|, |t_2|, |w_2|\}$. *Let*

$$\Delta = \gcd(t_1 g^{k_1} - w_1, t_2 g^{k_2} - w_2)$$

and let X be any real number with $X \geq \max\{k_1, k_2, 3\}$. If $t_1 g^{k_1}/w_1$ and $t_2 g^{k_2}/w_2$ are multiplicatively independent, then

$$\Delta \leq 2C^{2+5\sqrt{X}}.$$

The proof of this lemma depends, among other things, on the following result whose proof is based on the pigeon-hole principle and appears explicitly in [15].

Lemma 2.2 ([15, Claim 1]). *Let m, n and X be non-negative integers such that m and n are not both zero, and $X \geq \max\{3, m, n\}$. Then there exist integers $(u, v) \neq (0, 0)$ such that*

$$\max\{|u|, |v|\} \leq \sqrt{X} \quad \textit{and} \quad 0 \leq mu + nv \leq 2\sqrt{X}.$$

Proof of Lemma 2.1. Set $\lambda_i = \gcd(t_i g^{k_i}, w_i)$ for $i = 1, 2$. We have

$$t_i g^{k_i} - w_i = \lambda_i (t_i g^{k_i}/\lambda_i - w_i/\lambda_i) \quad (i = 1, 2).$$

Then $\Delta = \lambda_1 \lambda_2 \Delta_1$, with

$$\Delta_1 = \gcd(t_1 g^{k_1}/\lambda_1 - w_1/\lambda_1, t_2 g^{k_2}/\lambda_2 - w_2/\lambda_2).$$

Since $|\lambda_i| \leq |w_i| \leq C$ for $i = 1, 2$, we get the upper bound

$$\Delta \leq C^2 \Delta_1. \tag{2.1}$$

Thus, it remains to bound Δ_1.

Now, consider the pair of congruences

$$t_i g^{k_i}/\lambda_i \equiv w_i/\lambda_i \pmod{\Delta_1} \quad (i = 1, 2) \tag{2.2}$$

and note that w_i/λ_i and $t_i g^{k_i}/\lambda_i$ are invertible modulo Δ_1. Indeed, by (2.2) there exists an integer q such that

$$t_i g^{k_i}/\lambda_i - w_i/\lambda_i = q\Delta_1.$$

If w_i/λ_i and Δ_1 have a common prime factor p, then $p \mid t_i g^{k_i}/\lambda_i$, contradicting the fact that $t_i g^{k_i}/\lambda_i$ and w_i/λ_i are coprime.

By Lemma 2.2, we can find a pair of integers $(u_1, u_2) \neq (0, 0)$ such that

$$\max\{|u_1|, |u_2|\} \leq \sqrt{X} \quad \text{and} \quad 0 \leq u_1 k_1 + u_2 k_2 \leq 2\sqrt{X}.$$

Since both sides of (2.2) are invertible modulo Δ_1, it makes sense to take the u_ith powers on both sides of (2.2) for $i = 1, 2$. Multiplying the resulting two congruences, we get

$$\frac{t_1^{u_1} t_2^{u_2} g^{k_1 u_1 + k_2 u_2}}{\lambda_1^{u_1} \lambda_2^{u_2}} - \frac{w_1^{u_1} w_2^{u_2}}{\lambda_1^{u_1} \lambda_2^{u_2}} \equiv 0 \pmod{\Delta_1}. \tag{2.3}$$

The rational number on the left-hand side of (2.3) is non-zero, since otherwise

$$\left(\frac{t_1 g^{k_1}}{w_1}\right)^{u_1}\left(\frac{t_2 g^{k_2}}{w_2}\right)^{u_2} = 1,$$

which implies that $t_1 g^{k_1}/w_1$ and $t_2 g^{k_2}/w_2$ are multiplicatively dependent because $(u_1, u_2) \neq (0, 0)$. But this is excluded by our hypothesis. Thus, the left-hand side of (2.3) is a non-zero rational number whose numerator is divisible by Δ_1.

Therefore we can write

$$\frac{t_1^{u_1} t_2^{u_2} g^{k_1 u_1 + k_2 u_2}}{\lambda_1^{u_1}\lambda_2^{u_2}} = \frac{AB_1B_2}{C_1C_2}, \tag{2.4}$$

where $A = g^{k_1u_1+k_2u_2}$ and $\{B_1, B_2, C_1, C_2\} = \{t_1^{|u_1|}, t_2^{|u_2|}, \lambda_1^{|u_1|}, \lambda_2^{|u_2|}\}$. Similarly, we have

$$\frac{w_1^{u_1} w_2^{u_2}}{\lambda_1^{u_1}\lambda_2^{u_2}} = \frac{D_1D_2}{E_1E_2},$$

where $\{D_1, D_2, E_1, E_2\} = \{w_1^{|u_1|}, w_2^{|u_2|}, \lambda_1^{|u_1|}, \lambda_2^{|u_2|}\}$. Clearly, $|A| \leq C^{2\sqrt{X}}$, whereas

