An extension of Lehman's theorem and ideal set functions

Tamás Király[a,*], Júlia Pap[a,b]

[a] MTA-ELTE Egerváry Research Group, Pázmány Péter sétány 1/C, Budapest, 1117 Hungary
[b] Present address: MTA SZTAKI, Lágymányosi u. 11, Budapest, 1111 Hungary

Abstract

Lehman's theorem on the structure of minimally nonideal clutters is a fundamental result in polyhedral combinatorics. One approach to extending it has been to give a common generalization with the characterization of minimally imperfect clutters [15, 8]. We give a new generalization of this kind, which combines two types of covering inequalities and works well with the natural definition of minors. We also show how to extend the notion of idealness to unit-increasing set functions, in a way that is compatible with minors and blocking operations.

Keywords: packing, covering, ideal clutter, integer polyhedron, perfect graph

1. Introduction

A set family $\mathcal{C}$ on a ground set V of size n is called a *clutter* if no set in $\mathcal{C}$ is a subset of another. We will refer to elements of V simply as *elements*, while elements of $\mathcal{C}$ will be referred to as *members* of $\mathcal{C}$. Let $\mathcal{C}^{\uparrow}$ denote the uphull of $\mathcal{C}$, that is, $\mathcal{C}^{\uparrow} = \{U \subseteq V : U \supseteq C \text{ for some } C \in \mathcal{C}\}$. The *blocker* $b(\mathcal{C})$ of a clutter $\mathcal{C}$ is defined as the family of the (inclusionwise) minimal sets that intersect each member of $\mathcal{C}$. It is easy to check that $b(b(\mathcal{C})) = \mathcal{C}$, see e.g. [3, Theorem 1.3].

One of the most well-studied objects of polyhedral combinatorics is the *covering polyhedron* of a clutter, which we consider in the following bounded version:

$$P(\mathcal{C}) = \{x \in \mathbb{R}^V : \mathbf{0} \leq x \leq \mathbf{1},\ x(C) \geq 1 \text{ for every } C \in \mathcal{C}\},$$

where $x(C)$ denotes $\sum_{v \in C} x_v$. The integer points of $P(\mathcal{C})$ correspond to the sets in $b(\mathcal{C})^{\uparrow}$. A clutter $\mathcal{C}$ is called *ideal* if the polyhedron $P(\mathcal{C})$ is integer. By a result of Lehman [9], a clutter is ideal if and only if its blocker is.

Deciding whether a clutter is ideal is hard (see e.g. [6], where it is shown to include the co-NP-complete problem of recognizing quasi-bipartite graphs). However, interesting structural properties can be proved for clutters which are minimally nonideal (*mni*) in the sense that any facet of P defined by setting a variable to 0 or 1 is integer. A simple infinite family of mni clutters is the family of finite degenerate projective planes, defined as $\mathcal{J}_t = \{\{1, 2, \dots t\}, \{0, 1\}, \{0, 2\}, \dots \{0, t\}\}$ on ground set $\{0, 1, \dots t\}$, where $t \geq 2$. It is easy to check that the blocker of $\mathcal{J}_t$ is itself. The following theorem of Lehman [9, 10], which shows that all other mni clutters have a regular structure, is considered to be one of the fundamental results on covering polyhedra.

Theorem 1.1 (Lehman [9, 10]). *Let $\mathcal{C}$ be a minimally nonideal clutter nonisomorphic to a finite degenerate projective plane. Then $P(\mathcal{C})$ has a unique noninteger vertex, namely $\frac{1}{r}\mathbf{1}$, where r is the minimum size of an edge in $\mathcal{C}$. There are exactly n sets of size r in $\mathcal{C}$ and each element of V is contained in exactly r of them. The blocker $b(\mathcal{C})$ also has exactly n sets of minimum size, which correspond to the vertices of $P(\mathcal{C})$ adjacent to the noninteger vertex.*

*Corresponding author
Email addresses: `tkiraly@cs.elte.hu` (Tamás Király), `papjuli@cs.elte.hu` (Júlia Pap)

An important consequence of the theorem, observed by Seymour [16], is that the problem of deciding idealness of a clutter is in co-NP, provided that we have a membership oracle for $\mathcal{C}^{\uparrow}$. After Lehman's groundbreaking result, there have been several attempts to better understand the structure of minimally nonideal clutters (see [12] for an enumeration of mni matrices of small dimension, [5] for a characterization of mni circulants, [3] for a survey, and [4, 18] for more recent developments).

There have been successful efforts to combine Lehman's theorem with another fundamental result, the co-NP characterization of minimally imperfect clutters by Lovász and Padberg [11, 13]. A clutter $\mathcal{D}$ is *perfect* if the packing polyhedron $\{x \in \mathbb{R}^V : \mathbf{0} \le x \le \mathbf{1},\ x(D) \le 1 \text{ for every } D \in \mathcal{D}\}$ is integral. It is minimally imperfect if it is not perfect, but any face of the polyhedron obtained by setting some variable to 0 is integral. Note that it is unnecessary to consider faces obtained by setting a variable to 1, because if the face $x_v = 0$ is integral, then so is the face $x_v = 1$.

Theorem 1.2 (Lovász [11], Padberg [13]). *If a clutter $\mathcal{D}$ is minimally imperfect, then it either consists of all $(n-1)$-element subsets of V (the* non-Helly clutter*), or it consists of the maximal cliques of a minimally imperfect graph. In both cases, $\mathcal{D}$ has n maximum size members, and they form a regular hypergraph.*

Of course, we can claim much stronger properties for $\mathcal{D}$ using the Strong Perfect Graph Theorem of Chudnovsky, Robertson, Seymour, and Thomas [2].

Theorem 1.3 (Strong Perfect Graph Theorem [2]). *A graph is perfect if and only if it contains no odd hole (an induced subgraph isomorphic to an odd cycle of length at least 5) and no odd antihole (an induced subgraph isomorphic to the complement of an odd cycle of length at least 5).*

It follows that if $\mathcal{D}$ is minimally imperfect, then it is either a non-Helly clutter, or a clutter formed by the inclusionwise maximal cliques of an odd cycle of length at least 5 or of the complement of an odd cycle of length at least 5.

Sebő [15], and Gasparyan, Preissmann and Sebő [8] considered polyhedra defined by both packing and covering constraints, and gave an extension of Lehman's theorem that includes Theorem 1.2. An inconvenience in their approach is that the class of polyhedra they consider is not closed under taking facets defined by setting variables to 0 or 1, and there is no natural way to define a blocker.

In this paper we present two different approaches that address these issues. In the first part of the paper, in Section 2, we prove an extension of Lehman's theorem to another class of polyhedra that includes both packing and covering polyhedra as a subclass. Let $\mathcal{C}$ and $\mathcal{D}$ be clutters on the same ground set V. We consider polyhedra of the form

$$P(\mathcal{C},\mathcal{D}) = \{x \in \mathbb{R}^V : \mathbf{0} \le x \le \mathbf{1},\ x(C) \ge 1 \text{ for every } C \in \mathcal{C},\ x(D) \ge |D| - 1 \text{ for every } D \in \mathcal{D}\}.$$

If $\mathcal{D}$ is empty, then this is the same as $P(\mathcal{C})$. On the other hand, if $\mathcal{C}$ is empty, then $\{x \in \mathbb{R}^V : \mathbf{1} - x \in P(\mathcal{C},\mathcal{D})\}$ is the packing polyhedron of $\mathcal{D}$. Clearly, this polyhedron is integral if and only if $P(\mathcal{C},\mathcal{D})$ is integral. We will see that faces obtained by setting some variables to 0 or 1 are also polyhedra in this class, defined by appropriate pairs of clutters (these pairs will be called the minors of the pair $(\mathcal{C},\mathcal{D})$). Our main result is that if $P(\mathcal{C},\mathcal{D})$ is non-integral but the faces considered above are all integral, then one of the following holds: a) $\mathcal{D}$ is empty and $\mathcal{C}$ is a minimally nonideal clutter, b) $\mathcal{C}$ is empty and $\mathcal{D}$ is a minimally imperfect clutter, or c) $\mathcal{D}$ has only members of size 2, and $\mathcal{C} \cup \mathcal{D}$ is an odd cycle or a degenerate projective plane.

As a corollary, we derive a new characterization of integrality of a polytope associated with the vertex cover problem in hypergraphs. Let $H = (V, \mathcal{E})$ be a hypergraph, and let G_H be the graph consisting of the hyperedges of size two in H. Lehman's theorem is a characterization of the integrality of the fractional vertex cover polyhedron for H. A weakness of this LP relaxation is that the polyhedron is automatically non-integer if G_H contains a triangle. To fix this, let us consider the polyhedron obtained by adding the clique inequalities of G_H:

$$P = \{x \in \mathbb{R}^V : \mathbf{0} \le x \le \mathbf{1},\ x(e) \ge 1 \text{ for every } e \in \mathcal{E},\ x(K) \ge |K| - 1 \text{ for every clique } K \text{ in } G_H.\}$$

We give a Lehman-type characterization of the integrality of P. This implies that integrality is in co-NP even if the hypergraph is given implicitly by an oracle that outputs whether a given set $X \subseteq V$ induces a hyperedge or not.

An integer-valued set function f on ground set V is *unit-increasing* if $f(U) \leq f(U+v) \leq f(U)+1$ for every $U \subseteq V$ and $v \notin U$ [1]. In the second part of the paper, in Section 3, we extend the notion of idealness to unit-increasing set functions. To a clutter $\mathcal{C}$ we can associate the unit-increasing function

$$f_{\mathcal{C}}(U) = \begin{cases} 1 & \text{if } U \in \mathcal{C}^{\uparrow}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We show that it is possible to associate $(n+1)$-dimensional polyhedra to unit-increasing set functions in such a way that the notions of minor, blocker, and idealness are natural extensions of these notions for clutters, so the blocker of $f_{\mathcal{C}}$ is $f_{b(\mathcal{C})}$, and $f_{\mathcal{C}}$ is ideal if and only if $\mathcal{C}$ is ideal. Furthermore, the property that idealness is equivalent to the idealness of the blocker remains true for any unit-increasing function. For matroids this means that both the rank function (which is submodular) and the co-rank function (which is supermodular) are ideal.

