RIGID REPRESENTATIONS OF THE MULTIPLICATIVE COALESCENT WITH LINEAR DELETION

JAMES B. MARTIN[1] AND BALÁZS RÁTH[2]

Abstract. We introduce the *multiplicative coalescent with linear deletion*, a continuous-time Markov process describing the evolution of a collection of blocks. Any two blocks of sizes x and y merge at rate xy, and any block of size x is deleted with rate λx (where $\lambda \geq 0$ is a fixed parameter). This process arises for example in connection with a variety of random-graph models which exhibit self-organised criticality. We focus on results describing states of the process in terms of collections of excursion lengths of random functions. For the case $\lambda = 0$ (the coalescent without deletion) we revisit and generalise previous works by authors including Aldous, Limic, Armendariz, Uribe Bravo, and Broutin and Marckert, in which the coalescence is related to a "tilt" of a random function, which increases with time; for $\lambda > 0$ we find a novel representation in which this tilt is complemented by a "shift" mechanism which produces the deletion of blocks. We describe and illustrate other representations which, like the tilt-and-shift representation, are "rigid", in the sense that the coalescent process is constructed as a projection of some process which has all of its randomness in its initial state. We explain some applications of these constructions to models including mean-field forest-fire and frozen-percolation processes.

Keywords: multiplicative coalescent, Erdős-Rényi random graph, frozen percolation
AMS MSC 2010: 60J99, 60B12, 05C80

1. Introduction

1.1. The multiplicative coalescent and MCLD(λ).

The *multiplicative coalescent* (or briefly MC) is a continuous-time Markov process describing the evolution of a collection of blocks (components). The dynamics are as follows: for each pair of components with masses m_i and m_j, the pair coalesces at rate $m_i m_j$ to form a single component of mass $m_i + m_j$.

The process is simple to construct from any initial state with finitely many components. In [2, Section 1.5], Aldous uses a graphical construction to show that the process is well-defined starting from any initial state in which the sum of the squares of the masses is finite. Writing the masses in decreasing order, define the space

$$\ell_2^{\downarrow} = \{\underline{m} = (m_1, m_2, \dots) : m_1 \geq m_2 \geq \cdots \geq 0,\ \sum_{i=1}^{\infty} m_i^2 < \infty\}, \tag{1.1}$$

with the distance

$$\mathrm{d}(\underline{m}, \underline{m}') = \|\underline{m} - \underline{m}'\|_2 = \left(\sum_{i \geq 1} (m_i - m_i')^2 \right)^{1/2}. \tag{1.2}$$

Then Proposition 5 of [2] says that the MC is a Feller process in $(\ell_2^{\downarrow}, \mathrm{d})$.

[1]Department of Statistics, University of Oxford, United Kingdom. Email: martin@stats.ox.ac.uk
[2]MTA-BME Stochastics Research Group, Hungary. Email: rathb@math.bme.hu

We generalise the multiplicative coalescent to include *deletion* as well as coalescence. Let $\lambda \geq 0$. Now we want the dynamics of the process to satisfy:

$$\begin{array}{l}\text{(i) any two components of mass } m_i \text{ and } m_j \text{ merge with rate } m_i m_j,\\ \text{(ii) any component of mass } m_i \text{ is deleted with rate } \lambda m_i.\end{array} \tag{1.3}$$

We call such a process a *multiplicative coalescent with linear deletion*, with deletion rate λ, and denote it by MCLD(λ).

Again, if the initial state has finitely many components, the process is easily seen to be well-defined. For initial states in $\ell_2^{\downarrow}$, a graphical construction is given in [30], which gives rise to a well-behaved continuous-time Markov process taking values in $\ell_2^{\downarrow}$. This process has the Feller property with respect to the distance $\mathrm{d}(\cdot,\cdot)$ – see Theorem 1.2 of [30].

The motivation of our study of the MCLD process is twofold.

Firstly, ideas involving the representation of the state of a MC process in terms of the set of excursion lengths of some random function began with Aldous [2], and have been developed in a series of works including those of Aldous and Limic [5], Armendáriz [8], Uribe Bravo [35], Broutin and Marckert [18] and Limic [23]. We find that this theory has a surprising and elegant extension to processes involving deletion, which we build in this paper.

Secondly, the MCLD arises as a scaling limit of certain discrete processes of coalescence and fragmentation or deletion, such as the mean-field forest-fire model introduced by Ráth and Tóth in [32] and studied by Crane, Freeman and Tóth in [19], and the mean-field frozen percolation process introduced by Ráth in [31]. These processes are of particular interest because of their *self-organised criticality* properties. These scaling limits are the subject of a future paper [24], but we discuss related properties here in Section 6.

We further discuss related literature in Section 1.4.

1.2. **Tilt representations of MC.** We start by revisiting previous results giving representations of the MC in terms of excursions of random functions.

We first define *Brownian motion with parabolic drift.*

Definition 1.1 (Brownian motion with parabolic drift, BMPD(u))**.** Let $B(x), x \geq 0$ denote standard Brownian motion. Given $u \in \mathbb{R}$, define

$$h(x) = B(x) - \frac{1}{2}x^2 + ux, \quad x \geq 0. \tag{1.4}$$

Let us denote by BMPD(u) the law of the random function $h(x), x \geq 0$.

For a function $h(x), x \geq 0$, let $\mathcal{E}^{\downarrow}(h)$ denote the lengths of the excursions of h above its running minimum, written in non-increasing order (see Section 2 below for a formal definition). Aldous [2] showed that there is an eternal multiplicative coalescent process, whose marginal distributions can be written in terms of excursions of a Brownian motion with parabolic drift.

Proposition 1.2 (Aldous)**.** *There exists a MC process* $(\mathbf{m}_t, t \in \mathbb{R})$ *such that for each* t*, the marginal distribution* $\mathbf{m}_t$ *is the same as that of* $\mathcal{E}^{\downarrow}(h_t)$ *where* $h_t \sim \mathrm{BMPD}(t)$*.*

This process is known as the *standard multiplicative coalescent.* In her PhD thesis, Armendáriz [7, 8] showed that in fact the whole process can be constructed as a function of a single realisation of Brownian motion.

Proposition 1.3 (Armendáriz)**.** *Let* $h_0 \sim \mathrm{BMPD}(0)$*, and define* $h_t(x) = h_0(x) + tx$ *for all* $t \in \mathbb{R}$ *(so that* $h_t \sim \mathrm{BMPD}(t)$ *for all* t*). Then the process* $\mathcal{E}^{\downarrow}(h_t), t \in \mathbb{R}$ *is the standard MC.*

We call this representation a *tilt* representation; for each t, the function h_t is obtained from h_0 by adding the linear function tx. It is easy to see that the tilt representation indeed produces a coalescent process as t increases. Also note that only adjacent excursions can coalesce under the tilt representation.

Part of the approach of Armendáriz (described in [8, Section 4]) is further elaborated in Chapter 4 of the PhD thesis [35] of Uribe Bravo.

In [18, Corollary 4], Broutin and Marckert give an alternative proof of Proposition 1.3. Their proof involves considering the connected components of an Erdős-Rényi graph process on n vertices, and then taking the weak limit as $n \to \infty$. The key idea is to define an ordering of the vertices (the *Prim ordering*, related to invasion percolation and to the minimal spanning tree) which is consistent with the coalescent process in the sense that at all times, the components are intervals of the Prim order (and thus only adjacent intervals can coalesce); nonetheless, for each fixed time, exploring the graph in the Prim ordering yields a random walk with the same distribution as is obtained from a standard traversal in, say, depth-first or breadth-first order.

Aldous and Limic [5] characterized the set of all eternal multiplicative coalescents, i.e. those defined for all times $t \in (-\infty, \infty)$. They defined a three-parameter set of random processes $W^{\kappa,\tau,\underline{c}}(x), x \geq 0$, such that if $(\mathbf{m}_t, t \in \mathbb{R})$ is a non-constant eternal MC which is extremal (that is, its law is not the mixture of the laws of other eternal MC processes) then for each $t \in \mathbb{R}$, the marginal distribution of $\mathbf{m}_t$ is the same as that of the sequence $\mathcal{E}^{\downarrow}(h_t)$ of excursion lengths of $h_t(x) := W^{\kappa,\tau,\underline{c}}(x) + tx$ for some $(\kappa, \tau, \underline{c})$. The processes $W^{\kappa,\tau,\underline{c}}$ have been called *Lévy processes without replacement*; see Section 6.1 for their definition and the precise statement of the cited result of [5].

We will further discuss the links between our approach and those of Armendáriz [8], Uribe Bravo [35], Broutin and Marckert [18] and Limic [23] in Section 1.4.

1.3. Contributions of this paper.

In this section we summarize the main results of our paper. The central results are the following:

- **MC admits a tilt representation from any initial state:** In Theorem 2.8 we show that for any possible initial state $\underline{m} \in \ell_2^{\downarrow}$, there exists a random function $f_0 : [0, \infty) \to [-\infty, 0]$ such that if we define
$$f_t(x) = f_0(x) + tx \tag{1.5}$$
then $\mathcal{E}^{\downarrow}(f_t), t \in \mathbb{R}$ is a multiplicative coalescent process with initial state $\underline{m}$.
- **MCLD admits a "tilt-and-shift" representation:** Let $\underline{m} \in \ell_2^{\downarrow}$, and define f_t for all $t \geq 0$ as in (1.5). Let $\lambda > 0$. In Theorem 2.13 we show that there exists a $\sigma(f_0)$-measurable non-decreasing function $\Phi : [0, \infty) \to [0, \infty)$ such that if we define
$$g_t(x) = f_t(x + \Phi(t)) + \lambda t - \int_0^t \Phi(s)\, \mathrm{d}s, \quad t \geq 0 \tag{1.6}$$
then $\mathcal{E}^{\downarrow}(g_t), t \geq 0$ is a MCLD(λ) process with initial state $\underline{m}$.