$$\max_{i=1,2}\{|B_i|, |C_i|, |D_i|, |E_i|\} \leq C^{\sqrt{X}}.$$

First, assume that $u_1u_2 \geq 0$. Then u_1 and u_2 have the same sign and

$$\max\{k_1, k_2\} < k_1|u_1| + k_2|u_2| = |k_1u_1 + k_2u_2| \leq 2\sqrt{X},$$

which yields

$$\Delta_1 \leq \max\{|t_1g^{k_1} - w_1|, |t_2g^{k_2} - w_2|\} \leq 2C^{1+2\sqrt{X}} \leq 2C^{5\sqrt{X}}. \tag{2.5}$$

Next, assume that $u_1u_2 < 0$, which immediately implies that $\{C_1, C_2\}$ and $\{E_1, E_2\}$ have a common element. Without loss of generality, we may assume that $u_1 > 0$ and $u_2 < 0$. Then we can choose $\lambda_1^{u_1} = C_1 = E_1$ and Δ_1 divides the numerator of

$$\frac{AB_1B_2}{C_1C_2} - \frac{D_1D_2}{C_1E_2} = \frac{AB_1B_2E_2 - C_2D_1D_2}{C_1C_2E_2}.$$

That is, $\Delta_1 \mid AB_1B_2E_2 - D_1D_2C_2$. Since $AB_1B_2E_2 - D_1D_2C_2 \neq 0$, we obtain

$$\Delta_1 \leq 2C^{5\sqrt{X}}. \tag{2.6}$$

Therefore, we conclude by (2.5) and (2.6), together with (2.1), that

$$\Delta \leq 2C^{2+5\sqrt{X}}. \blacksquare$$

3. Proof of Theorem 1.1. Assume that $(a,b,c) \in \mathcal{D}_g$. By the definition of $\mathcal{D}_g$, we have

$$
\begin{aligned}
ab+1 &= d_3\frac{g^{n_3}-1}{g-1},\\
ac+1 &= d_2\frac{g^{n_2}-1}{g-1}, \qquad (3.1)\\
bc+1 &= d_1\frac{g^{n_1}-1}{g-1},
\end{aligned}
$$

where $d_i \in \{1,\dots,g-1\}$ and $n_i \geq 2$ for $i=1,2,3$. It is clear that $n_1 \leq n_2 \leq n_3$. Further, we may assume that $g \geq 3$, since if $g=2$, then $d_1=d_2=d_3=1$,

$$
\begin{aligned}
ab &= 2^{n_3}-2 = 2(2^{n_3-1}-1),\\
ac &= 2^{n_2}-2 = 2(2^{n_2-1}-1),\\
bc &= 2^{n_1}-2 = 2(2^{n_1-1}-1),
\end{aligned}
$$

and by multiplying the above equations we get

$$(abc)^2 = 8(2^{n_3-1}-1)(2^{n_2-1}-1)(2^{n_1-1}-1),$$

which yields a contradiction since the left-hand side is a square and the right-hand side is divisible by 8 but not by 16.

Next, we claim that

$$n_3 \leq 2n_2. \qquad (3.2)$$

In order to prove (3.2), we note that

$$a < ac+1 \leq g^{n_2}-1,$$

and therefore

$$g^{n_3-1}+g^{n_3-2}+\cdots+1 \leq \frac{d_3(g^{n_3}-1)}{g-1} = ab+1 < a^2 < (g^{n_2}-1)^2 < g^{2n_2}.$$

Thus, $n_3 < 2n_2+1$, and (3.2) is proved. Furthermore, note that

$$a > (ab+1)^{1/2} \geq (g^{n_3-1}+g^{n_3-2}+\cdots+1)^{1/2} > g^{(n_3-1)/2}. \qquad (3.3)$$

Let us fix some notation for the rest of this section. We rewrite (3.1) as:

$$
\begin{aligned}
ab &= \frac{\lambda_3}{g-1}\left(\frac{d_3g^{n_3}}{\lambda_3}-\frac{d_3+g-1}{\lambda_3}\right) =: \frac{\lambda_3}{g-1}(x_3-y_3),\\
ac &= \frac{\lambda_2}{g-1}\left(\frac{d_2g^{n_2}}{\lambda_2}-\frac{d_2+g-1}{\lambda_2}\right) =: \frac{\lambda_2}{g-1}(x_2-y_2), \qquad (3.4)\\
bc &= \frac{\lambda_1}{g-1}\left(\frac{d_1g^{n_1}}{\lambda_1}-\frac{d_1+g-1}{\lambda_1}\right) =: \frac{\lambda_1}{g-1}(x_1-y_1),
\end{aligned}
$$

where

$$\lambda_i = \gcd(d_ig^{n_i}, d_i+g-1), \quad x_i = \frac{d_ig^{n_i}}{\lambda_i}, \quad y_i = \frac{d_i+g-1}{\lambda_i} \qquad (i=1,2,3).$$

Note that $\gcd(x_i, y_i) = 1$ for $i = 1, 2, 3$. Hence, the fractions x_i/y_i are reduced. Note also that $x_i > y_i$ for $i = 1, 2, 3$.