Another attractive characteristic of this approach is the existence of a "twisting" operation on unit-increasing set functions that preserves idealness. For example, the degenerate projective plane on n elements (that can be considered as the exceptional case in Lehman's theorem) is a twisting of the set function corresponding to the exceptional non-Helly clutter in the theorem of Lovász.

One caveat is that this approach does not offer a direct generalization of packing polyhedra. However, we will show that to a clutter $\mathcal{D}$ one can associate a set function $g_{\mathcal{D}}$ such that $g_{\mathcal{D}}$ is minimally nonideal if and only if $\mathcal{D}$ is minimally imperfect.

It seems that Lehman's theorem cannot be fully extended to this setting, and this gives rise to several open question that are presented in Section 4. We show an example of an mni function where the fractional vertex of the polyhedron is not simple. However, we are unaware of any example where the polyhedron of an mni set function has more than one non-integer vertex. Note that Lehman's theorem implies that idealness of clutters (i.e. functions of type $f_{\mathcal{C}}$) is in co-NP if we have a function evaluation oracle. An interesting open question is whether this is true for arbitrary unit-increasing set functions.

1.1. Preliminaries on clutters

As several definitions in the paper are derived from the same notions used in the theory of clutters, it is useful to describe the clutter versions first. There are two types of minor operations for a clutter $\mathcal{C}$ on ground set V, corresponding to including or excluding an element $v \in V$ in the blocker:

- the *deletion minor* is the clutter $\mathcal{C} \backslash v$ on ground set $V - v$ with members $\{C \in \mathcal{C} : v \notin C\}$,
- the *contraction minor* is the clutter $\mathcal{C}/v$ on ground set $V - v$ whose members are the inclusionwise minimal sets in $\{C - v : C \in \mathcal{C}\}$.

A *minor* of $\mathcal{C}$ is a clutter obtained by repeated application of these two operations. It can be seen that the order of the operations does not matter. For disjoint subsets $U, W \subseteq V$, the minor obtained by deleting the elements of U and contracting the elements of V is denoted by $(\mathcal{C} \setminus U)/W$. It is easy to see that the covering polyhedron of this minor is the (perhaps empty) face of $P(\mathcal{C})$ obtained by setting $x_v = 1$ for every $v \in U$ and $x_v = 0$ for every $v \in W$ (more precisely, the covering polyhedron is obtained from this face by projecting out the variables in $U \cup W$). A clutter is *minimally nonideal* (or *mni* for short) if it is not ideal but all of its minors are ideal.

Minimally imperfect clutters are not defined through these minor operations. Instead, one can say that a clutter $\mathcal{D}$ is *minimally imperfect* if its packing polyhedron $\{x \in \mathbb{R}^V : \mathbf{0} \leq x \leq \mathbf{1},\ x(D) \leq 1 \text{ for every } D \in \mathcal{D}\}$ is integral, but for any $v \in V$ the packing polyhedron of the clutter formed by the inclusionwise maximal sets in $\{D - v : D \in \mathcal{D}\}$ is not integral.

[1]Throughout the paper, we use $U + v$ and $U - v$ as shorthand for $U \cup \{v\}$ and $U \setminus \{v\}$, respectively.

2. Generalization of Lehman's theorem to pairs of clutters

Let $\mathcal{C}$ and $\mathcal{D}$ be clutters on ground set V of size n. We consider the polyhedron

$$P(\mathcal{C},\mathcal{D}) = \{x \in \mathbb{R}^V : \mathbf{0} \le x \le \mathbf{1},\ x(C) \ge 1\ \forall C \in \mathcal{C},\ x(D) \ge |D|-1\ \forall D \in \mathcal{D}\}.$$

As mentioned in the introduction, we would like minors to correspond to faces obtained by fixing some variables to 0 or 1. This can be achieved by defining minors of a pair $(\mathcal{C},\mathcal{D})$ the following way.

- The *deletion* minor for $v \in V$ is a pair $(\mathcal{C}\setminus v, \mathcal{D}\setminus v)$ on ground set $V-v$, where $\mathcal{C}\setminus v = \{C \in \mathcal{C} : v \notin C\}$ and $\mathcal{D}\setminus v$ consists of the inclusionwise maximal members of $\{D - v : D \in \mathcal{D}\}$.
- The *contraction* minor for $v \in V$ is a pair $(\mathcal{C}/v, \mathcal{D}/v)$ on ground set $V-v$, where $\mathcal{C}/v$ consists of the inclusionwise minimal members of $\{C - v : C \in \mathcal{C}\} \cup \{\{w\} : \exists D \in \mathcal{D} : v, w \in D\}$, and $\mathcal{D}/v = \{D \in \mathcal{D} : v \notin D\}$.

Deletion corresponds to setting $x_v = 1$, while contraction is obtained by setting $x_v = 0$. To see the latter, observe that if $v \in D$ for some $D \in \mathcal{D}$ and we set $x_v = 0$, then the value of x must be 1 on all other elements in D in order to satisfy $x(D) \ge |D|-1$.

We call a pair $(\mathcal{C},\mathcal{D})$ *ideal* if $P(\mathcal{C},\mathcal{D})$ is an integer polyhedron. Thus $(\mathcal{C},\emptyset)$ is ideal if and only if $\mathcal{C}$ is an ideal clutter, and $(\emptyset,\mathcal{D})$ is ideal if and only if $\mathcal{D}$ is a perfect clutter. The pair $(\mathcal{C},\mathcal{D})$ is *minimally nonideal* if every minor is ideal but $(\mathcal{C},\mathcal{D})$ itself is not. By the correspondence between minors and faces of $P(\mathcal{C},\mathcal{D})$, this means that every non-integer vertex has only non-integral components.

We may assume that $\mathcal{D}$ contains no singletons, because these give redundant conditions. If $D \in \mathcal{D}$ has size 2, then the condition it defines is $x(D) \ge 1$, the same as if it was in $\mathcal{C}$. Therefore we can assume that $\mathcal{D}$ has no members of size 2. If $\mathcal{C}$ contains a singleton $\{v\}$, then $x_v = 1$ for any $x \in P(\mathcal{C},\mathcal{D})$, so $(\mathcal{C},\mathcal{D})$ is not mni. If $|C \cap D| \ge 2$ for some $C \in \mathcal{C}$ and $D \in \mathcal{D}$, then the condition $x(C) \ge 1$ is redundant because $x(C \cap D) \ge 1$ is implied by $x(D) \ge |D|-1$ and $x \le \mathbf{1}$. We can also assume $n \ge 3$, since there are no mni clutters on two elements. To summarize, the following can be assumed when mni pairs are concerned.

$$n \ge 3,\ |C| \ge 2\ \forall C \in \mathcal{C},\ |D| \ge 3\ \forall D \in \mathcal{D}, \text{ and } |C \cap D| \le 1\ \forall C \in \mathcal{C}\ \forall D \in \mathcal{D}. \tag{2}$$

Let $(\mathcal{C},\mathcal{D})$ be a minimally nonideal pair that satisfies (2), and let $\mathbf{0} < x^* < \mathbf{1}$ be a non-integral vertex of $P(\mathcal{C},\mathcal{D})$. We introduce the following notation.

$$\mathcal{C}^* = \{C \in \mathcal{C} : x^*(C) = 1\}, \tag{3}$$
$$\mathcal{D}^* = \{D \in \mathcal{D} : x^*(D) = |D|-1\}, \tag{4}$$
$$\mathcal{C}^*_v = \{C \in \mathcal{C}^* : v \notin C\} \quad \forall v \in V, \tag{5}$$
$$\mathcal{D}^*_v = \{D \in \mathcal{D}^* : v \notin D\} \quad \forall v \in V. \tag{6}$$

Before proving the main theorem of this section, we prove a sequence of propositions that are analogous to ones used in various proofs of Lehman's theorem (see e.g. [16]). By a slight abuse of notation, we sometimes identify a set $X \subseteq V$ and its characteristic vector $\chi_X \in \{0,1\}^V$, so if $\mathcal{F}$ is a family of sets, then $\langle \mathcal{F} \rangle$ denotes the subspace of $\mathbb{R}^V$ generated by the characteristic vectors of the members. Unless otherwise stated, a characteristic vector χ_X is an n-dimensional vector.

Proposition 2.1. *If $(\mathcal{C},\mathcal{D})$ is mni and x^* is a non-integral vertex of $P(\mathcal{C},\mathcal{D})$, then for any $v \in V$ and any $Z \in \mathcal{C}^*_v \cup \mathcal{D}^*_v$ we have* $\dim\langle \mathcal{C}^*_v \cup \mathcal{D}^*_v \rangle \le n - |Z|$.

PROOF. Fix a node $v \in V$. Let x^*_{-v} denote the vector x^* restricted to $V-v$. This vector is in the integer polyhedron $P(\mathcal{C}\setminus v, \mathcal{D}\setminus v)$, so x^*_{-v} is a convex combination $\sum_{j=1}^t \lambda_j z^j$ of integer vertices of $P(\mathcal{C}\setminus v, \mathcal{D}\setminus v)$, where $\lambda_j > 0$ for every j. Let $X_j \subseteq V - v$ be the subset corresponding to z^j.

Consider a set $C \in \mathcal{C}^*_v$. For every $u \in C$ we have $x^*_u > 0$, so there exists j_u such that $u \in X_{j_u}$. Since $x^*(C) = 1$, we have $X_{j_u} \cap C = \{u\}$. Therefore the vectors $\{\chi_{X_{j_u}} : u \in C\}$ are linearly independent. Moreover, $x^* \notin \langle \chi_{X_{j_u}} : u \in C \rangle$, since $x^*_v > 0$ and $X_j \subseteq V - v$ for every j. Thus $\dim\langle \chi_{X_{j_u}} - x^* : u \in C \rangle = |C|$.

On the other hand, each X_j is tight for the inequalities corresponding to sets in $\mathcal{C}_v^* \cup \mathcal{D}_v^*$. This means that $(\chi_{X_{j_u}} - x^*)(C') = 0$ for every $u \in C$ and every $C' \in \mathcal{C}_v^*$. Furthermore, $(\chi_{X_{j_u}} - x^*)(D') = 0$ for every $u \in C$ and every $D' \in \mathcal{D}_v^*$. Thus $\dim\langle \mathcal{C}_v^* \cup \mathcal{D}_v^* \rangle \leq n - |C|$.