Here Φ is a non-decreasing pure jump process; $\Phi(t)$ is the total weight of components deleted by time t. In addition to the "tilt" given by (1.5), we now have a "shift" since in (1.6), the function f_t is applied at $x + \Phi(t)$. If a component of size a is deleted, then $\Phi(t) - \Phi(t-) = a$, and we see a shift to the left of size a at time t; that is, the graph of g_{t-} on $[a, \infty]$ becomes the graph of g_t on $[0, \infty]$.

Note that the tilt representation of the MC is the $\lambda = 0$ case of the tilt-and-shift representation of the MCLD. Under the tilt-and-shift representation, only adjacent excursions coalesce and only the "leftmost" excursion gets deleted. We call the above representations "rigid" because all of the randomness is contained in the initial state f_0, and the rest of the evolution of f_t (resp. g_t) is deterministic.

The function f_0 above is constructed so that it has the *exponential excursion levels* property: conditional on the sequence of excursion lengths, the levels of the excursions are independent, and an excursion of length m occurs at level $-E$ where $E \sim \text{Exp}(m)$. One of the central observations behind the rigid representation results outlined above is that f_t and g_t for any $t \geq 0$ inherit the exponential excursion levels property.

We prove these main results in two stages: we first consider the case of finitely many blocks in Section 3, then we extend the tilt-and-shift representation to $\ell_2^{\downarrow}$ using truncation and approximation arguments in Section 5.

The proof of the finite tilt-and-shift representation result in Section 3 involves various other representations which are also of independent interest. Their constructions rely heavily on the notion of size-biased orderings, and on the realisation of such orderings using independent exponential random variables, c.f. Section 3.1.

Now we outline these representations and how they relate to each other.

- **Interval coalescent representation:** Given an initial state $\underline{m}$ with finitely many blocks, we consider a process whose states are finite sequences of lengths (depicted as intervals arranged in some order). In the initial state, the interval lengths are the masses of blocks of $\underline{m}$, and they are arranged in a size-biased random order. An interval I merges with the interval on its right at a rate equal to the product of the length of I with the combined length of all of the intervals to the right of I. The leftmost interval is deleted at a rate equal to λ times the total length of intervals. For an illustration, see Figure 1.1. In Section 3.2 we show that the decreasing rearrangement of interval lengths evolves like MCLD(λ). (If $\lambda = 0$, this gives the MC.)

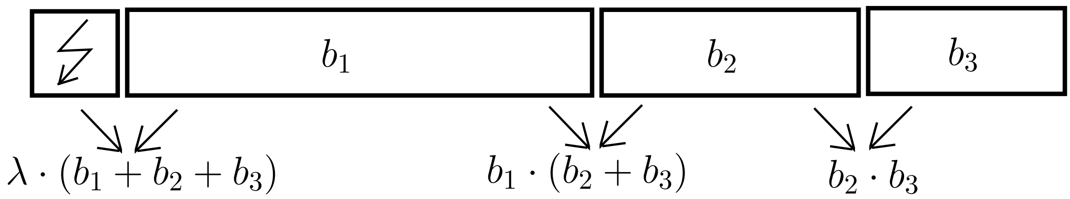

FIGURE 1.1. An illustration of the rates of the interval coalescent process: the lengths of the three intervals (from left to right) are b_1, b_2 and b_3. The leftmost block marked with a lightning represents the "cemetery". The rate at which the first interval gets deleted is $\lambda \cdot (b_1 + b_2 + b_3)$. The rate at which the first interval merges with the second one is $b_1 \cdot (b_2 + b_3)$. The rate at which the second and third intervals merge is $b_2 \cdot b_3$.

- **Particle representation:** Given an initial state $\underline{m} = (m_1, \ldots, m_n)$ as above, we initially put a particle of mass m_i at initial height $-E_i$, where $E_i \sim \text{Exp}(m_i), 1 \leq i \leq n$ are independent. Now the particles start to move up until they reach height 0 and die. The speed of a particle is equal to λ plus the total weight of particles strictly above it and strictly below 0. Note that once a particle reaches the one above it, they stick together until they die. Accordingly, we group the particles that share the same height into time-t blocks. In Section 3.3 we show that the the vector of sizes of the time-t blocks of the particle system, in decreasing order of their height, evolves like the above described interval coalescent process. See Figure 1.2 and Figure 1.3 for simulations; in the first, λ is positive and the system realises an MCLD, while in the second, λ is zero and the system realises a multiplicative coalescent without deletion.

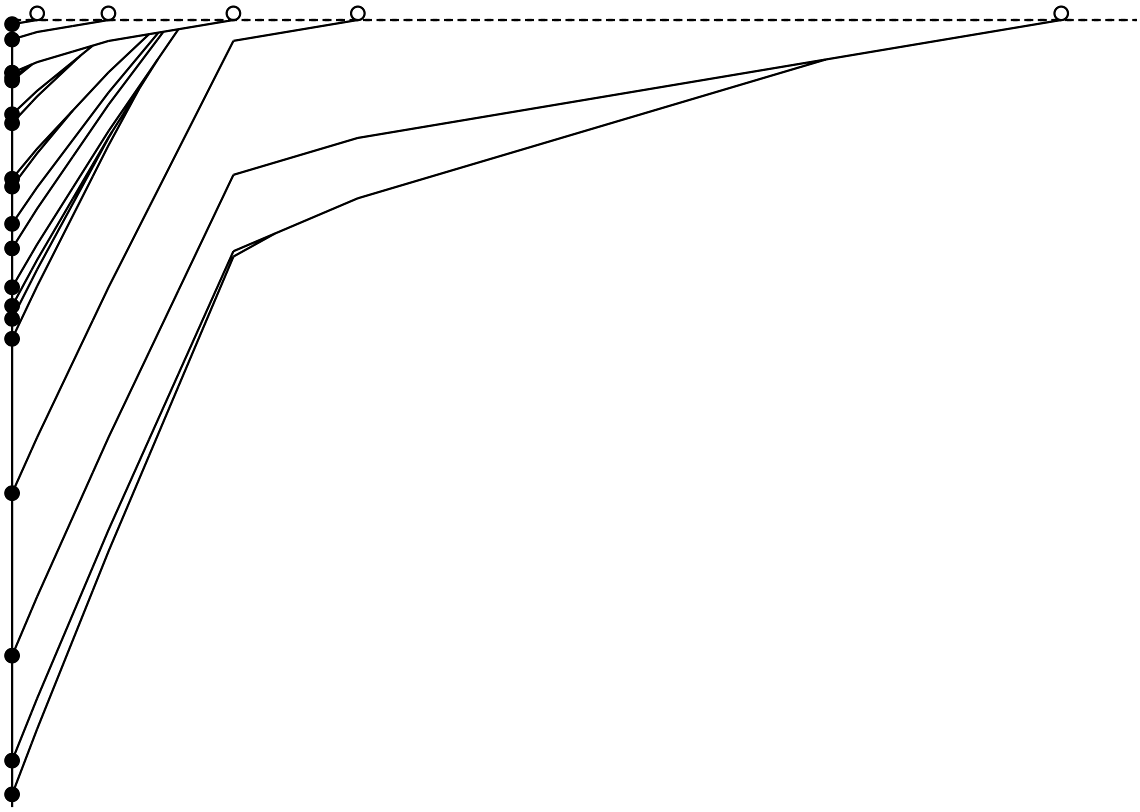

FIGURE 1.2. A simulation of the particle system which realises the MCLD. Each particle moves upwards at rate equal to λ plus the mass of particles strictly above it. When a particle reaches 0 it dies and is removed from the system. Particles at the same height at time t form a time-t block. The figure shows a system with $n = 20$ particles and $\lambda = 1.3$, evolving on the time interval $[0, 0.5]$. We see 5 deletion events, and the deleted blocks have sizes 1, 1, 14, 1, 3 respectively. Compare to Figure 1.3, where $\lambda = 0$ (realising the multiplicative coalescent without deletion), and to Figure 6.1, where deleted particles reenter at independent exponential heights (realising the forest fire model).

Note that the particle representation outlined above is also "rigid".

In Section 3.4 we explain how particles in the particle representation correspond to excursions of g_t in the tilt-and-shift representation, completing the proof of the validity of the tilt-and-shift representation of MCLD(λ) with finitely many blocks.