In order to prove Theorem 1.1, we consider several cases.

CASE 1: *x_1/y_1 and x_2/y_2 are multiplicatively dependent and so are x_1/y_1 and x_3/y_3.* In this case all the fractions x_i/y_i with $i = 1, 2, 3$ belong to the same cyclic subgroup of $\mathbb{Q}_+^*$. Let $\alpha/\beta > 1$ be a generator of this subgroup, where α, β are coprime integers. Since $x_i/y_i > 1$ for $i = 1, 2, 3$, there exist positive integers r_i for $i = 1, 2, 3$ such that

$$x_i = \alpha^{r_i} \quad \text{and} \quad y_i = \beta^{r_i}, \qquad i = 1, 2, 3.$$

We split this case up into further subcases and start with:

CASE 1.1: *There exist $i \neq j$ such that $r_i = r_j$.* Let us start with the case $r_3 = r_2$. We then get

$$\alpha^{r_3} = \frac{d_3 g^{n_3}}{\lambda_3} = \frac{d_2 g^{n_2}}{\lambda_2} = \alpha^{r_2}.$$

Hence,

$$g^{n_3 - n_2} = \frac{d_2 \lambda_3}{d_3 \lambda_2}.$$

We claim that $n_3 - n_2 \in \{0, 1\}$. Note that $d_2 \leq g - 1$ and $\lambda_3 \leq 2(g - 1)$, which yield $d_2 \lambda_3 \leq 2(g - 1)^2$. In case $d_3 \lambda_2 \geq 2$, we obtain

$$g^{n_3 - n_2} \leq \frac{2(g-1)^2}{2} = (g - 1)^2 < g^2,$$

so we have $n_3 - n_2 \in \{0, 1\}$. Therefore, we are left with the case when $d_3 \lambda_2 = 1$, i.e. $d_3 = \lambda_2 = 1$. But in this case,

$$\lambda_3 = \gcd(d_3 g^{n_3}, d_3 + g - 1) = \gcd(g^{n_3}, g) = g,$$

so

$$g^{n_3 - n_2} = \frac{d_2 \lambda_3}{d_3 \lambda_2} = d_2 \lambda_3 \leq g(g - 1) < g^2.$$

Thus, in all cases we have $n_3 - n_2 \in \{0, 1\}$.

Consider now the case $n_3 - n_2 = 0$. This means that

$$\frac{d_3}{\lambda_3} = \frac{d_2}{\lambda_2}. \tag{3.5}$$

But we also have

$$\beta^{r_3} = \frac{d_3 + g - 1}{\lambda_3} = \frac{d_2 + g - 1}{\lambda_2} = \beta^{r_2}. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain $(g - 1)/\lambda_3 = (g - 1)/\lambda_2$, so $\lambda_2 = \lambda_3$. Now we deduce by (3.5) that $d_2 = d_3$. Altogether this yields $ab + 1 = ac + 1$, contradicting our assumption that $b > c$.

Now, consider the case $n_3 - n_2 = 1$. Instead of (3.5), we now have

$$\frac{d_3 g}{\lambda_3} = \frac{d_2}{\lambda_2}. \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$\frac{d_2 + g - 1}{d_3 + g - 1} = \frac{\lambda_2}{\lambda_3} = \frac{d_2}{d_3 g},$$

which leads to

$$d_3 g(d_2 + g - 1) = d_2(d_3 + g - 1). \tag{3.8}$$

Assuming that $d_3 \geq 2$, equation (3.8) yields

$$2g^2 \leq d_3 g(d_2 + g - 1) = d_2(d_3 + g - 1) \leq 2(g-1)^2,$$

a contradiction, so we may assume that $d_3 = 1$. Inserting $d_3 = 1$ into (3.8) yields

$$g(d_2 + g - 1) = d_2 g,$$

or equivalently $g(g-1) = 0$, which is obviously false. In particular, we have proved that the case $r_2 = r_3$ yields no solution.

The same arguments hold if we replace the quantities $r_3, r_2, n_3, n_2, d_3, d_2$ by $r_2, r_1, n_2, n_1, d_2, d_1$ and $r_3, r_1, n_3, n_1, d_3, d_1$ respectively. Thus, Case 1.1 yields no solution and we assume from now on that r_1, r_2 and r_3 are pairwise distinct.