Now let $D \in \mathcal{D}_v^*$. We have $x_u^* < 1$ for every $u \in D$, so there exists j_u such that $u \notin X_{j_u}$. Since $x^*(D) = |D| - 1$ and $|X_{j_u} \cap D| \geq |D| - 1$ (the latter is because $D \in \mathcal{D} \setminus v$), we have $X_{j_u} \cap D = D - u$. Therefore the vectors $\{\chi_{X_{j_u}} : u \in D\}$ are affine independent. Moreover, $x^* \notin \text{aff}\{\chi_{X_{j_u}} : u \in D\}$, because $x_v^* > 0$ and $X_j \subseteq V - v$ for every j. Thus $\dim\langle \chi_{X_{j_u}} - x^* : u \in D \rangle = |D|$.

Here too we have $(\chi_{X_{j_u}} - x^*)(D') = 0$ for every $u \in D$ and every $D' \in \mathcal{D}_v^*$, and $(\chi_{X_{j_u}} - x^*)(C') = 0$ for every $u \in D$ and every $C' \in \mathcal{C}_v^*$. Thus $\dim\langle \mathcal{C}_v^* \cup \mathcal{D}_v^* \rangle \leq n - |D|$. □

Proposition 2.2. *If $(\mathcal{C}, \mathcal{D})$ is mni and x^* is a non-integral vertex of $P(\mathcal{C}, \mathcal{D})$, then the size of $\mathcal{C}^* \cup \mathcal{D}^*$ is n, and $|Z| = n - |\mathcal{C}_v^* \cup \mathcal{D}_v^*|$ for every $v \in V$ and every $Z \in \mathcal{C}_v^* \cup \mathcal{D}_v^*$. Also, every vertex of $P(\mathcal{C}, \mathcal{D})$ adjacent to x^* is integral.*

Proof. Let $\mathcal{B}$ be a base chosen from $\mathcal{C}^* \cup \mathcal{D}^*$, and for $v \in V$ let $\mathcal{B}_v$ denote $\{B \in \mathcal{B} : v \notin B\}$. The size of $\mathcal{B}$ is n, and by Proposition 2.1 we have $|\mathcal{B}_v| \leq n - |Z|$ for every $Z \in \mathcal{C}_v^* \cup \mathcal{D}_v^*$ and for every v. Let $U = \{u \in V : \exists B \in \mathcal{B} \text{ s.t. } u \notin B\}$. We can write

$$n = \sum_{B \in \mathcal{B}} 1 = \sum_{B \in \mathcal{B}} \sum_{v \in V \setminus B} \frac{1}{n - |B|} = \sum_{u \in U} \sum_{B \in \mathcal{B}_u} \frac{1}{n - |B|} \leq \sum_{u \in U} \sum_{B \in \mathcal{B}_u} \frac{1}{|\mathcal{B}_u|} = \sum_{u \in U} 1 = |U| \leq n.$$

Therefore there is equality throughout, so $U = V$, and $|B| = n - |\mathcal{B}_v|$ for every v and every $B \in \mathcal{B}_v$.

Let $H = (V, \mathcal{E})$ be the hypergraph with hyperedges $\mathcal{E} = \{V \setminus Z : Z \in \mathcal{C}^* \cup \mathcal{D}^*\}$, and let $H' = (V, \mathcal{E}')$ be the subhypergraph corresponding to $\mathcal{B}$. Let $H_1 = (V_1, \mathcal{E}_1), \ldots, H_k = (V_k, \mathcal{E}_k)$ denote the components of H'. By the above, H' has no isolated node, and there are numbers $r_1, \ldots, r_k$ such that component H_j is r_j-regular and r_j-uniform. If $H \neq H'$, then there is a set $B' \in \mathcal{B}$ and a set $B'' \in (\mathcal{C}^* \cup \mathcal{D}^*) \setminus \mathcal{B}$ such that $\mathcal{B}'' = \mathcal{B} - B' + B''$ is also a base. Let H'' be the corresponding sub-hypergraph. This must also have regular and uniform components, but since we replaced only one hyperedge, this is only possible if $B' = B''$, a contradiction. Thus we have $H = H'$, and $|\mathcal{E}| = n$.

We can also show by a similar argument that every vertex of $P(\mathcal{C}, \mathcal{D})$ adjacent to x^* is an integer vertex. Indeed, a non-integer adjacent vertex would satisfy with equality all but one of the inequalities corresponding to $\mathcal{C}^* \cup \mathcal{D}^*$. Furthermore, together with a new tight inequality we would obtain a hypergraph with the same kind of structure (because what we proved up to now is true for any non-integer vertex). This is impossible because we cannot have regular and uniform components after replacing a single hyperedge. □

Now we are ready to prove the main theorem of this section.

Theorem 2.3. *If $(\mathcal{C}, \mathcal{D})$ is an mni pair that satisfies* (2)*, then either $\mathcal{D}$ is empty and $\mathcal{C}$ is a minimally nonideal clutter, or $\mathcal{C}$ is empty and $\mathcal{D}$ is a minimally imperfect clutter.*

Proof. Let $\mathbf{0} < x^* < \mathbf{1}$ be a non-integer vertex of $P(\mathcal{C}, \mathcal{D})$ and let $\mathcal{C}^*$ and $\mathcal{D}^*$ be defined as in (3) and (4). As in the proof of Proposition 2.2, $H = (V, \mathcal{E})$ denotes the hypergraph with hyperedges $\mathcal{E} = \{V \setminus Z : Z \in \mathcal{C}^* \cup \mathcal{D}^*\}$, and its components are $H_1, \ldots, H_k$, where H_i is r_i-uniform and r_i-regular. We assume that $r_1 \leq r_2 \leq \cdots \leq r_k$. The vertex x^* is simple because $|\mathcal{C}^* \cup \mathcal{D}^*| = n$ by Proposition 2.2. The proof of the theorem is divided into three cases.

Case 1: $\mathcal{D}^* = \emptyset$. It can be seen that

$$x_v^* = \frac{1}{(-1 + \sum_{j=1}^k \frac{|V_j|}{r_j}) r_l} \text{ if } v \in V_l,$$

because this is the unique solution of the equation system given by $\mathcal{C}^*$. If $k = 1$ or $k \geq 3$, then $x_v^* \leq \frac{1}{2}$ for every v, which implies that $\mathcal{D}$ is empty. If $k = 2$, then $x_v^* \leq \frac{1}{2}$ for every v unless $|V_1| = 1$. In this case $x_v^* = \frac{r_2}{n-1}$ if $v = V_1$ and $x_v^* = \frac{1}{n-1}$ otherwise, which implies that $x^*(Z) < |Z| - 1$ for every set Z of size at least 3. Thus $\mathcal{D}$ is empty again, and therefore $\mathcal{C}$ is a minimally nonideal clutter.

Case 2: $\mathcal{C}^* = \emptyset$. Since $|C \cap D| \leq 1$ for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$, $\mathcal{C}$ must be empty in case of $k \geq 3$ because every pair of elements is in some $D \in \mathcal{D}^*$. Thus $\mathcal{D}$ is minimally imperfect. If $k = 2$, then all members of $\mathcal{C}$ have size 2. In this case $\mathcal{C} \cup \mathcal{D}$ is a minimally imperfect clutter; however, by Theorem 1.3, a minimally imperfect clutter does not contain members of size 2 unless it is an odd cycle, which contradicts the assumption that members of $\mathcal{D}$ have size at least 3.

If $k = 1$, then $\mathcal{D}^*$ is r-regular and r-uniform, so $x^*_v = \frac{r-1}{r}$ for every $v \in V$. If $D \in \mathcal{D} \setminus \mathcal{D}^*$, then $x^*(D) > |D| - 1$, which implies that $|D| < r$. Thus $\mathcal{D}^*$ consists precisely of the maximum size elements of $\mathcal{D}$, and x^* is the only non-integer vertex for which Case 2 holds. Also, there is no other non-integer vertex for which Case 1 holds either, because we have seen that $\mathcal{D}$ is empty in Case 1.

We claim that $\mathcal{D}$ is minimally imperfect. First, $\mathcal{D}$ is not perfect, because $\mathbf{1} - x^*$ is a non-integer vertex of the polyhedron $\{x \in \mathbb{R}^V : \mathbf{0} \leq x \leq \mathbf{1},\ x(D) \leq 1 \text{ for every } D \in \mathcal{D}\}$. Suppose that $\mathcal{D}$ is not minimally imperfect; then there is a set $U \subseteq V$ such that the inclusionwise maximal members of $\mathcal{D} \backslash U = \{D \backslash U : D \in \mathcal{D}\}$ form a minimally imperfect clutter. By Theorem 1.2, the polyhedron $\{x \in \mathbb{R}^{V \backslash U} : \mathbf{0} \leq x \leq \mathbf{1},\ x(Z) \leq 1 \text{ for every } Z \in \mathcal{D} \setminus U\}$ has a non-integer vertex x' whose components are at most $\frac{1}{2}$. This means that $\mathbf{1} - x'$ is a vertex of $P(\mathcal{C} \setminus U, \mathcal{D} \setminus U)$, because every member of $\mathcal{C} \setminus U$ has size at least 2. This contradicts the assumption that $(\mathcal{C}, \mathcal{D})$ is minimally nonideal, so we obtained that $\mathcal{D}$ is minimally imperfect.

Now we prove that $\mathcal{C} = \emptyset$. By Theorem 1.3, $\mathcal{D}$ is either a non-Helly clutter or a clutter formed by the inclusionwise maximal cliques of an odd antihole. In the former case, any two elements are in a member of $\mathcal{D}$, so $\mathcal{C}$ is empty because of (2). In the latter case, $\mathcal{C}$ can only have members of size 2 (the edges of the complement of the odd antihole), because all other sets have two common elements with at least one member of $\mathcal{D}$. Thus $\mathcal{C} \cup \mathcal{D}$ is a minimally imperfect clutter, which means that $\mathcal{C} = \emptyset$.

We proved the theorem for the cases when $\mathcal{C}^*$ or $\mathcal{D}^*$ is empty. Therefore we are done if we prove that Case 3 below is impossible.