In Section 6 we give some applications of our rigid representation results:

- In Section 6.1 we give a sketch proof of the fact that the Lévy processes without replacement $W^{\kappa,\tau,\underline{c}}$ have the exponential excursion levels property, and conclude that all eternal multiplicative coalescents have a tilt representation. This gives an alternative way to approach the main result of Limic [23] (see Section 1.4).
- In Section 6.2 we apply the tilt-and-shift representation to Brownian motion with parabolic drift. What we find can be non-rigorously summarized as follows: if we start from BMPD(u) then at time $t \geq 0$ we see BMPD($u + t - \Phi(t)$). We show that the resulting MCLD(λ) process is the scaling limit of the list of component sizes in the mean field frozen percolation model of [31] started from a near-critical

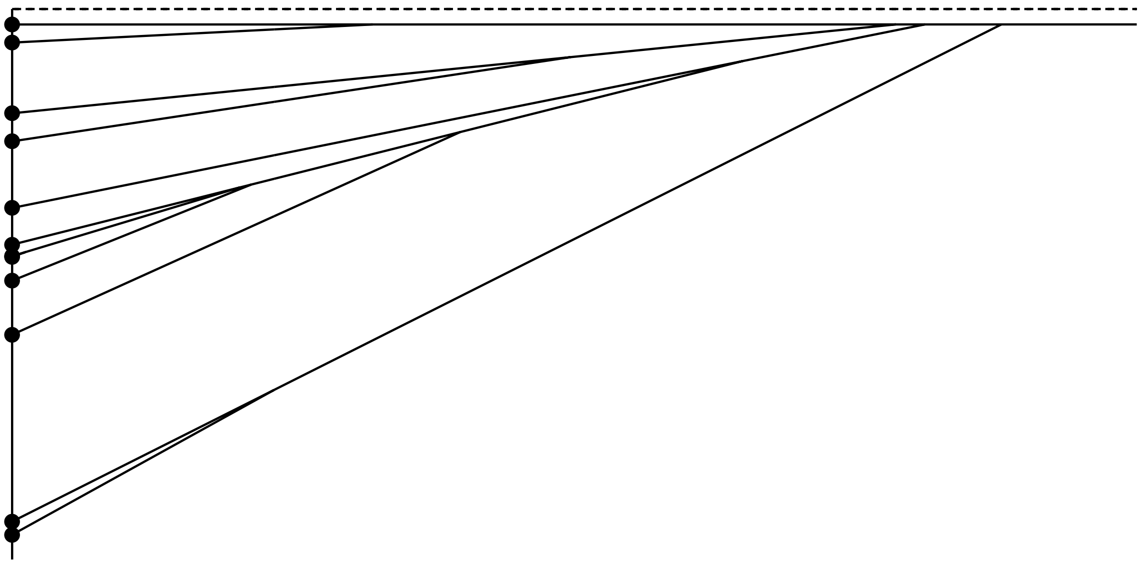

FIGURE 1.3. This shows a similar system to the one in Figure 1.2, but now with $\lambda = 0$ so that the system realises the multiplicative coalescent, with no deletion. In this case $n = 12$ and the figure shows the time interval $[0, 0.25]$. By the end, all particles have coalesced into a single block.

Erdős-Rényi graph, thus extending the result [2, Corollary 24], which is the $\lambda = 0$ case.

- In Section 6.3 we give a particle representation of the mean field forest fire model of [32]. We demonstrate how this representation allows us to give a new probabilistic interpretation of the non-linear controlled PDE problem (and the associated characteristic curves) which played a central role in the theory developed in [32] and [19].

1.4. **Related work.**

1.4.1. *Rigid representations of coalescent processes.* In this subsection we discuss links to previous work on rigid representations for the multiplicative coalescent, which is somewhat scattered (and in some cases unpublished). The various approaches and methods of Armendáriz [8], Uribe Bravo [35], Broutin and Marckert [18] and Limic [23] resulting in versions of the "tilt" representation are all related to ours, although these similarities are often implicit. Let us now try to sketch some of these connections.

In [8] Armendáriz draws attention to the fact that Brownian motion with parabolic drift has the *exponential excursion levels* property which we informally described in Section 1.3. This observation had been made by Aldous in [2, equation (72)] (see also Bertoin [9]). Based on this, Armendáriz constructs a representation of the MC with finitely many blocks in [8, Section 4] – this representation appears to be equivalent to the $\lambda = 0$ case of our *particle representation* (see Section 3.3), although the methods are different.

In Chapter 4.2 of his PhD thesis [35], Uribe Bravo recalls and further elaborates the representation of [8, Section 4] (without proofs). In particular, he points out that (what we call) the initial heights of particles are in size-biased order and formulates the crucial property corresponding to our (3.16) about the height gaps of particles. Again, the representation outlined in [35, Chapter 4.2] is equivalent to the $\lambda = 0$ case of our particle representation; moreover the representation stated (without proof) in [35, Chapter 4.3] is equivalent to our tilt representation of MC (c.f. Section 2.1) in the case of finitely many blocks. In [35, Chapter 4.4] Uribe Bravo gives a sketch proof of Proposition 1.2 which is quite similar to our short sketch proof of Proposition 1.3 given in Section 6.1.1.

At first, the approach of Broutin and Marckert in [18] seems quite different from ours. Their construction relies on exploration processes for random graphs, defined using the *Prim ordering* for a graph with edge-weights; our methods make no explicit reference to the graph structure underlying the MC (or MCLD). However, one can observe that the components of the dynamic Erdős-Rényi graph process, arranged in Prim order, give the *interval coalescent* process (see Section 3.2) without deletion, started from an initial state with equal-sized blocks. Broutin and Marckert also extend the representation to obtain a construction of the *standard augmented multiplicative coalescent* of Bhamidi, Budhiraja and Wang [11].

In a paper available in interim form at [23], Limic obtains tilt representations of all eternal multiplicative coalescents. The main result can be written as follows. Let $f_t(x) = W^{\tau,\kappa,\underline{c}}(x) + tx$, where $W^{\tau,\kappa,\underline{c}}$ is a Lévy processes without replacement. Then the process $\mathcal{E}^{\downarrow}(f_t), t \in \mathbb{R}$ is an eternal multiplicative coalescent. Limic's approach involves the generalisation of the *breadth-first walks* used in [2] and [5], to give a system of *simultaneous breadth-first walks* relating to the state of the multiplicative coalescent at different times. Limic recalls from [8, Section 4] and [35, Chapter 4.2] the definition of *Uribe's diagram* (which is equivalent to the $\lambda = 0$ case of our particle representation) and proves (via a calculation involving iterated integration) that this representation is indeed a valid representation of MC. In [23, Section 4] Limic explores the connection between Uribe's diagram and simultaneous breadth-first walks. We describe an alternative approach to the tilt representation of the set of eternal coalescents in Section 6.1. Note that the connection between Prim's algorithm [18] and simultaneous breadth-first walks [23] is the topic of ongoing research [34], c.f. Open question 3 of [23, Section 7].

Finally we mention that also in the case of the *standard additive coalescent* [6], a tilt representation can be given. Let $\mathrm{e}(x)$ be a standard Brownian excursion on $[0, 1]$, and for $\lambda > 0$ define the function h_λ on $[0, 1]$ by $h_\lambda(x) = \mathrm{e}(x) - \lambda x$. A consequence of the results of Bertoin [9] is that the process $\mathcal{E}^{\downarrow}(h_{e^{-t}}), t \in \mathbb{R}$ is a version of the standard additive coalescent. An alternative proof of this result is given by Broutin and Marckert [18].

1.4.2. *Mean-field graph models of self-organized criticality (SOC).* The mean-field frozen percolation model [31] is a dynamic random graph model where the initial number of vertices is n, two connected components of size k and l merge at rate $\frac{kl}{n}$ and a connected component of size k disappears at rate $\lambda(n)k$. Note that up to scaling, this is MCLD(λ). If $1/n \ll \lambda(n) \ll 1$ then the evolution of the densities of small components is asymptotically described by Smoluchowski's coagulation differential equations with multiplicative kernel [31, Theorem 1.2], the solutions of which exhibit SOC [31, Theorem 1.5].

The definition of the mean-field forest fire model [32] is the same as that of the above described frozen percolation model, with one difference: instead of removing a component, we only remove its edges, with the vertices remaining as singletons. A variant of Smoluchowski's system of equations describes the asymptotic densities of small components [32, Section 1.3], leading to SOC. The proof of the well-posedness of this infinite system of differential equations involves a non-linear PDE which is a controlled variant of the Burgers equation.

In [19] Crane, Freeman and Tóth describe the time evolution of the size of the component of a fixed vertex in the above forest fire model in the $n \to \infty$ limit. They also give a probabilistic meaning to the characteristic curves of the controlled Burgers equation in [19, Remark 3.11]. In Section 6.3.2 we give a different probabilistic meaning to these characteristic curves by comparing them to particle trajectories in the particle representation.

The model proposed by Aldous in [4, Section 5.5] is studied by Merle and Normand in [26]: two connected components of size k and l merge at rate $\frac{kl}{n}$ and connected components disappear if their size exceeds a threshold $\omega(n)$. In order to achieve SOC, one chooses $1 \ll \omega(n) \ll n$. Theorem 1.1 of [26] states that the densities of small components converge to the solution of Smoluchowski's coagulation equations with multiplicative kernel. Note that this is a special case of the main result of Fournier and Laurencot in [21], who study discrete models of Smoluchowski's coagulation equation with more general coagulation kernels. Theorem 1.3 of [26] identifies the Benjamini-Schramm limit of the "threshold deletion" graph model as $n \to \infty$.