Case 1.2: $r_3 > \max\{r_1, r_2\}$. With our notation, we have

$$(g-1)ab = \lambda_3(\alpha^{r_3} - \beta^{r_3}) \quad \text{and} \quad (g-1)ac = \lambda_2(\alpha^{r_2} - \beta^{r_2}),$$

and obviously $a(g-1)$ is a common divisor of $\lambda_3(\alpha^{r_3} - \beta^{r_3})$ and $\lambda_2(\alpha^{r_2} - \beta^{r_2})$. Thus, we have

$$(g-1)a \mid \gcd\bigl(\lambda_3(\alpha^{r_3} - \beta^{r_3}), \lambda_2(\alpha^{r_2} - \beta^{r_2})\bigr).$$

Taking a closer look at the greatest common divisor on the right-hand side above, we obtain

$$(g-1)a \mid \lambda_2\lambda_3(\alpha^r - \beta^r),$$

where $r = \gcd(r_3, r_2)$. Similarly,

$$(g-1)b \mid \lambda_3\lambda_1(\alpha^s - \beta^s),$$

where $s = \gcd(r_3, r_1)$. Together, the last two inequalities give

$$(g-1)^2 ab < \lambda_1\lambda_2\lambda_3^2\alpha^{r+s}.$$

Write $r = r_3/\delta$ and $s = r_3/\lambda$ for some divisors $\delta > 1$ and $\lambda > 1$ of r_3. Note that we cannot have $\delta = \lambda = 2$: this would yield $r_2 = r_1 = r_3/2$, which was excluded by Case 1.1. Thus,

$$ab < \frac{\lambda_1\lambda_2\lambda_3^2\alpha^{r+s}}{(g-1)^2} \leq 16(g-1)^2\alpha^{r+s},$$

and therefore

$$ab < 16(g-1)^2\alpha^{r+s} = 16(g-1)^2\alpha^{r_3(1/\delta+1/\lambda)} \le 16(g-1)^2\alpha^{5r_3/6}.$$

On the other hand, we have

$$(g-1)ab = \lambda_3(\alpha^{r_3} - \beta^{r_3}) \ge \alpha^{r_3} - \beta^{r_3} \ge \alpha^{r_3} - 2(g-1),$$

where we have used the fact that $\beta^{r_3} = (d_3 + g - 1)/\lambda_3 \le 2(g-1)$. Hence,

$$\alpha^{r_3} - 2(g-1) \le (g-1)ab < 16(g-1)^3\alpha^{5r_3/6},$$

and a crude estimate now yields

$$\alpha^{r_3} < 16(g-1)^3\alpha^{5r_3/6} + 2(g-1) < 17(g-1)^3\alpha^{5r_3/6}.$$

Thus,

$$\alpha^{r_3} < 17^6(g-1)^{18}.$$

Now combining the various estimates we obtain

$$\begin{aligned} g^{n_3-1} < \frac{d_3(g^{n_3}-1)}{g-1} - 1 = ab = \frac{\lambda_3}{g-1}(\alpha^{r_3} - \beta^{r_3}) \\ < 2\alpha^{r_3} < 2 \times 17^6(g-1)^{18}. \end{aligned}$$

Since $g \ge 3$, the above inequality gives $n_3 \le 28$, and therefore this case does not yield any solution with $n_3 \ge 29$.

Case 1.3: $r_3 < \max\{r_1, r_2\}$. Assume for the moment that $r_3 < r_2$. Then

$$(g-1)ab = \lambda_3(\alpha^{r_3} - \beta^{r_3}) \quad \text{and} \quad (g-1)ac = \lambda_2(\alpha^{r_2} - \beta^{r_2}). \tag{3.9}$$

Write $\gcd(r_2, r_3) = r_2/\delta$ with some integer $\delta > 1$. Then, as before,

$$(g-1)a \le \lambda_2\lambda_3(\alpha^{r_2/\delta} - \beta^{r_2/\delta})$$

and by the second equation of (3.9), we get

$$c \ge \frac{\lambda_2(\alpha^{r_2} - \beta^{r_2})}{\lambda_2\lambda_3(\alpha^{r_2/\delta} - \beta^{r_2/\delta})} > \frac{\alpha^{r_2(\delta-1)/\delta}}{2(g-1)}. \tag{3.10}$$

The above bound yields

$$2\alpha^{r_3} \ge \frac{\lambda_3}{g-1}(\alpha^{r_3} - \beta^{r_3}) = ab > c^2 > \frac{\alpha^{2r_2(\delta-1)/\delta}}{4(g-1)^2}. \tag{3.11}$$

If we assume that $\delta \ge 3$, then since $r_2 > r_3$, we get

$$2r_2(\delta-1)/\delta > 4r_3/3.$$

If we assume that $\delta = 2$, then

$$2r_2(\delta-1)/\delta = r_2 = 2r_3 > 4r_3/3.$$

In both cases inequality (3.11) implies

$$\alpha^{r_3/3} < 8(g-1)^2.$$

Hence,

$$g^{n_3-1} < \frac{d_3(g^{n_3}-1)}{g-1} - 1 = ab = \frac{\lambda_3}{g-1}(\alpha^{r_3} - \beta^{r_3}) < 2\alpha^{r_3} < 2^{10}(g-1)^6,$$

which has no solution for $n_3 \geq 12$ and $g \geq 3$.