Case 3: $\mathcal{C}^* \neq \emptyset$, $\mathcal{D}^* \neq \emptyset$. Consider a set $D \in \mathcal{D}^*$. If $k \geq 3$, then any $C \in \mathcal{C}^*$ intersects D in at least 2 elements, which is impossible because $|C \cap D| \leq 1$ for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$ by (2). If $k = 2$, then for the same reason the only possibility is that $|V_1| = 1$, $\mathcal{D}^* = \{V_2\}$ and $\mathcal{C}$ has only sets of size 2. As in Case 2, we can argue that this must correspond to a minimally imperfect clutter. Therefore we can assume that $k = 1$, and $\mathcal{C}^* \cup \mathcal{D}^*$ is r-regular and r-uniform, where $r \geq 3$.

We now prove that it is impossible to have $\mathcal{C}^* \neq \emptyset$ and $\mathcal{D}^* \neq \emptyset$. Let $D \in \mathcal{D}^*$ and let χ_X be the vertex adjacent to x^* that is not tight for D. Thus $D \subseteq X$ and X is tight for all other inequalities corresponding to $\mathcal{C}^* \cup \mathcal{D}^*$. Let M be the matrix whose rows are the characteristic vectors of $\mathcal{C}^* \cup \mathcal{D}^*$, with the last row being χ_D, and let e_n denote the n-th unit vector. Then $M\chi_X = Mx^* + e_n$ and $\mathbf{1}^T M = r\mathbf{1}^T$, so $|X| = \frac{1}{r}\mathbf{1}^T M \chi_X = \frac{1}{r}\mathbf{1}^T(Mx^* + e_n) = \mathbf{1}^T x^* + \frac{1}{r}$. Thus the fractional part of $\mathbf{1}^T x^*$ is $\frac{r-1}{r}$. If $x^*_v > \frac{r-1}{r}$ for some $v \in V$, then $\mathbf{1}^T x^*_{-v} < \lfloor \mathbf{1}^T x^* \rfloor = |X| - 1$. In this case $P(\mathcal{C} \setminus v, \mathcal{D} \setminus v)$ has an integer vertex χ_Y with $|Y| < |X| - 1$. The vector $\chi_Y + \chi_v$ is in $P(\mathcal{C}, \mathcal{D})$ and there is a member of $\mathcal{C}^* \cup \mathcal{D}^*$ for which it is not tight, so

$$|X| > |Y + v| = \frac{1}{r}\mathbf{1}^T M(\chi_Y + \chi_v) \geq \mathbf{1}^T x^* + \frac{1}{r} = |X|,$$

a contradiction. We obtained that $x^*_v \leq \frac{r-1}{r}$ for every v, which implies that for every $D \in \mathcal{D}^*$ and $v \in D$ we have $x^*_v = \frac{r-1}{r}$.

Suppose that there exist $C \in \mathcal{C}^*$ and $D \in \mathcal{D}^*$ such that $|C \cap D| = 1$, and let v be the intersection. Since $\mathbf{1}^T x^*_{-v} = \lfloor \mathbf{1}^T x^* \rfloor$, x^*_{-v} must be a convex combination of integer vertices $z^1, \ldots, z^t$ of $P(\mathcal{C} \setminus v, \mathcal{D} \setminus v)$ that all satisfy $\mathbf{1}^T z^j = \mathbf{1}^T x^*_{-v}$. Let $X_j \subseteq V - v$ be the set corresponding to z^j. As $|X_j + v| = |X|$, $X_j + v$ satisfies all but one of the inequalities corresponding to $\mathcal{C}^* \cup \mathcal{D}^*$ with equality, and the slack of the remaining inequality is 1. Consequently, $X_j + v$ is a vertex adjacent to x^* in $P(\mathcal{C}, \mathcal{D})$ for every j. We can now get a contradiction using the fact that $|C| \geq 3$. Indeed, x^*_{-v} is positive on the elements of $C - v$, and each X_j contains at most 1 such element because $|C \cap (X_j + v)| \leq 2$, so there are at least two sets X_{j_1} and X_{j_2} each containing an element of $C - v$. Thus $X_{j_1} + v$ and $X_{j_2} + v$ are not tight for C, hence they are tight for all other members of $\mathcal{C}^* \cup \mathcal{D}^*$. But this is impossible because there is only one integer vertex of $P(\mathcal{C}, \mathcal{D})$ that is tight for all of those sets.

The only remaining case is when $C \cap D = \emptyset$ for any $C \in \mathcal{C}^*$ and $D \in \mathcal{D}^*$. Let U be the union of the members of $\mathcal{C}^*$. Since $\mathcal{C}^* \cup \mathcal{D}^*$ is a base, $\mathcal{C}^*$ must be a base on U. The vector $x^*|_U$ is in $P(\mathcal{C} \setminus (V \setminus U), \mathcal{D} \setminus$

$(V \setminus U)$), and it satisfies the inequalities corresponding to $\mathcal{C}^*$ with equality, so it is a non-integer vertex of $P(\mathcal{C} \setminus (V \setminus U), \mathcal{D} \setminus (V \setminus U))$. But this contradicts the assumption that the polyhedron is integer, so this case is also impossible. □

We now prove that with the appropriate oracles it is in co-NP to decide whether a pair $(\mathcal{C}, \mathcal{D})$ is ideal.

Theorem 2.4. *We assume that, given a set Z, we have an oracle that outputs whether $\mathcal{C}$ contains a subset of Z, and another oracle that outputs whether $\mathcal{D}$ contains a superset of Z. With these oracles, it is in co-NP to decide if a pair $(\mathcal{C}, \mathcal{D})$ is ideal.*

Proof. One can check by $O(n^2)$ oracle calls whether the characteristic vector of a set X is in the polyhedron $P(\mathcal{C}, \mathcal{D})$. Furthermore, it is easy to see that oracles for any given minor of $(\mathcal{C}, \mathcal{D})$ can be obtained using the oracles for $(\mathcal{C}, \mathcal{D})$. Thus it is enough to certify non-idealness for an mni pair $(\mathcal{C}, \mathcal{D})$.

If we are in Case 1 of Theorem 2.3, then the emptiness of $\mathcal{D}$ can be checked by the oracle, and a certificate for non-idealness of $\mathcal{C}$ can be given as in [16]. In Case 2, we know that $\mathcal{D}^*$ is r-regular and r-uniform for some r, and that the vertices adjacent to x^* are integral. The certificate for non-idealness is $\mathcal{D}^*$ and the set of vertices adjacent to x^*. Since the latter are integral, we can check using the oracles that they are indeed in the polyhedron; we can also check that the members of $\mathcal{D}^*$ are indeed in $\mathcal{D}$. Although we cannot check that x^* itself is in the polyhedron, we can be sure that the simplex formed by x^* and the adjacent vertices contains at least one non-integer vertex of $P(\mathcal{C}, \mathcal{D})$, so the pair is nonideal. □

The following property of nonideal pairs can also be derived from Theorem 2.3.

Corollary 2.5. *If $(\mathcal{C}, \mathcal{D})$ is a nonideal pair, then $\mathcal{C}$ is a nonideal clutter or $\mathcal{D}$ is an imperfect clutter.*

Proof. If $(\mathcal{C}, \mathcal{D})$ is nonideal, then it has an mni minor, obtained by deleting a set U and contracting a set W. In the contraction minor, some singletons are added to $\mathcal{C}$; however, an mni minor does not contain singletons, so all these singletons are in U. This means that the minor is $(\mathcal{C}', \mathcal{D}')$, where $\mathcal{C}'$ consists of the inclusionwise minimal members of $\{C \setminus W : C \in \mathcal{C},\ C \cap U = \emptyset\}$, while $\mathcal{D}'$ consists of the inclusionwise maximal members of $\{D \setminus (U \cup W) : D \in \mathcal{D}\}$. Similarly to the proof of Theorem 2.4, we can see that if Case 1 holds in Theorem 2.3, then $\mathcal{C}'$ is mni, thus $\mathcal{C}$ is nonideal. If Case 2 holds, then $\mathcal{D}'$ is minimally imperfect, so $\mathcal{D}$ is imperfect too. □

2.1. Vertex cover in hypergraphs

Using Theorem 2.3, we can prove a new result on the vertex cover problem in hypergraphs. We give a characterization of the integrality of the fractional vertex cover polyhedron strengthened by clique inequalities for the edges of size two. Let $H = (V, \mathcal{E})$ be a hypergraph, and let G_H be the graph consisting of the hyperedges of size two in H. Let

$$P = \{x \in \mathbb{R}^V : \ \mathbf{0} \le x \le \mathbf{1},\ x(e) \ge 1 \text{ for every } e \in \mathcal{E},\ x(K) \ge |K| - 1 \text{ for every clique } K \text{ in } G_H.\} \qquad (7)$$

A *minor* of H is obtained by deletion of a node set U_1 and contraction of a node set U_2: we remove all hyperedges incident to U_1, and remove the nodes of U_2 from all remaining hyperedges. We assume that U_2 does not induce any hyperedge. A minor is called *triangle-free* if U_1 covers every triangle of G_H, and it is *mni* if the clutter formed by the inclusionwise minimal hyperedges is mni.

Theorem 2.6. *The polyhedron P is integer if and only if H has no triangle-free mni minor and G_H is perfect.*

Proof. The proof of necessity is straightforward: first, if G_H contains an odd hole or an odd antihole, then the corresponding non-integer vertex of the fractional vertex cover polyhedron of G_H is a vertex of P. Second, if H' is a triangle-free mni minor and $\mathcal{C}'$ is the clutter of inclusionwise minimal hyperedges, then the non-integer vertex of $P(\mathcal{C}')$, supplemented by $\mathbf{1}$ on U_1 and $\mathbf{0}$ on U_2, is a vertex of P.

To prove sufficiency, let $\mathcal{C}$ be the clutter formed by the inclusionwise minimal elements of $\mathcal{E}$, and let $\mathcal{D}$ be the clutter formed by the inclusionwise maximal cliques of G_H of size at least 3. It is easy to see that P is integer if and only if $P(\mathcal{C},\mathcal{D})$ is integer. Suppose that the pair $(\mathcal{C},\mathcal{D})$ has a minimally nonideal minor $(\mathcal{C}',\mathcal{D}')$, obtained by deleting U_1 and contracting U_2. In order to satisfy the assumptions of Theorem 2.3, we remove the two-element sets from $\mathcal{D}'$ (these are also present in $\mathcal{C}'$), and remove all elements from $\mathcal{C}'$ that intersect some member of $\mathcal{D}'$ in more than one element. Let $(\mathcal{C}_0,\mathcal{D}_0)$ denote the resulting mni pair; by Theorem 2.3, one of the following two cases holds.