Note that by comparing [31, Theorem 1.2] and [26, Theorem 1.1] one sees that (in the SOC regime) the densities of small components converge to the same hydrodynamic limit in the model with linear deletion and the model with threshold deletion. However, if one is interested in the scaling limit of big component dynamics, the exact deletion mechanism does crucially enter the picture. We discuss scaling limits of the model with linear deletion in Section 6.2. The scaling limit of the threshold deletion model is not yet known, but in Remark 6.16 we give a particle representation of it.

In [27] Merle and Normand identify the Benjamini-Schramm limit of a different self-organized critical model of aggregation, where vertices can only connect to a fixed number of other vertices.

1.4.3. *Scaling limits of critical random graph models.* In [2, Corollary 2] Aldous identifies the scaling limit of component sizes in a near-critical Erdős-Rényi graph in terms of the excursion lengths of BMPD and in [2, Corollary 24] he shows that the standard MC is the scaling limit of the evolution of component sizes of the dynamic Erdős-Rényi graph in the critical window. Let us now list some related results.

The papers [10, 15, 20, 22, 29, 33] explore the universality class of graph models whose scaling limits are described by BMPD and the standard MC.

The family of eternal multiplicative coalescent processes are characterized in [5]. The scaling limits of some classes of inhomogeneous random graph models are given by Lévy processes without replacement and non-standard eternal MC processes, see [5, 16, 22].

The continuum scaling limit of the metric structure of critical random graphs is studied in the "BMPD" universality class in [1, 12, 13] and in the "Lévy processes without replacement" universality class in [14].

Martin and Yeo [25] study the Erdős-Rényi random graph within the critical window, conditioned to be acyclic; alternatively expressed, this is the uniform distribution over forests with a given number of vertices and edges. Analogously to Proposition 1.2, they obtain a scaling limit for the collection of component sizes which is described by the sequence of excursions of an appropriate reflected diffusion; here the drift of the diffusion depends on space as well as on time.

In the future paper [24] we describe the possible scaling limits that can arise from a frozen percolation process with lightning rate $\lambda n^{1/3}$, started from an empty graph. The possible limit objects are eternal MCLD(λ) processes. The "arrival" of the process at the critical window gives rise to a non-stationary eternal MCLD(λ) scaling limit, while the scaling limit in the "self-organized critical" regime is a stationary MCLD(λ).

1.5. **Plan of the paper.** In Section 2 we give the main results about tilt representations for MC and tilt-and-shift representations for MCLD, and discuss their interpretation.

In Section 3 we prove the tilt-and-shift representation result of MCLD in the case of finitely many blocks. Along the way, we introduce the interval coalescent representation and the particle representation and show how all these representations relate to each other.

In Section 4 we collect some preparatory results about random point measures and excursions that we will use in Section 5.

In Section 5 we extend our rigid representation results to any initial state $\underline{m} \in \ell_2^\downarrow$ by approximating $\underline{m}$ with a sequence of truncated initial states. Since the particle representation is essentially the same as the tilt-and-shift representation, our proof involves a careful analysis of the effect of the insertion of a new particle on the death times of other particles.

In Section 6 we present the applications of the theory of rigid representations that we mentioned in Section 1.3.

2. Main results

We begin by introducing some notation. Let

$$\ell_\infty^\downarrow = \{\, \underline{m} = (m_1, m_2, \dots) \;:\; m_1 \geq m_2 \geq \dots \geq 0 \,\}$$

$$\ell_p^\downarrow = \{\, \underline{m} \in \ell_\infty^\downarrow \;:\; \sum_{i=1}^{\infty} m_i^p < \infty\} \text{ for } 0 < p < \infty\}$$

$$\ell_0^\downarrow = \{\, \underline{m} \in \ell_\infty^\downarrow \;:\; \exists\, i_0 \in \mathbb{N} \;:\; m_i = 0 \;\text{ for all }\; i \geq i_0 \,\}$$

We will use the topology of coordinate-wise convergence on $\ell_\infty^\downarrow$.

For $\underline{m} \in \ell_0^\downarrow$ with n non-zero entries, we will sometimes ignore the infinite trailing string of zeros and regard $\underline{m}$ as an element of $\mathbb{R}^n_{>0}$.

For $\underline{m}, \underline{m}' \in \ell_2^\downarrow$ we define the distance $\mathrm{d}(\underline{m}, \underline{m}')$ by (1.2). The metric space $\left(\ell_2^\downarrow, \mathrm{d}(\cdot,\cdot)\right)$ is complete and separable.

Next, we introduce some definitions relating to excursions.

Definition 2.1. Let $g : [0,\infty) \to \mathbb{R} \cup \{-\infty\}$ be a function which is càdlàg and lower semi-continuous (i.e. all jumps are downards). For $0 \leq l < r < \infty$, the interval $[l, r)$ is an *excursion above the minimum* of g if:

(i) $g(x) > g(l)$ for all $x < l$.
(ii) $r = \inf\{\, x : g(x) < g(l) \,\}$.

We say that $r - l$ is the *length* of the excursion and $g(l)$ is the *level* of the excursion. We say that the excursion is *strict* if $g(x) > g(l)$ for any $x \in (l, r)$.

From now on, we say simply "excursion" to mean excursion above the minimum, and we say that l is a "minimum" if l is the left endpoint of an excursion of g.

Definition 2.2. Suppose that for any $\varepsilon > 0$, g has only finitely many excursions with length greater than ε. Then let $\mathcal{E}^\downarrow(g) \in \ell_\infty^\downarrow$ be the sequence of the lengths of the excursions of g, arranged in non-increasing order.

We write $\bar{g}$ for the function defined by

$$\bar{g}(x) = \inf_{0 \leq u \leq x} g(u). \tag{2.1}$$

Note that if g is a lower semi-continuous càdlàg function, then the excursions of g and $\bar{g}$ have the same lengths and levels; in particular,

$$\mathcal{E}^\downarrow(g) = \mathcal{E}^\downarrow(\bar{g}). \tag{2.2}$$

2.1. **Tilt representation of multiplicative coalescent.** The aim of this section is to formulate the "rigid" representation of the MC process from any initial condition in Theorem 2.8.

Definition 2.3. Given a locally finite measure μ on $(-\infty, 0]$, we define the *inverse cumulative distribution function* $f_\mu : [0, +\infty) \to [-\infty, 0]$ of μ by

$$f_\mu(x) = \sup\{\, y \le 0 \;:\; \mu[y, 0) > x \,\}, \qquad x \ge 0. \tag{2.3}$$

In particular, $f_\mu(x) = -\infty$ for any $x \ge \mu(-\infty, 0)$.

Note that f_μ is non-increasing, lower semi-continuous and càdlàg.

Definition 2.4. Given $\underline{m} = (m_1, m_2, \dots) \in \ell_2^\downarrow$, we define the independent exponential random variables

$$E_i \sim \mathrm{Exp}(m_i),\ i = 1, 2, \dots. \tag{2.4}$$

(If $m_i = 0$, we formally define $E_i = +\infty$.) We say that the random measure μ has $\mathrm{Exp}(\underline{m})$ distribution if μ is a point measure with point masses of weight m_i at locations $-E_i$, $i \in \mathbb{N}$:

$$\mu = \sum_{i=1}^{\infty} m_i \cdot \delta_{Y_i}, \quad Y_i = -E_i. \tag{2.5}$$

If $\mu \sim \mathrm{Exp}(\underline{m})$, then the total mass $\mu(-\infty, 0] = \sum_i m_i$ is infinite if $\underline{m} \notin \ell_1^\downarrow$. However, as long as $\underline{m} \in \ell_2^\downarrow$, the mass distribution is locally finite: in Lemma 4.1 we will show that almost surely $\mu(A) < \infty$ for every bounded set $A \subseteq (-\infty, 0]$.

Definition 2.5. Let $\underline{m} \in \ell_2^\downarrow$ and $\mu_0 \sim \mathrm{Exp}(\underline{m})$. Let $f_0 : [0, +\infty) \to [-\infty, 0]$ be the inverse cdf of μ_0, i.e.,

$$f_0(x) \overset{(2.3)}{=} f_{\mu_0}(x). \tag{2.6}$$

Remark 2.6. Let f_0 be defined by Definition 2.5.

(i) An alternative characterization of the function f_0 is as follows: f_0 is the non-increasing càdlàg function such that the interval I_j on which it takes the value $-E_j$ has length m_j, moreover the Lebesgue measure of the complement of $\cup_{j=1}^\infty I_j$ is zero.

(ii) If $\underline{m} = (m_1, \dots, m_n) \in \ell_0^\downarrow$ and $E_{\sigma_1} < \dots < E_{\sigma_n}$ is the increasing rearrangement of E_i, $1 \le i \le n$, then an equivalent way to write the function f_0 is

$$f_0(x) = \begin{cases} -E_{\sigma_k} & \text{if } \sum_{l=1}^{k-1} m_{\sigma_l} \le x < \sum_{l=1}^{k} m_{\sigma_l}, \quad 1 \le k \le n, \\ -\infty & \text{if } x \ge \sum_{i=1}^{n} m_{\sigma_l}. \end{cases} \tag{2.7}$$

For an illustration of (2.7), see Figure 2.1.

(iii) Recalling Definitions 2.1 and 2.2 we see that the excursion lengths of f_0 are given by the entries of $\underline{m} \in \ell_2^\downarrow$; that is, $\mathcal{E}^\downarrow(f_0) = \underline{m}$.