The case when $r_1 > r_3$ can be dealt with similarly. In particular, instead of (3.10) we obtain

$$b \geq \frac{\alpha^{r_1(\delta-1)/\delta}}{2(g-1)},$$

where $r_1/\delta = \gcd(r_1, r_3)$. Using the inequality $ab > b^2$ instead of $ab > c^2$ in the middle of (3.11), we obtain the same bound for n_3.

Case 2: *x_3/y_3 and x_2/y_2 are multiplicatively independent.* By (3.1), we have

$$\begin{aligned}(g-1)ab &= d_3 g^{n_3} - (d_3 + g - 1),\\ (g-1)ac &= d_2 g^{n_2} - (d_2 + g - 1).\end{aligned}$$

Hence, we get an upper bound for a:

$$(g-1)a \leq \gcd\bigl(d_3 g^{n_3} - (d_3+g-1), d_2 g^{n_2} - (d_2+g-1)\bigr). \tag{3.12}$$

Since, by assumption,

$$\frac{x_3}{y_3} = \frac{d_3 g^{n_3}}{d_3+g-1} \quad \text{and} \quad \frac{x_2}{y_2} = \frac{d_2 g^{n_2}}{d_2+g-1}$$

are multiplicatively independent, we may apply Lemma 2.1 with the parameters

$$(t_1, w_1, t_2, w_2, k_1, k_2) = (d_3, d_3+g-1, d_2, d_2+g-1, n_3, n_2),$$

where

$$\max\{|t_1|, |w_1|, |t_2|, |w_2|\} \leq 2(g-1) \quad \text{and} \quad \max\{k_1, k_2, 3\} \leq n_3.$$

Thus, by Lemma 2.1 and (3.12), for a we get the upper bound

$$a \leq 4(2g-2)^{5\sqrt{n_3}+1}. \tag{3.13}$$

On the other hand, we have an upper bound for n_3 given by (3.3):

$$n_3 < \frac{2\log a}{\log g} + 1. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$n_3 < \frac{(10\sqrt{n_3}+2)\log(2g-2) + \log 16}{\log g} + 1. \tag{3.15}$$

From (3.15), we get

$$n_3 \leq 178, \tag{3.16}$$

which actually occurs when $g = 4$. Note that if $g = 3$ then (3.15) yields $n_3 \leq 171$, while for larger values of g we obtain better upper bounds for n_3. In particular, we have $n_3 \leq 105$ provided g is large enough. If we only assume $g \geq 200$ and $g \geq 10^6$ we find that $n_3 \leq 135$ and $n_3 \leq 116$, respectively.

CASE 3: *x_3/y_3 and x_2/y_2 are multiplicatively dependent and x_3/y_3 and x_1/y_1 are not.* As in Case 1, we may write

$$x_3 = \alpha^{r_3},\ y_3 = \alpha^{r_3} \quad \text{and} \quad x_2 = \alpha^{r_2},\ y_2 = \alpha^{r_2}.$$

Note that in the proof of Case 1.3 we never used the quantity r_1 when we considered the case $r_2 < r_3$. Therefore, we may assume $r_3 > r_2$.

Similarly to Case 2, we find an upper bound for b, but we use

$$\begin{aligned}(g-1)ab &= d_3 g^{n_3} - (d_3 + g - 1),\\ (g-1)bc &= d_2 g^{n_1} - (d_1 + g - 1)\end{aligned}$$

instead. Therefore, by Lemma 2.1, we obtain the upper bound

$$b \leq 4(2g-2)^{5\sqrt{n_3}+1}. \tag{3.17}$$

Next we want to find an upper bound for a. To this end, we consider

$$ab = \frac{\lambda_3}{g-1}(\alpha^{r_3} - \beta^{r_3}) \quad \text{and} \quad ac = \frac{\lambda_2}{g-1}(\alpha^{r_2} - \beta^{r_2}). \tag{3.18}$$

Hence, we obtain

$$(g-1)a \mid \lambda_3\lambda_2 \gcd(\alpha^{r_3} - \beta^{r_3}, \alpha^{r_2} - \beta^{r_2}) < 4(g-1)^2\alpha^r,$$

where $r = \gcd(r_2, r_3)$. Thus,

$$a < 4(g-1)\alpha^r. \tag{3.19}$$

On the other hand,

$$ab + 1 = \frac{d_3(g^{n_3} - 1)}{g-1} \geq g^{n_3-1} + g^{n_3-2} + \cdots + 1,$$

that is, $a \geq g^{n_3-1}/b$, whence by (3.17) we get

$$a \geq \frac{g^{n_3-1}}{4(2g-2)^{5\sqrt{n_3}+1}}. \tag{3.20}$$

By using (3.17) and the fact that $d_3 \geq 1$ and $d_2 \leq g-1$, we find the following lower bound for b:

$$b \geq \frac{b}{c} = \frac{ab}{ac} > \frac{ab+1}{ac+1} = \frac{d_3(g^{n_3} - 1)}{d_2(g^{n_2} - 1)} \geq \frac{g^{n_3-n_2}}{g-1},$$

which yields

$$g^{n_3-n_2} < (g-1)b. \tag{3.21}$$

Recall that

$$\alpha^{r_2} = \frac{d_2 g^{n_2}}{\lambda_2} \geq \frac{g^{n_2}}{2(g-1)} \quad \text{and} \quad \alpha^{r_3} = \frac{d_3 g^{n_3}}{\lambda_3} \leq (g-1)g^{n_3}.$$