Case 1: $\mathcal{D}_0$ is empty, and $\mathcal{C}_0$ is an mni clutter. The emptiness of $\mathcal{D}_0$ means that U_1 covers all triangles of G_H, therefore we have a triangle-free mni minor of H.

Case 2: $\mathcal{C}_0$ is empty and $\mathcal{D}_0$ is a minimally imperfect clutter. Let $U_3 = V\setminus(U_1\cup U_2)$, and let $G = G_H[U_3]$, i.e. the graph induced by U_3. As $\mathcal{C}_0$ cannot contain a singleton, every edge of G_H is either induced by $U_1\cup U_3$ or goes between U_1 and U_2. This means that $\mathcal{D}_0$ consists of the cliques of G of size at least 3, so G is an odd antihole. □

By Theorem 2.4, we obtain the following corollary.

Corollary 2.7. *Let the hypergraph $H = (V,\mathcal{E})$ be given implicitly by an oracle that outputs whether a given set $X \subseteq V$ induces a hyperedge or not. Then it is in co-NP to decide if the polyhedron P defined in (7) is integral.*

3. Ideal set functions

Unit-increasing set functions are ubiquitous in combinatorics; well-known examples include matroid rank functions, clique number and chromatic number of an induced subgraph, etc. The aim of this section is to extend the notions of the blocking relation and idealness from clutters to unit-increasing set functions. We show that several properties of ideal clutters can be maintained: idealness is preserved under taking minors and blockers. In addition, we describe a transformation, called twisting of the set function at a subset, that preserves idealness. In Lehman's theorem, the degenerate projective planes are the exceptional, irregular structures. We show that although they are irregular, they have a twisting that has a regular structure; this gives hope for a possible generalization of Lehman's theorem where no exceptional case is needed.

Let V be a ground set of size n, and let $f : 2^V \to \mathbb{Z}$ be an integer-valued unit-increasing set function. We define the following two minor operations on unit-increasing functions for a given $v \in V$:

- the *deletion minor* is the function on ground set $V - v$, denoted by $f\backslash v$, given by $f\backslash v(X) = f(X)$ for every $X \subseteq V - v$,
- the *contraction minor* is the function on ground set $V - v$, denoted by f/v, given by $f/v(X) = f(X+v)$ for every $X \subseteq V - v$.

A function f' is a *minor* of f if it can be obtained from f by deletions and contractions. It is easy to see that the order of the operations does not affect the minor we get, and the minors are unit-increasing functions.

The *blocker* $b(f) : 2^V \to \mathbb{Z}$ of a unit-increasing set function f is the set function defined by

$$b(f)(X) = -f(V \setminus X)$$

for any set $X \subseteq V$.

Proposition 3.1. *The blocker $b(f)$ has the following properties.*

(i) $b(f)$ *is unit-increasing,*

(ii) $b(b(f)) = f$,

(iii) $b(f\backslash v) = b(f)/v$,

(iv) $b(f/v) = b(f)\backslash v$.

PROOF. (i) f is unit-increasing, so $-f(V\setminus X) \le -f(V\setminus(X+v)) \le -f(V\setminus X)+1$. (ii) $b(b(f)) = -(-f(V\backslash(V\backslash X))) = f(X)$. (iii) If $X\subseteq V-v$, then $b(f\backslash v)(X) = -f((V-v)\backslash X) = b(f)(X+v) = b(f)/v(X)$. (iv) If $X\subseteq V-v$, then $b(f/v)(X) = -f(((V-v)\setminus X)+v) = b(f)(X) = b(f)\backslash v(X)$. □

We call functions f_1 and f_2 *equivalent* if there is a constant c such that $f_2(X) = f_1(X)+c$ for every $X\subseteq V$; we will use the notation $f_1 \cong f_2$.

3.1. Polyhedra and idealness

In this section we show that it is possible to associate polyhedra to unit-increasing set functions in such a way that minors correspond to faces, blockers to integer vertices, and the notion of idealness can be defined in terms of integrality of polyhedra. The trick is to move to an $(n+1)$-dimensional space. For a function f, let

$$P(f) = \{(y,\beta)\in\mathbb{R}^{n+1} : \mathbf{0}\le y\le \mathbf{1},\ y(X)-\beta\ge f(X) \text{ for every } X\subseteq V\}.$$

Proposition 3.2. *The following hold for the minors of f:*

$$P(f\backslash v) = \{(y,\beta)\in\mathbb{R}^{n-1+1} : (y,1,\beta)\in P(f)\},\ \textit{and}$$
$$P(f/v) = \{(y,\beta)\in\mathbb{R}^{n-1+1} : (y,0,\beta)\in P(f)\},$$

in particular, both $P(f\backslash v)$ and $P(f/v)$ are facets of $P(f)$.

PROOF. It is easy to see that for a vector $(y,1,\beta)\in P(f)$, (y,β) satisfies the inequalities of $P(f\backslash v)$, since they are present in the system of $P(f)$ too.

If $(y,\beta)\in P(f\backslash v)$ and $X\subseteq V-v$, then on one hand we have $(y,1)(X)-\beta = y(X)-\beta \ge f\backslash v(X) = f(X)$, and on the other hand $(y,1)(X+v)-\beta = y(X)+1-\beta \ge f\backslash v(X)+1 = f(X)+1\ge f(X+v)$, since f is unit-increasing. So $(y,1,\beta)\in P(f)$.

It is easy to see that for a vector $(y,0,\beta)\in P(f)$, (y,β) satisfies the inequalities of $P(f/v)$, since $y(X)-\beta = (y,0)(X+v)-\beta\ge f(X+v) = f/v(X)$.

If $(y,\beta)\in P(f/v)$ and $X\subseteq V-v$, then on one hand we have $(y,0)(X)-\beta = y(X)-\beta\ge f/v(X) = f(X+v)\ge f(X)$, since f is unit-increasing, and on the other hand $(y,0)(X+v)-\beta = y(X)-\beta\ge f/v(X) = f(X+v)$, thus $(y,0,\beta)\in P(f)$. □

A unit-increasing set function f is called *ideal* if the polyhedron $P(f)$ is integral. As expected, idealness is preserved under taking minors.

Proposition 3.3. *If f is ideal, then any minor of it is also ideal.*

PROOF. It follows from Proposition 3.2. □

This enables us to call a unit-increasing function f *minimally nonideal* (*mni*) if it is not ideal but every minor is ideal. Before showing that this is a direct extension of the same notion for clutters, we prove that we get the same notion of idealness if we remove the upper bound or both bounds on y in the polyhedron. Let

$$Q(f) = \{(y,\beta)\in\mathbb{R}^{n+1} : y\ge\mathbf{0},\ y(X)-\beta\ge f(X) \text{ for every } X\subseteq V\},$$
$$R(f) = \{(y,\beta)\in\mathbb{R}^{n+1} : y(X)-\beta\ge f(X) \text{ for every } X\subseteq V\}.$$

Let C be the cone generated by $\{e_i : i\in[n]\}\cup\{-e_i-e_{n+1} : i\in[n]\}$. We call a set X *tight* with respect to f and a vector (y,β) if $y(X)-\beta = f(X)$.

Lemma 3.4. *If f is a unit-increasing set function, then $Q(f) = P(f)+\mathbb{R}^n_+$, the characteristic cone of $R(f)$ is C, and $R(f) = P(f)+C = Q(f)+C$.*

PROOF. The $Q(f) \supseteq P(f) + \mathbb{R}^n_+$ inclusion is easy, since the matrix describing $Q(f)$ has nonnegative coefficients in the first n variables.

For the $Q(f) \subseteq P(f) + \mathbb{R}^n_+$ inclusion, let $(y, \beta) \in Q(f)$. We want to show that there is a $(y', \beta) \in P(f)$ for which $y' \leq y$. Let $y'_i = \min(y_i, 1)$. Then $y' \leq y$ and $0 \leq y' \leq 1$ hold, so it remains to show that $y'(X) - \beta \geq f(X)$ for each $X \subseteq V$. We have $y'(X) = |X \cap \{i : y_i > 1\}| + y(X \cap \{i : y_i \leq 1\}) \geq |X \cap \{i : y_i > 1\}| + f(X \cap \{i : y_i \leq 1\}) + \beta \geq f(X) + \beta$, since f is unit-increasing.

Next we show that the characteristic cone of $R(f)$ is C. It is easy to see that all the vectors e_i and $-e_i - e_{n+1}$ are in the characteristic cone of $R(f)$. If a vector (z, γ) is in the characteristic cone of $R(f)$, then for every $X \subseteq V$, $z(X) - \gamma \geq 0$ holds. For $X = \{i : z_i < 0\}$ we have $(z, \gamma) = \sum_{i \in X} -z_i(-e_i - e_{n+1}) + (z', \gamma')$, where $z' \geq 0$ and $\gamma' \leq 0$, and it is easy to see that $(z', \gamma') \in C$.

The next step is to show that every vertex (y^*, β^*) of $R(f)$ satisfies $0 \leq y^* \leq 1$. Suppose that $y^*_v < 0$. Then every tight set X contains v, because otherwise the inequality for $X + v$ would be violated since $f(X + v) \geq f(X)$. Now, if every tight set X contains v, then $(y^*, \beta^*) + \varepsilon(\chi_v, 1)$ is in $R(f)$ for some positive ε. This contradicts the fact that (y^*, β^*) is a vertex and $(-\chi_v, -1)$ is an extreme direction. Now suppose that $y^*_v > 1$ for a vertex (y^*, β^*). Then no tight set contains v, since otherwise the inequality for $X - v$ would be violated: $y^*(X - v) - \beta < y^*(X) - 1 - \beta = f(X) - 1 \leq f(X - v)$, a contradiction. This implies that for some positive ε, the vector $(y^*, \beta^*) - \varepsilon(\chi_v, 0)$ is in $R(f)$, which contradicts the fact that (y^*, β^*) is a vertex and e_v is an extreme direction.

We obtained that every vertex of $R(f)$ is in $P(f)$, thus $R(f) = P(f) + C$. As $\mathbb{R}^n_+ \subseteq C$, this implies $R(f) = Q(f) + C$. □

For a polyhedron P, let $\mathrm{vert}(P)$ denote the set of its vertices.