Definition 2.7. Let f_0 be defined by Definition 2.5. Let us define

$$f_t(x) = f_0(x) + tx, \qquad x \ge 0. \tag{2.8}$$

We say that the function f_t is a "tilt" of f_0.

In Lemma 4.4 we will show that, with probability 1, for all t the function f_t satisfies the criteria of Definition 2.2.

Theorem 2.8. *Let $\underline{m} \in \ell_2^\downarrow$. The process $\mathcal{E}^\downarrow(f_t), t \ge 0$ has the law of the multiplicative coalescent started from $\underline{m}$.*

We will prove Theorem 2.8 for $\underline{m} \in \ell_0^\downarrow$ in Section 3, and extend this result to $\underline{m} \in \ell_2^\downarrow$ in Section 5.

We say that Theorem 2.8 gives a "rigid" representation of the MC process, because all of the randomness is contained in the initial state of the representation and the rest of the dynamics is rigid, i.e., deterministic.

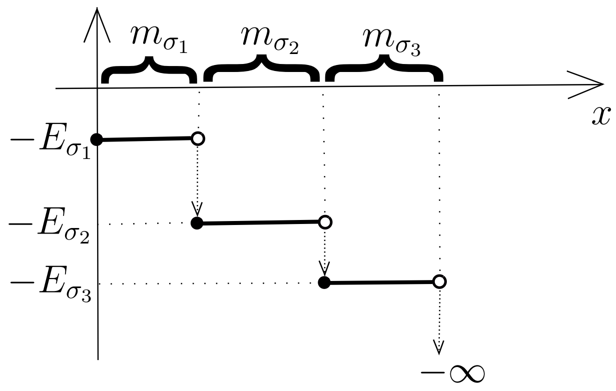

FIGURE 2.1. An illustration of the function f_0 defined in (2.7) when $n = 3$.

2.2. **Tilt-and-shift representation of** MCLD(λ)**.** Similarly to the rigid representation of the MC in terms of the excursion lengths of $(f_t(\cdot))$ in Theorem 2.8, we will give a rigid representation of the MCLD(λ) in terms of the excursion lengths of another function $(g_t(\cdot))$ in Theorem 2.13 below. We begin with the case of finitely many components.

Definition 2.9. Given $\underline{m} \in \ell_0^{\downarrow}$, we define $g_0(x) \equiv f_0(x)$, were $f_0(x)$ is defined by (2.6) (or, equivalently, (2.7)). We will now define $g_t(x)$ for $t, x \geq 0$ such that for all $x \geq 0$, the function $t \mapsto g_t(x)$ is càdlàg. The time evolution of $g_t(\cdot)$ consists of two parts:

(1) *Tilt:* If $g_{t-}(0) < 0$ then we let $\frac{\mathrm{d}}{\mathrm{d}t} g_t(x) = \lambda + x$.
(2) *Shift:* If $g_{t-}(0) = 0$, then we let $g_t(x) = g_{t-}(x + x^*(t))$, where

$$x^*(t) = \inf\{\, x > 0 \;:\; g_{t-}(x) < 0\} \tag{2.9}$$

is the length of the *first excursion* of $g_{t-}(\cdot)$ (see Definition 2.1).

Let us define ν to be the measure on $[0, \infty)$ given by

$$\nu = \sum_{0 \leq t < \infty} x^*(t) \cdot \delta_t \tag{2.10}$$

where $x^*(t) > 0$ is the size of the shift to the left at time t (see (2.9)); and if no shift occurred at time t, then we let $x^*(t) = 0$. Let us also define

$$\Phi(t) = \nu[0, t], \tag{2.11}$$

the total amount of left shifts up to time t.

For an illustration of Definition 2.9 see Figure 2.2.

Recall the definition of the MCLD(λ) from (1.3) and the notion of $\mathcal{E}^{\downarrow}$ from Definition 2.2.

Proposition 2.10. *Let $\underline{m} \in \ell_0^{\downarrow}$, $\lambda > 0$ and let f_0 be defined by Definition 2.5. If we define $g_t(\cdot)$ by Definition 2.9, then the process $\mathcal{E}^{\downarrow}(g_t), t \geq 0$ has the law of the* MCLD(λ) *process $\mathbf{m}_t, t \geq 0$ started from $\mathbf{m}_0 = \underline{m}$.*

We will prove Proposition 2.10 in Section 3.

Remark 2.11. In the MCLD(λ) interpretation, $\Phi(t)$ corresponds to the total amount of mass deleted up to time t.

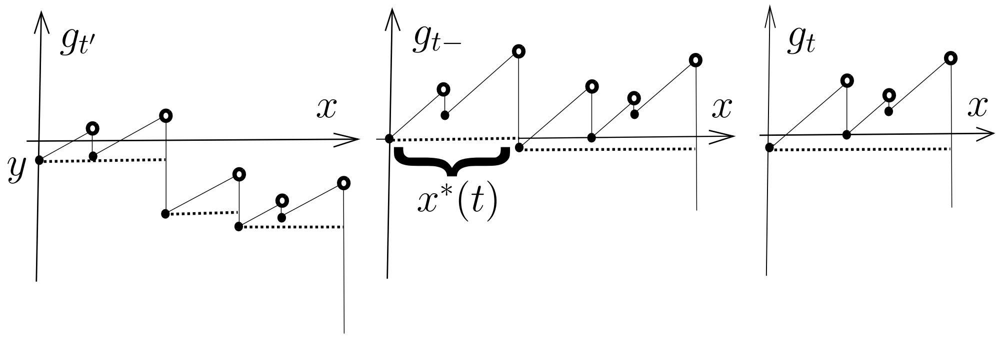

FIGURE 2.2. An illustration of Definition 2.9. Dashed lines are used to denote the running minimum. The second graph g_{t-} is obtained from the first graph $g_{t'}$ using the *tilt* operation. Observe that some excursions of $g_{t'}$ got merged with the tilt operation. The third graph g_t is obtained from the second graph by *shifting* it to the left by $x^*(t)$, i.e., the length of the first excursion of the second graph. If we denote by y the level of the first excursion of $g_{t'}$ then $t = t' + |y|/\lambda$ is the time instant when this excursion reaches level zero and thus gets shifted.

From Definition 2.9 it follows that we have

$$g_t(x) = g_0\big(x + \Phi(t)\big) + \lambda t + \int_0^t \big(x + \Phi(t) - \Phi(s)\big)\,\mathrm{d}s \tag{2.12}$$
$$= g_0\big(x + \Phi(t)\big) + \big(x + \Phi(t) + \lambda\big)t - \int_0^t \Phi(s)\,\mathrm{d}s.$$

Remark 2.12. Extending the dynamics of g_t for initial states in $\underline{m} \in \ell_2^\downarrow$ will amount to finding $\Phi(\cdot)$ corresponding to $g_0 := f_0$, c.f. Definition 2.5. We will then define g_t using the formula (2.12): the question is how to define $\Phi(\cdot)$ in a way that will appropriately extend Definition 2.9 from $\underline{m} \in \ell_0^\downarrow$ to $\underline{m} \in \ell_2^\downarrow$. The technical issue that we have to overcome is that for a typical $t \geq 0$ our functions $g_{t-}(\cdot)$ does not have a "first excursion" (c.f. (2.9)) if $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$. For example, a Brownian motion with parabolic drift does not have a first excursion, yet in Section 6.2 we will apply our tilt-and-shift representation to BMPD(u).

We are ready to state the main result of the paper.

Theorem 2.13. *For any $\underline{m} \in \ell_2^\downarrow$ let us define $g_0(x) \equiv f_0(x)$, where f_0 is constructed using Definition 2.5. There exists a random measure ν such that if we define $\Phi(t) = \nu[0,t]$ and $g_t(x)$ by (2.12) then*

(i) the process $\mathcal{E}^\downarrow(g_t), t \geq 0$ has the law of the MCLD(λ) *process started from $\underline{m}$,*
(ii) if $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$, then $g_t(0) = 0$ for any $t \geq 0$,
(iii) for any $t, x \geq 0$, the event $\{\Phi(t) \leq x\}$ is measurable with respect to the σ-algebra $\mathcal{F}_x^+ := \bigcap_{\varepsilon > 0} \sigma\left(g_0(x'), 0 \leq x' \leq x + \varepsilon\right)$.

We will prove Theorem 2.13 in Section 5.

Remark 2.14. (i) Put another way, Theorem 2.13(iii) says that for any fixed t, the random variable $\Phi(t) := \nu[0,t]$ is a stopping time with respect to the filtration $(\mathcal{F}_x^+)_{x \geq 0}$. We will make good use of this together with the strong Markov property of Brownian motion in Section 6.2.
(ii) By Theorem 2.13(iii), The control function $\Phi(\cdot)$ is measurable with respect to the σ-algebra $\sigma\left(g_0(x), x \geq 0\right)$. Therefore we have a "rigid" representation of MCLD(λ), since the function $g_t(\cdot)$ defined by (2.12) is determined by the initial state $g_0(\cdot)$.

(iii) We construct the measure ν that appears in Theorem 2.13 by extending our earlier construction given in Definition 2.9 from $\ell_0^\downarrow$ to $\ell_2^\downarrow$ in the sense that we obtain ν as the weak limit as $n \to \infty$ of the measures $\nu^{(n)}$ corresponding to initial conditions $\underline{m}^{(n)}$ truncated at index n, see Lemma 5.8 and Corollary 5.9.