As in Case 1, let $r = \gcd(r_2, r_3)$. We then find that

$$\alpha^r \leq \alpha^{r_3-r_2} \leq 2(g-1)^2 g^{n_3-n_2} < 2(g-1)^3 b, \tag{3.22}$$

where the last inequality is due to (3.21). We combine (3.17), (3.19), (3.20) and (3.22) to obtain

$$\begin{aligned} \frac{g^{n_3-1}}{4(2g-2)^{5\sqrt{n_3}+1}} \leq \frac{g^{n_3-1}}{b} &\leq a < 4(g-1)\alpha^r \\ &< 8(g-1)^4 b \leq 32(g-1)^4(2g-2)^{5\sqrt{n_3}+1}. \end{aligned}$$

Taking logarithms we obtain a similar inequality for n_3 to the one in Case 2:

$$n_3 < \frac{(10\sqrt{n_3}+2)\log(2g-2) + 4\log(g-1) + \log 128}{\log g} + 1. \tag{3.23}$$

The above yields

$$n_3 \leq 186. \tag{3.24}$$

Note that we obtain $n_3 \leq 186$ if $g \in \{4, 5\}$, whereas in all other cases we obtain better bounds. In particular, if we assume that $g \geq 200$, then we obtain $n_3 \leq 143$, and if we assume that $g \geq 10^6$, then we get $n_3 \leq 124$. Finally, if g is large enough, then we may even assume that $n_3 \leq 113$.

Let us summarize our results so far:

Proposition 3.1. *Assume equations* (3.1) *hold. Then* $n_3 \leq 186$. *If we assume that* $g \geq 200$ *or that* $g \geq 10^6$, *then we have* $n_3 \leq 143$ *and* $n_3 \leq 124$, *respectively. Moreover, we may even assume that* $n_3 \leq 113$ *if* g *is large enough* ($g > 10^{153}$).

Now a simple combinatorial argument concludes the proof of the first part of our theorem. Indeed, the distinct tuples $(n_1, d_1), (n_2, d_2), (n_3, d_3)$ may be selected from a set of cardinality $185(g-1)$ and altogether in

$$(185g - 185)(185g - 186)(185g - 187)$$

ways. Since only those results are acceptable where

$$d_1 \frac{g^{n_1}-1}{g-1} < d_2 \frac{g^{n_2}-1}{g-1} < d_3 \frac{g^{n_3}-1}{g-1},$$

we are left with

$$\frac{(185g - 185)(185g - 186)(185g - 187)}{6}$$

possibilities for the tuple $(d_1, n_1, d_2, n_2, d_3, n_3)$. Further, for a given sextuple $(d_1, n_1, d_2, n_2, d_3, n_3)$, the system of equations (3.1) has at most one solution in positive integers (a, b, c). Additionally, since $b \geq 2$, $d_3 \leq g-1$, $n_3 \leq 186$ and (3.1), the estimate for a is trivial. This concludes the proof of the first part of Theorem 1.1.

4. Counting the number of triples. We are left with the proof of the last statement of Theorem 1.1. The main purpose of this section is to prove Theorem 4.1 below. Let $\tilde{\mathcal{R}}_g$ be the set of repdigits together with the integers of digit length 1 in base g. Denote by $\tilde{\mathcal{D}}_g$ the set of triples $(a,b,c) \in \mathbb{N}^3$ such that $1 \le c < b < a$ and $ab+1$, $ac+1$ and $bc+1$ are elements of $\tilde{\mathcal{R}}_g$. We prove the following theorem:

Theorem 4.1. *We have*

$$\#\tilde{\mathcal{D}}_g \asymp g^{3/2} \qquad (g \to \infty),$$
$$\#\mathcal{D}_g \ll g^{1+o(1)} \qquad (g \to \infty).$$

Since g is fixed throughout this section, we will omit the index of $\mathcal{D}_g$ and $\tilde{\mathcal{D}}_g$ and write only $\mathcal{D}$ and $\tilde{\mathcal{D}}$, respectively. In the course of the proof of Theorem 4.1, we consider several subsets of $\tilde{\mathcal{D}}$ which will be denoted by $\mathcal{D}_1, \dots, \mathcal{D}_4$. We emphasize that in the following a subscript of $\mathcal{D}$ does not refer to the base g, but instead to a certain subset of $\tilde{\mathcal{D}}$.