Corollary 3.5. $\mathrm{vert}(P(f)) \supseteq \mathrm{vert}(Q(f)) \supseteq \mathrm{vert}(R(f))$ *for any unit-increasing function* f. □

The corollary implies that if f is ideal, that $Q(f)$ and $R(f)$ are integral. In the next section, we will prove the reverse statement (Theorem 3.8). To give some preliminary intuition on why this equivalence is useful, we show how it can be used to show that a clutter is ideal if and only if its set function is.

Recall that to a clutter $\mathcal{C}$ we associate the set function $f_\mathcal{C}$ defined in (1). It is easy to check that this works well with the minor operations: for any $v \in V$, $f_{\mathcal{C}\setminus v} = f_\mathcal{C} \setminus v$ and $f_{\mathcal{C}/v} = f_\mathcal{C}/v$. Likewise, one can check that the blocker $b(f_\mathcal{C})$ is equivalent to the set function corresponding to the blocker of $\mathcal{C}$ (they differ by 1).

Proposition 3.6. *A clutter* $\mathcal{C}$ *is ideal if and only if* $f_\mathcal{C}$ *is ideal.*

PROOF. It is easy to see that

$$\begin{aligned} Q(f_\mathcal{C}) &= \{(y, \beta) \in \mathbb{R}^{n+1} : \; y \geq \mathbf{0}, \;\; y(X) - \beta \geq f_\mathcal{C}(X) \;\; \forall X \subseteq V\} = \\ &= \{(y, \beta) \in \mathbb{R}^{n+1} : \; y \geq \mathbf{0}, \;\; \beta \leq 0, \; y(C) - \beta \geq 1 \;\; \forall C \in \mathcal{C}\}, \end{aligned} \tag{8}$$

where $\beta \leq 0$ is implied by the inequality $y(\emptyset) - \beta \geq f_\mathcal{C}(\emptyset)$. It follows that the face of $Q(f_\mathcal{C})$ in the hyperplane $\beta = 0$ is the polyhedron $Q(\mathcal{C}) = \{y \in \mathbb{R}^n : y \geq \mathbf{0}, y(C) \geq 1 \; \forall C \in \mathcal{C}\}$, which is integral if and only if $\mathcal{C}$ is an ideal clutter.

To see the other direction, note that in (8) all inequalities but $\beta \leq 0$ are satisfied at equality by the vector $(\mathbf{0}, -1)$. Therefore all vertices of $Q(f_\mathcal{C})$ apart from $(\mathbf{0}, -1)$ are in the hyperplane $\beta = 0$, so they correspond to the vertices of $Q(\mathcal{C})$. It follows that if $\mathcal{C}$ is ideal, then $Q(f_\mathcal{C})$ is integral, so $f_\mathcal{C}$ is ideal by Theorem 3.8. □

Corollary 3.7. *A clutter* $\mathcal{C}$ *is mni if and only if* $f_\mathcal{C}$ *is mni.*

We note that Lehman's Theorem 1.1 has the consequence that if $\mathcal{C}$ is mni, then the polyhedron $P(f_\mathcal{C})$ has a unique fractional vertex and it is simple (here a vertex is simple if it lies on $n + 1$ facets). Indeed, if x^* is the unique fractional vertex of $P(\mathcal{C})$, then $(x^*, 0)$ is the unique fractional vertex of $P(f_\mathcal{C})$, and it lies on the facet $\beta \leq 0$ in addition to the facets determined by the minimum size members of $\mathcal{C}$.

3.2. Blockers and idealness

For a unit-increasing set function f, let us define the following finite set of vectors in $\mathbb{R}^{n+1}$:

$$S(f) = \{(\chi_X, f(X)) : \; X \subseteq V\}.$$

We denote the set $S(f) - \mathrm{cone}\{(\mathbf{0}, -1)\}$ by $S^{\downarrow}(f)$. We note that the idealness of f is equivalent to $P(f) = \mathrm{conv}\{S^{\downarrow}(b(f))\}$. Indeed, a vector $(\chi_Y, \beta) \in \{0,1\}^n \times \mathbb{Z}$ is in $P(f)$ if and only if $\beta \leq |X \cap Y| - f(X)$ for every $X \subseteq V$. As f is unit-increasing, the minimum of $|X \cap Y| - f(X)$ is $b(f)(Y)$, attained at $X = V \setminus Y$. Thus the integer vectors in $P(f)$ are the vectors of the form (χ_Y, β) where $\beta \leq b(f)(Y)$.

Theorem 3.8. *For a unit-increasing set function f, the following are equivalent:*

(i) *f is ideal, that is,* $P(f) = \mathrm{conv}\{S^{\downarrow}(b(f))\}$

(ii) *$b(f)$ is ideal, that is,* $P(b(f)) = \mathrm{conv}\{S^{\downarrow}(f)\}$

(iii) *$R(f)$ is an integer polyhedron*

(iv) *$R(b(f))$ is an integer polyhedron*

(v) *$Q(f)$ is an integer polyhedron*

(vi) *$Q(b(f))$ is an integer polyhedron*

Proof. In the proof we will use an operation B on polyhedra in $\mathbb{R}^{n+1}$, which is similar to taking the blocker of a polyhedron, it differs only in the last coordinate. For a polyhedron $P \subseteq \mathbb{R}^{n+1}$, let us define $B(P)$ as follows:

$$B(P) = \{(y, \beta) \in \mathbb{R}^{n+1} : x^{\mathsf{T}} y \geq \alpha + \beta \text{ for every } (x, \alpha) \in P\}.$$

Note that $B(P)$ is indeed a polyhedron, since using standard polyhedral techniques one can prove that if $P = \mathrm{conv}\{S\} + \mathrm{cone}\{T\}$ for finite vector sets S and T in $\mathbb{R}^{n+1}$, then

$$B(P) = \{(y, \beta) \in \mathbb{R}^{n+1} : s_{[n]}^{\mathsf{T}} y \geq s_{n+1} + \beta \;\forall s \in S \text{ and } t_{[n]}^{\mathsf{T}} y \geq t_{n+1} \;\forall t \in T\}. \tag{9}$$

Suppose that the polyhedron $P \subset \mathbb{R}^{n+1}$ has the following properties:

(a) $\exists \bar{\alpha} : (\mathbf{0}, \bar{\alpha}) \in P$

(b) P is bounded from above in the last coordinate

(c) $(\mathbf{0}, -1)$ is in the characteristic cone of P

Proposition 3.9. *If P satisfies properties (a)-(c) then so does $B(P)$.*

Proof. To see property (a), we can observe that if $P = \mathrm{conv}\{S\} + \mathrm{cone}\{T\}$, then from (9) we get that for $\bar{\beta} = \min(-s_{n+1} : s \in S)$, $(\mathbf{0}, \bar{\beta}) \in P$. For property (b) we can take an $\bar{\alpha}$ such that $(\mathbf{0}, \bar{\alpha}) \in P$ which implies that $\beta \leq \mathbf{0}^{\mathsf{T}} y - \bar{\alpha} = -\bar{\alpha}$. For property (c) we need that $x^{\mathsf{T}} \mathbf{0} \geq -1$ which is obvious, and that $B(P)$ is nonempty which follows from (a). □

Lemma 3.10. *If P satisfies properties (a)-(c) then $B(B(P)) = P$.*

Proof. For every $(x, \alpha) \in P$ and $(y, \beta) \in B(P)$ we have $x^{\mathsf{T}} y \geq \alpha + \beta$ which shows that $P \subseteq B(B(P))$.

Suppose that there is a vector $(x^*, \alpha^*) \in B(B(P))$ which is not in P. Then there is a vector (z, γ) and a number ξ such that ${x^*}^{\mathsf{T}} z + \alpha^* \gamma < \xi$, but for every $(x, \alpha) \in P$, $x^{\mathsf{T}} z + \alpha\gamma \geq \xi$. From (c) it follows that $\gamma \leq 0$.

Case 1: $\gamma = 0$. We show that there is an $\varepsilon > 0$ such that ${x^*}^{\mathsf{T}} z + \alpha^*(-\varepsilon) < x^{\mathsf{T}} z + \alpha(-\varepsilon)$ for each $(x, \alpha) \in P$. Because of (b) we know that there is an $a \in \mathbb{R}$ such that $\alpha \leq a$ for every $(x, \alpha) \in P$. We can assume that $a > \alpha^*$. If $\varepsilon < \frac{\xi - x^{*\mathsf{T}} z}{a - \alpha^*}$, then for every $(x, \alpha) \in P$, $\varepsilon(\alpha - \alpha^*) \le \varepsilon(a - \alpha^*) < \xi - x^{*\mathsf{T}} z \le x^{\mathsf{T}} z - x^{*\mathsf{T}} z$. Since $x^{\mathsf{T}} z + \alpha(-\varepsilon)$ attains its minimum on P, we have an instance of Case 2.

Case 2: $\gamma < 0$. We can assume that $\gamma = -1$, since we can scale the inequalities with a positive multiplier. So we have $x^{*\mathsf{T}} z - \alpha^* < \xi$, and for each $(x, \alpha) \in P$, $x^{\mathsf{T}} z - \alpha \ge \xi$. That means the vector $(z, \xi) \in B(P)$ but for this vector (x^*, α^*) does not fulfil the required inequality to be in the blocker of $B(P)$, which contradicts $(x^*, \alpha^*) \in B(B(P))$. $\square$

Notice that for a unit-increasing function f, the polyhedron $P(f)$ satisfies properties (a)-(c).

Proposition 3.11. $B(P(f)) = \mathrm{conv}\{S(f)\} + C$ *and* $B(R(f)) = \mathrm{conv}\{S^{\downarrow}(f)\}$.

PROOF. First we prove that $B(\mathrm{conv}\{S(f)\} + C) = P(f)$, by Lemma 3.10 this implies the first equation. Using (9), we have

$$B(\mathrm{conv}\{S(f)\} + C) = \{\ (y, \beta) \in \mathbb{R}^{n+1} :\ y(X) \ge f(X) + \beta \quad \forall X \subseteq V,\\ y_i \ge 0 \quad \forall i \in [n],\ -y_i \ge -1 \quad \forall i \in [n]\ \},$$

which is equal to $P(f)$.