(iv) If $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$, then g_0 is a continuous function satisfying $g_0(x)/x \to -\infty$ as $x \to \infty$ (see Lemmas 4.4, 5.3). We conjecture that for any such function g_0, there is a unique measure ν such that the function $g_t(x)$ controlled by $\Phi(t) = \nu[0,t]$ according to (2.12) satisfies
- $g_t(0) \equiv 0$ for all $t \geq 0$,
- $g_{t-}(x) \geq 0$ for any $0 \leq x \leq \nu(\{t\})$ for all $t \geq 0$.

(v) The function g_0 is constant on each of its excursions. Suppose that h_0 is another càdlàg and lower semi-continuous function such that $g_0 = \bar{h}_0$, that is, the excursions of h_0 have the same lengths and levels as those of g_0. Define, analogously to (2.12),

$$h_t(x) = h_0\big(x + \Phi(t)\big) + \lambda t + \int_0^t \big(x + \Phi(t) - \Phi(s)\big)\,\mathrm{d}s, \tag{2.13}$$

so that h_t is constructed using the same tilt-and-shift procedure as g_t (and using the same control function Φ). Then it is straightforward to see that $\bar{h}_t = \bar{g}_t$, and so, as at (2.2), $\mathcal{E}^\downarrow(g_t)$ and $\mathcal{E}^\downarrow(h_t)$ are the same process. We will make use of this observation in Section 6.2 when we consider the tilt-and-shift construction started from Brownian motion with a parabolic drift.

3. Rigid representations: finite state space

In this section we restrict the MCLD(λ) process to the space $\ell_0^\downarrow$ of states with only finitely many blocks. The ultimate goal of this section is to prove Proposition 2.10.

In Section 3.1 we recall the notion of a size-biased rearrangement of $\underline{m} \in \ell_0^\downarrow$ and how this notion can be extended to $\ell_2^\downarrow$ using an appropriate family of independent exponential random variables.

In Section 3.2 we define the interval coalescent with linear deletion (or briefly ICLD(λ)) and show that an ICLD(λ) with a size-biased initial state gives a representation of the MCLD(λ).

In Section 3.3 we define our particle representation and show that if the initial heights of particles form an appropriate family of independent exponential random variables, then we obtain a representation of of the ICLD(λ) with a size-biased initial state.

In Section 3.4 we show that a copy of the particle system is embedded in the tilt-and-shift representation introduced in Definition 2.9.

3.1. Construction of size-biased sequences using independent exponential random variables.

Now we recall some useful definitions from [2, Section 3.3].

Definition 3.1. Let $\underline{m} = (m_1, m_2, \ldots, m_n) \in \ell_0^\downarrow$. A random total linear order $\prec$ on $[n]$ is a *size-biased order* (with respect to $\underline{m}$) if for each permutation $i_1, i_2, \ldots, i_n$ of $[n]$,

$$\mathbb{P}(i_1 \prec i_2 \prec \cdots \prec i_n) = \prod_{r=1}^n \frac{m_{i_r}}{m_{i_r} + m_{i_{r+1}} + \cdots + m_{i_n}}. \tag{3.1}$$

We say that $\underline{b} = (b_1, \ldots, b_n)$ is the size-biased reordering of $\underline{m}$ if $b_k = m_{i_k}$, where $i_1 \prec i_2 \prec \cdots \prec i_n$.

Definition 3.2. Suppose $\underline{m} \in \ell_\infty^\downarrow$. Let $E_i \sim \mathrm{Exp}(m_i)$ independently for each i. Define a random linear order on $\mathbb{N}$ by $i \prec j$ if and only if $E_i < E_j$.

The proof of the next claim follows from the memoryless property of exponential random variables and we omit it.

Claim 3.3. *The law of $\prec$ introduced in Definition 3.2 is size-biased (with respect to the sizes m_i), in the sense that for any $n \in \mathbb{N}$ the restriction of $\prec$ to $[n]$ is size-biased with respect to $(m_1, m_2, \dots, m_n)$, c.f. Definition 3.1.*

Remark 3.4. (i) There is a smallest element with respect to the order $\prec$ if and only if $\underline{m} \in \ell_1^{\downarrow}$. If $\underline{m} \in \ell_\infty^{\downarrow} \setminus \ell_1^{\downarrow}$ then the values E_i are dense in $\mathbb{R}_+$, see Lemma 4.2.
(ii) The excursions (see Definition 2.1) of the random function f_0 defined in Definition 2.5 appear in size-biased order.

3.2. **A size-biased interval representation of** MCLD(λ)**.** We introduce a related process $\mathbf{b}_t, t \geq 0$ in which only neighbouring blocks are allowed to merge, and only the leftmost block is allowed to be deleted.

Definition 3.5. With some abuse of notation, we denote by $\ell_0 = \bigcup_{n \geq 0} \mathbb{R}^n_{>0}$ the space of finite sequences with positive entries. Given $\underline{b} = (b_1, \dots, b_n) \in \ell_0$, let $R^{\downarrow}(\underline{b}) \in \ell_0^{\downarrow}$ denote the reordering of $\underline{b}$ into non-increasing order.

Definition 3.6 (Interval coalescent with linear deletion, ICLD(λ))**.** The state space of the continuous-time Markov process $(\mathbf{b}_t)$ is ℓ_0. The dynamics consist of coalescence and deletion:

(i) If $\underline{b}, \underline{b}' \in \ell_0$ where $\underline{b}' \in \mathbb{R}^{n-1}_{>0}$ arises from $\underline{b} \in \mathbb{R}^n_{>0}$ by merging the blocks b_k and b_{k+1} for some $1 \leq k < n$; that is

$$b'_i = \begin{cases} b_i, & i = 1, 2, \dots, k-1 \\ b_k + b_{k+1}, & i = k \\ b_{i+1}, & i = k, \dots, n-1 \end{cases} \tag{3.2}$$

then the rate of the transition from $\underline{b}$ to $\underline{b}'$ is

$$\mathcal{R}_{IC}(\underline{b}, \underline{b}') = b_k \cdot \sum_{i=k+1}^{n} b_i. \tag{3.3}$$

(ii) If $\underline{b}, \underline{b}' \in \ell_0$ where $\underline{b}' \in \mathbb{R}^{n-1}_{>0}$ arises from $\underline{b} \in \mathbb{R}^n_{>0}$ by deleting the leftmost block, i.e.,

$$b'_i = b_{i+1}, \qquad 1 \leq i \leq n-1$$

then the rate of this transition is

$$\mathcal{R}_{IC}(\underline{b}, \underline{b}') = \lambda \cdot \sum_{i=1}^{n} b_i. \tag{3.4}$$

All other rates of the ICLD(λ) are 0.

We are now ready to state the main result of Section 3.2.

Theorem 3.7. *Let $\lambda \geq 0$ and $\underline{m} \in \ell_0^{\downarrow}$. Let $\mathbf{b}_0$ be a random size-biased reordering (see Definition 3.1) of $\underline{m}$. Let $\mathbf{b}_t, t \geq 0$ be an* ICLD(λ) *started from the initial state $\mathbf{b}_0$. Then*

(i) the law of the process $R^{\downarrow}(\mathbf{b}_t), t \geq 0$ is that of the MCLD(λ) *process $\mathbf{m}_t, t \geq 0$ started from $\mathbf{m}_0 = \underline{m}$, and*
(ii) given the σ-algebra $\sigma(R^{\downarrow}(\mathbf{b}_s), 0 \leq s \leq t)$, the conditional distribution of $\mathbf{b}_t$ is that of a random size-biased reordering of $R^{\downarrow}(\mathbf{b}_t)$.

We will prove Theorem 3.7 in Section 3.2.1.

Remark 3.8. (i) To the best of our knowledge, even the $\lambda = 0$ case of Theorem 3.7 is novel, and it gives an interval coalescent representation of the multiplicative coalescent process (on the state space $\ell_0^{\downarrow}$).

(ii) Theorem 3.7 generalizes in a natural way to any initial condition $\underline{m} \in \ell_1^{\downarrow}$. For initial conditions with infinite total mass, the situation is more complicated since under the natural extension of the concept of size-biased order (see Definition 3.2) there is no smallest element of the order, and any two elements are separated by infinitely many other elements in the order, c.f. Remark 3.4(i).

3.2.1. *Proof of Theorem 3.7.*

Lemma 3.9. *Let $\underline{b} \in \ell_0$ and $\underline{m} = R^{\downarrow}(\underline{b})$. Then the total jump rate of the* MCLD(λ) *out of the state $\underline{m}$ is the same as the total jump rate of the* ICLD(λ) *out of the state $\underline{b}$.*

Proof. The total jump rate in the MCLD(λ) is $\lambda \sum m_i + \sum_{i<j} m_i m_j$. Since $\underline{m}$ is a reordering of $\underline{b}$, we obtain the same total rate for the ICLD(λ) by adding the deletion rate in (3.4) to the sum over $1 \le k < n$ of the coalescence rate in (3.3). □

Lemma 3.10. *Let $\underline{m} \in \ell_0^{\downarrow}$. The following two procedures give the same distribution of $\underline{b}'$:*

(1) *Let $\underline{b}$ be a size-biased reordering of $\underline{m}$, and then, given $\underline{b}$, obtain $\underline{b}'$ from $\underline{b}$ by performing a single step of the jump chain of the* ICLD(λ).
(2) *Obtain $\underline{m}'$ from $\underline{m}$ by performing a single step of the jump chain of the* MCLD(λ), *and then, given $\underline{m}'$, let $\underline{b}'$ be a size-biased reordering of $\underline{m}'$.*

In particular, in (2) it is the case that the conditional distribution of $\underline{b}'$ given $R^{\downarrow}(\underline{b}')$ is that of a size-biased reordering of $R^{\downarrow}(\underline{b}')$, so the same is also true in (1).