Proof. Clearly, $\tilde{\mathcal{D}}$ can also be identified with the set of all sextuples

$$(d_1, d_2, d_3, n_1, n_2, n_3) \quad \text{where} \quad 1 \le d_i \le g-1 \quad \text{for } i = 1,2,3,$$

such that there exist positive integers $c < b < a$ satisfying (3.1). Under this identification and using Proposition 3.1, for g large enough we have $n_1 \le n_2 \le n_3 \le 113$ and $1 \le d_i \le g-1$ for $i = 1,2,3$. So, trivially, $\#\tilde{\mathcal{D}} \ll g^3$. Let us improve this trivial bound. Let $\mathcal{D}_1$ be the subset of $\tilde{\mathcal{D}}$ such that $n_3 = 1$, and $\mathcal{D}_2 = \tilde{\mathcal{D}} \backslash \mathcal{D}_1$. We prove:

(i) $\#\mathcal{D}_1 \asymp g^{3/2}$ as $g \to \infty$.
(ii) $\#\mathcal{D}_2 \ll g^{1+o(1)}$ as $g \to \infty$.

The conclusion of Theorem 4.1 follows from (i), (ii) and the fact that

$$\#\tilde{\mathcal{D}} = \#\mathcal{D}_1 + \#\mathcal{D}_2.$$

First, let us deal with (i). For the lower bound, we choose $a > b > c$ all three in $\{1, \dots, \lfloor\sqrt{g-2}\rfloor\}$. For each of these choices,

$$ab + 1 \le \lfloor\sqrt{g-2}\rfloor^2 + 1 \le g - 1,$$

so $ab+1 = d_1 \in [1, g-1]$ and similarly $ac+1 = d_2$ and $bc+1 = d_3$. Thus, (a,b,c) is in $\mathcal{D}_1$, and we get

$$\#\mathcal{D}_1 \ge \binom{\lfloor\sqrt{g-2}\rfloor}{3} \gg g^{3/2}. \tag{4.1}$$

For the upper bound, note that we have to count the integers $a > b > c$

satisfying (3.1) with $n_1 = n_2 = n_3 = 1$. In particular, we have to count the triples (a, b, c) satisfying

$$(4.2) \qquad 1 \leq a \leq g-2 \quad \text{and} \quad 1 \leq c < b < \min\left\{a, \frac{g-1}{a}\right\}.$$

For fixed a there are $\ll \min\{a, g/a\}^2$ pairs (b, c) satisfying (4.2). Therefore

$$\#\mathcal{D}_1 \ll \int_1^g \min\{a, g/a\}^2\, da = \int_1^{\sqrt{g}} a^2\, da + \int_{\sqrt{g}}^{g} \left(\frac{g}{a}\right)^2 da \ll g^{3/2},$$

which is the desired upper bound.

For (ii), let $\mathcal{D}_3$ be the subset of $\mathcal{D}_2$ such that $n_3 \geq 3$. Due to Proposition 3.1, for g large enough we may assume that $n_3 \leq 113$ and $d_3 \leq g-1$. We look at

$$(4.3) \qquad ab = d_3 \frac{g^{n_3}-1}{g-1} - 1.$$

Clearly, since $n_3 \geq 3$, we have $a^2 > ab \geq (g^2+g+1) - 1 > g^2$, so $a > g$. Since d_3 and g are fixed and $n_3 \leq 113$, the number of ways of choosing (a, b) such that $a > b$ and (4.3) holds is

$$\tau\left(\frac{d_3(g^{n_3}-1)}{g-1} - 1\right) \ll g^{o(1)} \qquad \text{as } g \to \infty,$$

where $\tau(n)$ is the number of divisors of n. The asymptotic bound on the right side follows from a well-known upper bound for the divisor function (e.g. see [18, Theorem 2.11] or [4, Chapter 7.4]). It remains to find out in how many ways we can choose c. Well, let us also fix $n_2 \leq n_3$. Then $d_2 \in \{1, \ldots, g-1\}$ is such that

$$d_2 \frac{g^{n_2}-1}{g-1} \equiv 1 \pmod{a}.$$

This puts d_2 into a fixed arithmetic progression α_{n_2} modulo a, where α_{n_2} is the inverse of $(g^{n_2}-1)/(g-1)$ modulo a. We show that this progression contains at most one value for d_2. Assuming this is not the case, let d_2 and d_2' be both congruent to α_{n_2} and in the interval $[1, g-1]$. Assume that $d_2 < d_2'$; then $a \mid d_2' - d_2$, so

$$g < a \leq d_2' - d_2 \leq g-2,$$

which is false. This shows that indeed once d_3, n_3 and a (hence also b) are determined, then any choice of $n_2 \leq n_3$ determines d_2 (hence c) uniquely. Thus,

$$(4.4) \qquad \#\mathcal{D}_3 \leq \sum_{d_3=1}^{g-1} \sum_{n_3=3}^{113} \sum_{n_2=1}^{n_3} \tau\left(\frac{d_3(g^{n_3}-1)}{g-1} - 1\right) \ll g^{1+o(1)} \qquad (g \to \infty).$$