Now let us prove that $B(\mathrm{conv}\{S^{\downarrow}(f)\}) = R(f)$, which implies the second equation. Using (9), we have

$$B(\mathrm{conv}\{S^{\downarrow}(f)\}) = \{\ (y, \beta) \in \mathbb{R}^{n+1} :\ y(X) \ge f(X) + \beta\ \forall X \subseteq V\},$$

which is $R(f)$. $\square$

Now we are ready to prove Theorem 3.8. Using Proposition 3.11 and Lemmas 3.4 and 3.10 we have

$$P(f) = \mathrm{conv}\{S^{\downarrow}(b(f))\} \overset{+C}{\Longrightarrow} R(f) = \mathrm{conv}\{S^{\downarrow}(b(f))\} + C \overset{B(.)}{\Longrightarrow}\\ \overset{B(.)}{\Longrightarrow} \mathrm{conv}\{S^{\downarrow}(f)\} = P(b(f)) \overset{+C}{\Longrightarrow} \mathrm{conv}\{S^{\downarrow}(f)\} + C = R(b(f)),$$

which shows the equivalence of (i)-(iv). Corollary 3.5 implies that if $P(f)$ is integral then so is $Q(f)$, and if $Q(f)$ is integral then so is $R(f)$, which together with the above equivalences imply the equivalence of (v) (and also (vi)) and the other statements. $\square$

As an example, we show that the rank function and the co-rank function of a matroid are both ideal functions. This result can also be derived from the theory of bi-submodular polyhedra, see e.g. [1, Theorem 4.5]. We use the fact that the rank function is is submodular, while the co-rank function is supermodular.

Proposition 3.12. *Let $M = (V, r)$ be a matroid with rank function r, and let q be its co-rank function. Then (i) function q is equivalent to the blocker of r; (ii) both r and q are ideal functions.*

PROOF. First we prove that q is ideal. By Theorem 3.8, it is enough to show that the polyhedron

$$R(q) = \{(y, \beta) \in \mathbb{R}^{n+1} :\ y(X) - \beta \ge q(X)\ \forall X \subseteq V\}$$

is integer. Since q is supermodular, a standard dual uncrossing proof shows that the system is TDI, hence the polyhedron is integral (see e.g. [7, proof of Theorem 16.1.3]; the only observation needed is that dual uncrossing does not change the validity of the dual equation corresponding to variable β).

The blocker of r is $b(r)(X) = -r(V - X) = q(X) - r(V)$, which is equivalent to q, thus $b(r)$ is ideal, and r is also ideal by Theorem 3.8. $\square$

3.3. Twisting

In this section we introduce the twisting operation that preserves idealness. Let f be a unit-increasing set function on ground set V, and let U be a subset of V. The *twisting of f at U* is the set function f^U on ground set V defined by

$$f^U(X) = f(X\Delta U) + |X \cap U|.$$

It is easy to see that f^U is a unit-increasing set function. Note that $f^U(\emptyset) = f(U)$, and $f^U(U) = f(\emptyset) + |U|$. The behaviour with respect to minors is the following.

Proposition 3.13. *For a set $U \subseteq V$ and an element $v \in V$ the following hold.*

(i)

$$(f\backslash v)^{U-v} \cong \begin{cases} f^U/v & \text{if } v \in U, \\ f^U\backslash v & \text{if } v \notin U, \end{cases}$$

(ii)

$$(f/v)^{U-v} \cong \begin{cases} f^U\backslash v & \text{if } v \in U, \\ f^U/v & \text{if } v \notin U. \end{cases}$$

Proof. Suppose that $v \in U$ and take a set $X \subseteq V - v$. Then

$$\begin{aligned} (f\backslash v)^{U-v}(X) &= f\backslash v(X\Delta(U-v)) + |X \cap (U-v)| = \\ &= f((X+v)\Delta U) + |(X+v) \cap U| - 1 = \\ &= f^U(X+v) - 1 = f^U/v(X) - 1, \text{ and} \end{aligned}$$

$$\begin{aligned} (f/v)^{U-v}(X) &= f/v(X\Delta(U-v)) + |X \cap (U-v)| = \\ &= f(X\Delta U) + |X \cap U| = f^U(X) = f^U\backslash v(X). \end{aligned}$$

The other cases are similar. □

Proposition 3.14. *Every twisting of an ideal set function is also ideal.*

Proof. Let f be an ideal set function on V, and let U be a subset of V. Consider the following $(|V|+1) \times (|V|+1)$ matrix:

$$M_U = \begin{array}{cl} \begin{pmatrix} -1 & & & & & & & & \\ & -1 & & & & & & & \\ & & \ddots & & & & 0 & & \\ & & & -1 & & & & & \\ & & & & 1 & & & & \\ & & 0 & & & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \\ -1 & -1 & \dots & -1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} & \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} U \\ \vphantom{\begin{matrix} a \\ a \\ a \\ a \\ a \end{matrix}} \end{array} \end{array}$$

It is easy to check that $M_U^{-1} = M_U$, so M_U is unimodular. We claim that

$$R(f) = M_U R(f^U) + (\chi_U, |U|).$$

Indeed, if we denote by A the describing matrix of $R(f)$ (i.e. the matrix with rows $(\chi_X, -1)^\mathsf{T}$), then by $(\chi_X, -1)^\mathsf{T} M_U^{-1} = (\chi_{X\Delta U}, -1)^\mathsf{T}$, we have

$$\begin{aligned}
&M_U R(f^U) + (\chi_U, |U|) = \{M_U(y,\beta) : \ A(y,\beta) \geq f^U\} + (\chi_U, |U|) = \\
&= \{(z,\gamma) : \ AM_U^{-1}(z,\gamma) \geq f^U + AM_U^{-1}(\chi_U, |U|)\} = \\
&= \{(z,\gamma) : \ (\chi_{X\Delta U}, -1)^\mathsf{T}(z,\gamma) \geq f^U(X) + (\chi_{X\Delta U}, -1)^\mathsf{T}(\chi_U, |U|) \quad \forall X \subseteq V\} = \\
&= \{(z,\gamma) : \ z(X\Delta U) - \gamma \geq f(X\Delta U) + |X \cap U| + |U \setminus X| - |U| \quad \forall X \subseteq V\} = \\
&= \{(z,\gamma) : \ z(Y) - \gamma \geq f(Y) \quad \forall Y \subseteq V\} = R(f).
\end{aligned}$$

Hence we also have $R(f^U) = M_U^{-1}(R(f) - (\chi_U, |U|)) = M_U R(f) + (\chi_U, 0)$. Therefore $R(f)$ is integer if and only if $R(f^U)$ is integer. □

Corollary 3.15. *Every twisting of an mni set function is also mni.*

Proof. This follows from Propositions 3.13 and 3.14. □

As an example, consider the following set function on ground set V of size n:

$$\theta_n(X) = \begin{cases} 0 & \text{if } X = \emptyset, \\ n-2 & \text{if } X = V, \\ |X|-1 & \text{otherwise.} \end{cases}$$

This set function is equivalent to a twisting of the function corresponding to the degenerate projective plane:

$$\theta_n \cong f_{\mathcal{J}_{n-1}}^{V\setminus\{0\}}.$$

3.4. Further mni functions

It is a natural question whether the idealness introduced in this section generalizes the notion of idealness of clutter pairs defined in Section 2. The answer is no; in fact, it does not even generalize perfectness of clutters. It would be natural to associate to a clutter $\mathcal{D}$ the unit-increasing set function

$$g_\mathcal{D}(X) = \max\{0, \max\{|X \cap D| - 1 : D \in \mathcal{D}\}\}. \tag{10}$$

The problem is that $P(g_\mathcal{D})$ is not necessarily integral if $\mathcal{D}$ is the set of inclusionwise maximal cliques of a perfect graph. To see this, consider the perfect graph G obtained from a K_5 on node set $v_1, \dots, v_5$ by adding nodes $u_1, \dots, u_5$ and the following edges: u_iv_i, u_iv_{i+1} $(i = 1, \dots, 4)$, u_5v_5, u_5v_1. Let $\mathcal{D}$ be the clutter of the inclusionwise maximal cliques of G; thus $\mathcal{D}$ has one member of size 5 and 5 members of size 3. Let y be the vector defined by $y(u_i) = 0$ and $y(v_i) = \frac{2}{3}$ $(i = 1, \dots, 5)$. The vector $(y, -\frac{2}{3})$ is in $P(g_\mathcal{D})$, and all maximal cliques are tight with respect to it. It is easy to check that $(y, -\frac{2}{3})$ is the only vector with this property that also satisfies $y(u_i) = 0$ $(i = 1, \dots, 5)$, so it is a vertex of $P(g_\mathcal{D})$.

In this light, it is somewhat surprising that the following is true. An interesting question is whether one can prove it without using the Strong Perfect Graph Theorem.

Theorem 3.16. *The function $g_\mathcal{D}$ is minimally nonideal if and only if $\mathcal{D}$ is minimally imperfect.*

Proof. If $g_\mathcal{D}$ is ideal, then $\mathcal{D}$ is perfect, because the facet of $P(g_\mathcal{D})$ given by $\beta = 0$ is the same as the inversion through $\frac{1}{2}\mathbf{1}$ of the packing polyhedron of $\mathcal{D}$. Another observation is that if $\mathcal{D}$ is perfect, then $g_\mathcal{D}$ is not minimally nonideal. Suppose otherwise; the point $(\mathbf{1}, 1)$ is not in $P(g_\mathcal{D})$, but satisfies with equality all facet-defining inequalities of $P(g_\mathcal{D})$ except for $\beta \leq 0$ and $y \geq 0$. This means that any all-fractional vertex of $P(g_\mathcal{D})$ must satisfy $\beta = 0$. However, the face $\beta = 0$ is the inversion through $\frac{1}{2}\mathbf{1}$ of the packing polyhedron of $\mathcal{D}$, so it has only integer vertices, a contradiction.

These observations together imply that if $g_{\mathcal{D}}$ is minimally nonideal, then $\mathcal{D}$ is minimally imperfect. To prove the other direction, we resort to Theorems 1.2 and 1.3. According to these, $\mathcal{D}$ is minimally imperfect if and only if it is the non-Helly clutter (consisting of the complements of singletons), or the clutter of inclusionwise maximal cliques of an odd hole or odd antihole.