Before proving Lemma 3.10, we use it to prove Theorem 3.7.

Proof of Theorem 3.7. Assume that the initial state has n non-zero blocks. Then the process $\mathbf{b}_t$ will make n jumps before being absorbed in the all-0 state. Let $\tau_0 = 0$ and let $\tau_1, \dots, \tau_n$ be the jump times of the process, so that $(\mathbf{b}_{\tau_0}, \mathbf{b}_{\tau_1}, \dots, \mathbf{b}_{\tau_n})$ is the jump-chain.

Let $\mathbf{m}_t = R^{\downarrow}(\mathbf{b}_t)$. We claim that for $0 \le k \le n$, the following properties hold:

P1(k): $(\mathbf{m}_{\tau_0}, \mathbf{m}_{\tau_1}, \dots, \mathbf{m}_{\tau_k})$ has the distribution of the first k steps of the jump-chain of the MCLD(λ).

P2(k): Given $\mathbf{m}_{\tau_1}, \dots, \mathbf{m}_{\tau_k}$, the distribution of $\mathbf{b}_{\tau_k}$ is that of a random size-biased reordering of $\mathbf{m}_{\tau_k}$.

As soon as we prove these properties, the proof of Theorem 3.7 follows using Lemma 3.9 and the memoryless property.

We now prove P1(k) and P2(k) by induction on k. The case $k = 0$ is immediate from the definition of $\mathbf{b}_0$ in the statement of Theorem 3.7.

For the induction step, suppose that P1(k) and P2(k) hold for a given k with $0 \le k < n$. Now we condition on $\mathbf{m}_{\tau_1}, \dots, \mathbf{m}_{\tau_k}$ and apply Lemma 3.10 with the choices

$$\underline{m} = \mathbf{m}_{\tau_k}, \qquad \underline{b} = \mathbf{b}_{\tau_k}, \qquad \underline{m}' = \mathbf{m}_{\tau_{k+1}}, \qquad \underline{b}' = \mathbf{b}_{\tau_{k+1}}.$$

From P2(k), we know that given $\mathbf{m}_{\tau_1}, \dots, \mathbf{m}_{\tau_k}$, the distribution of $\mathbf{b}_{\tau_k}$ is a random size-biased reordering of $\mathbf{m}_{\tau_k}$; then, as in (1) of Lemma 3.10, we obtain $\mathbf{b}_{\tau_{k+1}}$ by taking a step of the ICLD(λ) chain from $\mathbf{b}_{\tau_k}$.

Hence, using the final observation of Lemma 3.10, the distribution of $\mathbf{b}_{\tau_{k+1}}$ given $\mathbf{m}_{\tau_1}, \dots, \mathbf{m}_{\tau_k}, \mathbf{m}_{\tau_{k+1}}$, is a random size-biased reordering of $\mathbf{m}_{\tau_{k+1}}$, and P2($k+1$) holds.

But also, comparing with (2) of Lemma 3.10, the distribution we obtain of $\mathbf{m}_{\tau_{k+1}} = R^{\downarrow}(\mathbf{b}_{\tau_{k+1}})$ is the same as we would have obtained by taking a step of the MCLD(λ) chain from $\mathbf{m}_{\tau_k}$. So indeed we can extend P1(k) to give P1($k+1$) also.

This completes the induction step. In this way we obtain that P1(n) and P2(n) hold, and this completes the proof of Theorem 3.7. □

Proof of Lemma 3.10. To avoid awkward notation, we will write the proof under the extra assumption that $\underline{m}$ has all its block sizes distinct, and so does $\underline{m}'$ for each $\underline{m}'$ that can be obtained from $\underline{m}'$ by a coalescence step. The general statement can then be easily obtained by continuity.

Under the above stated extra assumption we can write the transition rates of the MCLD(λ) chain as follows. If $\underline{m}'$ is obtained from $\underline{m}$ by merging the two blocks of size m_I and m_J, then the rate of the transition from $\underline{m}$ to $\underline{m}'$ is

$$\mathcal{R}_{MC}(\underline{m}, \underline{m}') = m_I m_J. \tag{3.5}$$

If $\underline{m}'$ is obtained from $\underline{m}$ by deleting the block of size m_I, then the rate of the transition from $\underline{m}$ to $\underline{m}'$ is

$$\mathcal{R}_{MC}(\underline{m}, \underline{m}') = \lambda m_I. \tag{3.6}$$

Let $\pi_{\underline{m}}$ denote the probability distribution on ℓ_0 which arises from the size-biased reordering of $\underline{m}$. Again if $\underline{m}$ has distinct block sizes, we can write

$$\pi_{\underline{m}}(\underline{b}) = \mathbb{1}[\, R^{\downarrow}(\underline{b}) = \underline{m}\,] \cdot \prod_{i=1}^{n} \frac{b_i}{\sum_{j=i}^{n} b_j}. \tag{3.7}$$

Now, to prove Lemma 3.10 it is enough to demonstrate the following claim: if $\underline{m} \in \ell_0^{\downarrow}$ and $\underline{b}' \in \ell_0$, with $\underline{m}' = R^{\downarrow}(\underline{b}')$, then

$$\sum_{\underline{b}:\, R^{\downarrow}(\underline{b}) = \underline{m}} \pi_{\underline{m}}(\underline{b}) \mathcal{R}_{IC}(\underline{b}, \underline{b}') = \mathcal{R}_{MC}(\underline{m}, \underline{m}') \pi_{\underline{m}'}(\underline{b}'). \tag{3.8}$$

(Rescaled by the total jump rate, which by Lemma 3.9 are the same for the ICLD(λ) and the MCLD(λ), the left side represents the probability of obtaining a given value $\underline{b}'$ using procedure (1) in Lemma 3.10, while the right side represents the same for procedure (2).)

To prove (3.8), there are two cases that we have to handle, corresponding to coalescence and deletion.

We first treat the case of coalescence, that is we assume that the state $R^{\downarrow}(\underline{b}') = \underline{m}'$ arises from $\underline{m}$ by merging some blocks with sizes m_I and m_J. Then $\underline{b}'$ has an interval of size $m_I + m_J$, say $b'_k = m_I + m_J$. There are exactly two reorderings $\underline{b}$ of $\underline{m}$ for which $\mathcal{R}_{IC}(\underline{b}, \underline{b}') > 0$, namely

$$\underline{b}^1 = (b'_1, \ldots, b'_{k-1}, m_I, m_J, b'_{k+1}, \ldots, b'_n),$$
$$\underline{b}^2 = (b'_1, \ldots, b'_{k-1}, m_J, m_I, b'_{k+1}, \ldots, b'_n).$$

Now let us rewrite the two sides of (3.8). The left side is

$$\sum_{\underline{b}: R^{\downarrow}(\underline{b}) = \underline{m}} \pi_{\underline{m}}(\underline{b}) \mathcal{R}_{IC}(\underline{b}, \underline{b}') = \pi_{\underline{m}}(\underline{b}^1) \mathcal{R}_{IC}(\underline{b}^1, \underline{b}') + \pi_{\underline{m}}(\underline{b}^2) \mathcal{R}_{IC}(\underline{b}^2, \underline{b}') \overset{(3.3),(3.7)}{=}$$

$$\left(\prod_{i=1}^{k-1} \frac{b'_i}{\sum_{j=i}^{n} b'_j} \right) \frac{m_I}{\sum_{j=k}^{n} b'_j} \frac{m_J}{\sum_{j=k}^{n} b'_j - m_I} \cdot$$
$$\left(\prod_{i=k+1}^{n} \frac{b'_i}{\sum_{j=i}^{n} b'_j} \right) \cdot m_I \cdot \left(\sum_{j=k}^{n} b'_j - m_I \right) +$$
$$\left(\prod_{i=1}^{k-1} \frac{b'_i}{\sum_{j=i}^{n} b'_j} \right) \frac{m_J}{\sum_{j=k}^{n} b'_j} \frac{m_I}{\sum_{j=k}^{n} b'_j - m_J} \cdot$$
$$\left(\prod_{i=k+1}^{n} \frac{b'_i}{\sum_{j=i}^{n} b'_j} \right) \cdot m_J \cdot \left(\sum_{j=k}^{n} b'_j - m_J \right).$$

Using (3.5) and (3.7), the right side can be rewritten as

$$\mathcal{R}_{MC}(\underline{m}, \underline{m}')\pi_{\underline{m}'}(\underline{b}') = m_I m_J \cdot \left(\prod_{i=1}^{k-1} \frac{b'_i}{\sum_{j=i}^n b'_j}\right) \frac{m_I + m_J}{\sum_{j=k}^n b'_j} \left(\prod_{i=k+1}^{n} \frac{b'_i}{\sum_{j=i}^n b'_j}\right).$$

These are the same, so (3.8) holds in the case of coalescence.