It remains to find an upper bound for the cardinality of $\mathcal{D}_4 := \mathcal{D}_2 \backslash \mathcal{D}_3$. These triples are the ones that have $n_3 = 2$. We fix d_3 and write

$$ab = d_3(g+1) - 1. \tag{4.5}$$

There are at most $\tau(d_3(g+1)-1) = g^{o(1)}$ possibilities for $a > b$ satisfying the above relation (4.5) as $g \to \infty$. It remains to determine the number of choices for c. Let us also fix $n_2 \le n_3 = 2$. Then determining c is equivalent to determining the number of choices for d_2 such that

$$d_2 \frac{g^{n_2} - 1}{g-1} \equiv 1 \pmod{a}, \qquad 1 \le d_2 \le g-1. \tag{4.6}$$

Congruence (4.6) puts d_2 in a certain fixed arithmetic progression modulo a, and the number of such numbers $1 \le d_2 \le g-1$ is at most

$$1 + \left\lfloor \frac{g-1}{a} \right\rfloor.$$

We assume that $a \le g-1$, otherwise there is at most one choice for d_2, and the counting function of such examples is at most $g^{1+o(1)}$ by the argument for $\#\mathcal{D}_3$. Then the number of choices for c is at most

$$1 + \left\lfloor \frac{g-1}{a} \right\rfloor \le 1 + \frac{g-1}{a} < \frac{2g}{a} \le \frac{2g}{\sqrt{d_3(g+1)-1}} \ll \frac{\sqrt{g}}{\sqrt{d_3}}.$$

This shows that

$$\begin{aligned}
\#\mathcal{D}_4 &\ll \sum_{n_2 \le 2} \sum_{d_3=1}^{g-1} \tau(d_3(g+1)-1) \frac{\sqrt{g}}{\sqrt{d_3}} \\
&\ll g^{1/2+o(1)} \sum_{1 \le d_3 \le g-1} \frac{1}{\sqrt{d_3}} \ll g^{1/2+o(1)} \int_1^{g-1} \frac{dt}{t^{1/2}} \\
&\ll g^{1/2+o(1)} \left(2t^{1/2} \Big|_{t=1}^{t=g-1} \right) \ll g^{1+o(1)} \qquad (g \to \infty).
\end{aligned}$$

Together with (4.4), we get

$$\#\mathcal{D}_2 \le \#\mathcal{D}_3 + \#\mathcal{D}_4 \le g^{1+o(1)} \qquad (g \to \infty),$$

which is (ii). ∎

5. The case of small bases g. For the bases $2 \le g \le 200$ we have computed all triples $(a, b, c) \in \mathcal{D}_g$. In particular we found the following triples:

g	a	b	c
23	65	17	7
42	136	93	6
104	292	187	32
171	5607	619	5
190	439	248	67

In our computations, we considered all values of $2 < g \leq 200$ one by one and we split our work depending on the size of a. If $g \leq 100$, we set $B := 1000$, and for $101 \leq g \leq 200$, we set $B := 10000$.

For every $a < B$ we proceed as follows: for $2 \leq b < a$ we check whether $ab + 1$ is a repdigit number in base g. If yes, we also check if we can find $c < b$ such that $ab + 1$, $ac + 1$ and $bc + 1$ are all repdigit numbers in base g.

For $a \geq B$ we proceed as follows: We use equations (3.1) and (3.2). For all integer values of $2 \leq n_2 \leq 186$, and all integer values of n_3 between n_2, and the minimum of 186 and $2n_2$, and for all possible digits d_2 and d_3, we compute

$$ab = d_3 \frac{g^{n_3} - 1}{g - 1} - 1, \qquad ac = d_2 \frac{g^{n_2} - 1}{g - 1} - 1.$$

Since $a \leq \gcd(ab, ac)$, the cases when $\gcd(ab, ac) < B$ are covered by the cases when $a < B$, so we only have further work to do if

$$\gcd\left(d_3 \frac{g^{n_3} - 1}{g - 1} - 1, d_2 \frac{g^{n_2} - 1}{g - 1} - 1\right) \geq B.$$

In this case, for every integer $2 \leq n_1 \leq n_2$ and every digit d_1, we check whether

$$\left(d_3 \frac{g^{n_3} - 1}{g - 1} - 1\right)\left(d_2 \frac{g^{n_2} - 1}{g - 1} - 1\right)\left(d_1 \frac{g^{n_1} - 1}{g - 1} - 1\right)$$

is a square, and if yes, then we check whether the corresponding values of a, b and c are integers. If yes, then we found a solution.

We implemented the above algorithm in Magma [2], and the running time was less than four days on an Intel(R) Core(TM) 960 3.2GHz processor.

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