The function associated to the non-Helly clutter is θ_n, which is mni by Corollary 3.15, because it is a twisting of the function of the mni clutter $\mathcal{J}_{n-1}$. If $\mathcal{D}$ is the odd hole clutter, then $g_{\mathcal{D}} = f_{\mathcal{D}}$, so it is mni because the clutter is mni. The following lemma completes the proof of the theorem.

Lemma 3.17. *If $\mathcal{D}$ is the clutter of inclusionwise maximal cliques of an odd antihole, then $g_{\mathcal{D}}$ (as defined in* (10)*) is minimally nonideal.*

The polyhedron $P(g_{\mathcal{D}})$ can be written as:

$$P(g_{\mathcal{D}}) = \{(y, \beta) \in \mathbb{R}^{n+1} : \ \mathbf{0} \leq y \leq \mathbf{1}, \ \beta \leq 0, \ y(K) - \beta \geq |K| - 1 \ \forall \text{ clique } K\}.$$

It will be more convenient to consider a transformed polyhedron for the complement graph and packing type constraints. For a graph $G = (V, E)$, let

$$P(G) = \{(x, t) \in \mathbb{R}^{|V|+1} : \mathbf{0} \leq x \leq \mathbf{1}, \ t \geq 0, \ x(S) \leq 1 + t \text{ for every stable set } S\}.$$

Clearly $P(G)$ is integer if and only if $g_{\mathcal{D}}$ is ideal for the clutter $\mathcal{D}$ of inclusionwise maximal stable sets of G. Thus the following proposition implies Lemma 3.17.

Proposition 3.18. *If G is a path, then $P(G)$ is an integral polyhedron. If G is an odd cycle, then $P(G)$ has a unique non-integral vertex.*

Proof. We use induction on $|V|$ and we consider both cases simultaneously. Let (x^*, t^*) be a non-integer vertex of $P(G)$.

First we claim that $\text{supp}(x^*) = V$. Suppose indirectly that $x^*(v) = 0$ for some $v \in V$. If G is a path, then let G_1 and G_2 be the two paths of $G - v$, and let $x_i = x^*|_{V(G_i)}$ (for $i = 1, 2$). Let t_1 and t_2 be minimal such that $(x_i, t_i) \in P(G_i)$. Then $t_1 + t_2 + 1 \leq t^*$, since there are stable sets S_1 and S_2 which are tight, so $t^* + 1 \geq x^*(S_1 \cup S_2) = x_1(s_1) + x_2(S_2) = t_1 + t_2 + 2$.

By induction, (x_1, t_1) and (x_2, t_2) can be written as convex combination of integer points in $P(G_1)$ and $P(G_2)$, respectively: $(x_1, t_1) = \sum \lambda_i(a_i, b_i)$, $(x_2, t_2) = \sum \mu_i(c_i, d_i)$. Then the convex combination $\sum_{i,j} \lambda_i \mu_j (a_i, 0, c_j, b_i + d_j + 1)$ (where the 0 component corresponds to v) produces $(x^*, t_1 + t_2 + 1)$, and it is easy to see that every vector used in the combination is in $P(G)$. This and $t^* \geq t_1 + t_2 + 1$ implies that (x^*, t^*) can not be a vertex.

In the case that G is a cycle, the proof is a similar reduction to the path case.

Next, suppose that x^* has an interval of consecutive ones, with odd length (and the neighboring values are smaller than 1). Let u and v be the neighboring nodes. Then every tight set S contains every other node in the interval (1st, 3rd etc.), and does not contain u or v (because otherwise we could obtain a stable set S' with $x^*(S') > x^*(S)$ by moving more elements of S to the interval). But then $(x^*, t^*) \pm \epsilon(\chi_u - \chi_v)$ would be still in the polyhedron $P(G)$, which contradicts that (x^*, t^*) is a vertex. In the case that G is a path and the interval of ones is at the beginning, we get a similar contradiction.

Now consider the case that (x^*, t^*) is such that every consecutive interval of ones is of even length. Let I denote the set of nodes in V where x^* is one and let $2k$ be its cardinality. We write (x^*, t^*) as the following convex combination for some λ close to 1:

$$(x^*, t^*) = (1 - \lambda)(\chi_I, k - 1) + \lambda \left(\max(\chi_I, \frac{x^*}{\lambda}), \frac{t^* - (1 - \lambda)(k - 1)}{\lambda} \right).$$

The vector $(\chi_I, k - 1)$ is in $P(G)$, because of the evenness property of I. Let (x', t') denote the second vector, about which we want to show that it is in $P(G)$ for λ close enough to 1. The nonnegativity constraints and the $x' \leq \mathbf{1}$ constraint hold around 1.

Let S be an arbitrary stable set. If $|S \cap I| < k$, then there is another stable set S' for which $x'(S') > x'(S)$, so we can assume that $|S \cap I| = k$. Then

$$x'(S) = k + \frac{x^*(S \setminus I)}{\lambda} = k + \frac{x^*(S) - k}{\lambda} \leq k + \frac{1 + t^* - k}{\lambda} = 1 + \frac{t^* - (1-\lambda)(k-1)}{\lambda},$$

which proves that $(x', t') \in P(G)$.

We remain with the case when x^* has only non-integer values. In this case, every node v has to be in a tight set. The vector $(\mathbf{0}, -1)$ satisfies all of these tight inequalities with equality too, except for $t \geq 0$. Thus $t^* = 0$, and x^* is a vertex of $\mathrm{QSTAB}(\overline{G})$. If G is a path, then $\mathrm{QSTAB}(\overline{G})$ is integer, while for odd circuits it has a unique non-integer vertex. This concludes the proof. □

□

We present one more mni function that shows the difficulty of extending Lehman's theorem. So far all mni functions we have seen satisfied the property that $P(f)$ has a unique non-integer vertex that is simple. The following mni set function f on ground set $\{1,2,3,4,5\}$ is an example where the unique fractional vertex of $P(f)$ is not simple. The properties were checked using the software Polymake.

$$f(X) = \begin{cases} 0 & \text{if } X = \emptyset, \\ 1 & \text{if } |X| = 1 \text{ or } X \in \{\{1,2\},\{2,3\},\{3,4\}\{4,5\}\}, \\ 2 & \text{if } |X| = 3 \text{ or } X \in \{\{1,3\},\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,5\}\} \\ & \quad \text{or } X \in \{\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,5\}\}, \\ 3 & \text{if } X \in \{\{1,2,3,4\},\{2,3,4,5\},\{1,2,3,4,5\}\}. \end{cases}$$

4. Open questions

There are several questions about ideal unit-increasing functions that we think may lead to a better understanding of the structure of 0-1 polyhedra. As Lehman's theorem turned out to be useful for proving sufficient conditions for integrality of various covering polyhedra, we hope that answers to these questions may help to prove similar results for problems beyond standard packing and covering.

- We are not aware of an example of an mni function that has more than one non-integer vertex, so one is tempted to conjecture that the non-integer vertex is always unique. In addition, in all known examples there is a value λ such that every component of the unique non-integer vertex (except for the last one) is either λ or $1 - \lambda$.
- Can one define a class of functions that contains all functions of type $f_{\mathcal{C}}$ and $g_{\mathcal{C}}$, is closed under taking minors, blockers, and twisting, and has the property that any minimally nonideal member of the class has a unique fractional vertex that is simple?
- In a model where functions are given by an evaluation oracle, is it in co-NP to decide if a unit-increasing function is ideal?

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References

[1] A. Bouchet, W.H. Cunningham, Delta-matroids, jump systems, and bisubmodular polyhedra, SIAM Journal on Discrete Mathematics Volume 8, Issue 1 (1995), 17–32.

[2] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, The strong perfect graph theorem, Annals of Mathematics 164 (2006), 51–229.

[3] G. Cornuéjols, B. Guenin, Ideal clutters, Discrete Applied Mathematics Volume 123, Issues 1-3 (2002), 303–338.

[4] G. Cornuéjols, B. Guenin, L. Tunçel, Lehman matrices, Journal of Combinatorial Theory Series B Volume 99, Issue 3, (2009), 531–556.

[5] G. Cornuéjols, B. Novick, Ideal $0,1$ matrices, J. Combin. Theory B Volume 60 (1994), 145–157.

[6] G. Ding, L. Feng, W. Zang, The complexity of recognizing linear systems with certain integrality properties, Mathematical Programming Volume114, Issue 2 (2008), 321–334.

[7] A. Frank, Connections in Combinatorial Optimization, Oxford Lecture Series in Mathematics and its Applications 38, Oxford University Press, 2011.

[8] G. Gasparyan, M. Preissmann, A. Sebő, Imperfect and Nonideal Clutters: A Common Approach, Combinatorica, Volume 23, Issue 2 (2003), 283–302.

[9] A. Lehman, On the width–length inequality, Mimeographic Notes, 1965, published: Math. Program. 17 (1979), 403–417.

[10] A. Lehman, The Width-Length Inequality and Degenerate Projective Planes, Cook, W. and P. D. Seymour (eds.), Polyhedral Combinatorics DIMACS vol.1 (1990), 101–105.

[11] L. Lovász, A characterization of perfect graphs, J. of Comb. Theory B, 13 (1972), 95–98.

[12] C Lütolf, F Margot, A catalog of minimally nonideal matrices, Mathematical Methods of Operations Research, Volume 47, Issue 2 (1998), 221–241.

[13] M. Padberg, Perfect zero-one matrices, Math. Programming, 6 (1974), 180–196.

[14] J. Pap, Integrality, complexity and colourings in polyhedral combinatorics, Ph.D. thesis, Eötvös Loránd University, 2012.

[15] A. Sebő, Characterizing Noninteger Polyhedra with 0–1 Constraints, Integer Programming and Combinatorial Optimization Lecture Notes in Computer Science Volume 1412 (1998), 37–52.

[16] P. D. Seymour, On Lehman's width-length characterization, DIMACS, vol.1, Polyhedral Combinatorics (1990), 107–117.

[17] F. B. Shepherd, Applying Lehman's theorems to packing problems, Mathematical Programming, 71 (1995), 353–367.

[18] J Wang, A new infinite family of minimally nonideal matrices, Journal of Combinatorial Theory, Series A Volume 118, Issue 2 (2011), Pages 365–372.