We now turn to the case of deletion; that is, we assume that the state $R^{\downarrow}(\underline{b}') = \underline{m}'$ arises from $\underline{m}$ by deleting a block of size m_I. There is one rearrangement $\underline{b}$ of $\underline{m}$ for which $\mathcal{R}_{IC}(\underline{b}, \underline{b}') > 0$, namely

$$\underline{b}^0 = (m_I, b'_1, \dots, b'_n). \tag{3.9}$$

Thus

$$\begin{aligned}\sum_{\underline{b}:\underline{m}=R^{\downarrow}(\underline{b})} \pi_{\underline{m}}(\underline{b})\mathcal{R}_{IC}(\underline{b}, \underline{b}') &= \pi_{\underline{m}}(\underline{b}^0)\mathcal{R}_{IC}(\underline{b}^0, \underline{b}') \\ &\overset{(3.4),(3.7)}{=} \frac{m_I}{m_I + \sum_{j=1}^n b'_j} \pi_{\underline{m}'}(\underline{b}') \cdot \lambda \cdot \left(m_I + \sum_{j=1}^n b'_j\right) \\ &= \lambda m_I \cdot \pi_{\underline{m}'}(\underline{b}') \\ &= \mathcal{R}_{MC}(\underline{m}, \underline{m}')\pi_{\underline{m}'}(\underline{b}')\end{aligned}$$

So (3.8) holds in this case also. This completes the proof of Lemma 3.10. □

3.3. **Particle representation.** The representation of the MCLD(λ) in Section 3.2 moved some of the randomness of the process into the choice of an initial condition (using a size-biased reordering). Thereafter the possible transitions of the process were restricted (only neighbouring blocks were allowed to merge, and only the first block could be deleted).

In this section we take this to an extreme by giving a natural construction of the process in which *all* the randomness is in the initial condition; the evolution of the process thereafter is entirely deterministic, but nonetheless the process projects to the MCLD(λ). Such processes might be called "rigid".

Definition 3.11 (Particle system)**.** Let $\underline{m} = (m_1, \dots, m_n) \in \ell_0^{\downarrow}$. Let $E_1, \dots, E_n$ be independent with $E_i \sim \text{Exp}(m_i)$. For $i = 1, \dots, n$, let $Y_i(0) = -E_i$ be the initial height of a particle i with mass m_i. The heights of particles evolve over time; we describe the joint evolution of the heights $Y_1(t), \dots, Y_n(t)$ using a system of ordinary differential equations.

Analogously to the definition (2.5), we define

$$\mu_t = \sum_{i=1}^n m_i \cdot \delta_{Y_i(t)}. \tag{3.10}$$

The system of differential equations governing $Y_1(t), \dots, Y_n(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_i(t) = \lambda \cdot \mathbb{1}[\, Y_i(t) < 0\,] + \mu_t(Y_i(t), 0). \tag{3.11}$$

We say that the particle i "dies" at time t_i, where t_i is defined by

$$t_i := \min\{\, t \,:\, Y_i(t) = 0\,\}. \tag{3.12}$$

A "time-t block" consists of the union of all the particles that share the same (strictly negative) height at time t.

In words, particles start at negative locations and move up. If a particle reaches zero then it stops there and dies. Before it dies, the speed of a particle is equal to λ plus the total weight of particles strictly above it and strictly below zero. Observe that if two particles ever reach the same height, then they stay together forever.

Recall the definition of the ICLD(λ) process from Definition 3.6.

Proposition 3.12. *Let $\mathbf{b}_t \in \ell_0$ be the vector of sizes of the time-t blocks of the particle system, in decreasing order of their height. Then the process $\mathbf{b}_t, t \geq 0$ has the law of* ICLD(λ)*, started from an initial state $\mathbf{b}_0$.*

Before proving Proposition 3.12, let us state the following corollary.

Corollary 3.13. *The process $\mathbf{m}_t = R^{\downarrow}(\mathbf{b}_t)$ has the law of MCLD(λ) started from the state $\underline{m}$.*

Proof of Corollary 3.13. It follows from Definition 3.11 and Claim 3.3 that $\mathbf{b}_0$ is a size-biased reordering of $\underline{m}$. Now the statement of Corollary 3.13 follows from Theorem 3.7 and Proposition 3.12. □

Before we prove Proposition 3.12, let us introduce a useful notation.

Definition 3.14. Given $\underline{b} = (b_1, \dots, b_n) \in \ell_0$ we say that the the random vector

$$(Y^{(1)}, \dots, Y^{(n)})$$

has law $\mathrm{Exp}^{\prec}(\underline{b})$ if $0 > Y^{(1)} > \cdots > Y^{(n)}$,

$$\begin{aligned} -Y^{(1)} &\sim \mathrm{Exp}(b_1 + b_2 + \cdots + b_n) \\ Y^{(1)} - Y^{(2)} &\sim \mathrm{Exp}(b_2 + \cdots + b_n) \\ &\vdots \\ Y^{(k)} - Y^{(k+1)} &\sim \mathrm{Exp}\left(\sum_{i=k+1}^{n} b_i\right) \qquad (3.13) \\ &\vdots \\ Y^{(n-1)} - Y^{(n)} &\sim \mathrm{Exp}(b_n), \end{aligned}$$

and all these gaps are independent.

Proof of Proposition 3.12. Suppose there are $n(t)$ time-t blocks. Let

$$\left(Y^{(1)}(t), \dots, Y^{(n(t))}(t)\right) \tag{3.14}$$

be the vector of their heights in decreasing order. Then μ_t, defined at (3.10), is also given by

$$\mu_t = \sum_{i=1}^{n(t)} (\mathbf{b}_t)_i \, \delta_{Y^{(i)}(t)}. \tag{3.15}$$

Observe that in particular, $\left(Y^{(1)}(0), \dots, Y^{(n)}(0)\right)$ is the decreasing rearrangement of the initial heights of the particles, and $Y^{(i)}(0)$ is the initial height of a particle of mass $(\mathbf{b}_0)_i$.

From here onwards, let us condition throughout on $\mathbf{b}_0 = \underline{b} = (b_1, \dots, b_n)$.

By repeatedly applying the memoryless property of the exponential distribution, we see that given $\mathbf{b}_0 = \underline{b}$, we have

$$\left(Y^{(1)}(0), \dots, Y^{(n)}(0)\right) \sim \mathrm{Exp}^{\prec}(\underline{b}), \tag{3.16}$$

where $\mathrm{Exp}^{\prec}(\cdot)$ was introduced in Definition 3.14.

Let us consider the time τ_1 at which the first jump of the process $(\mathbf{b}_t)$ occurs. This jump may be either a deletion (if a block reaches height 0 and so dies) or it may be a coalescence (if two blocks reach the same height). Between times 0 and τ_1, the kth highest particle moves at speed $\lambda + \sum_{i=1}^{k-1} b_i$. Hence the gap $Y^{(k)}(t) - Y^{(k+1)}(t)$ decreases at rate b_k for $1 \le k \le n-1$, and the gap $-Y^{(1)}(0)$ decreases at rate λ.

Let us therefore define

$$t^{(0)} := \frac{Y^{(1)}(0)}{\lambda} \sim \text{Exp}\left(\lambda(b_1 + \cdots + b_n)\right) \tag{3.17}$$

$$t^{(k)} := \frac{Y^{(k)}(0) - Y^{(k+1)}(0)}{b_k} \sim \text{Exp}\left(b_k \sum_{i=k+1}^{n} b_i\right) \text{ for } 1 \le k \le n-1. \tag{3.18}$$

These variables are all independent.

Then $\tau_1 = \min_{0 \le k \le n-1} t^{(k)}$, and we have

$$\tau_1 \sim \text{Exp}\left(\lambda \sum_{i=1}^{n} b_i + \sum_{1 \le i < j \le n} b_i b_j\right).$$

Further, let $\kappa = \text{argmin}_{0 \le k \le n-1} t^{(k)}$. This argmin is uniquely defined with probability 1. If $\kappa = 0$, then the event at time τ_1 is a deletion of the highest block, while if $\kappa = k \ge 1$ then the event at time τ_1 is a coalescence of the kth and $(k+1)$st highest blocks. Then κ and τ_1 are independent; the probability that $\kappa = k$ is proportional to $\lambda \sum_{i=1}^{n} b_i$ for $k = 0$, and to $b_k \sum_{i=k+1}^{n} b_i$ for $1 \le k \le n-1$.

Comparing the previous paragraph with the rates of the ICLD(λ) at (3.3) and (3.4), we see that the distribution of the new state $\underline{b}' = \mathbf{b}_{\tau_1}$, together with the time τ_1 at which it occurs, is the same as for the ICLD(λ) process.

From here on, let us condition further on τ_1 and $\mathbf{b}_{\tau_1} = \underline{b}'$ as well as on $\mathbf{b}_0 = \underline{b}$. Again applying the memoryless property, the conditional distribution of the remaining height gaps just before τ_1, excluding the one which reaches 0 at that time, is unchanged from what it was at time 0.

The heights and masses of blocks at time τ_1 are given by

$$Y^{(i)}(\tau_1) = \begin{cases} Y^{(i)}(\tau_1-) & \text{for } i \le \kappa, \\ Y^{(i+1)}(\tau_1-) & \text{for } i > \kappa, \end{cases} \tag{3.19}$$

and

$$b'_i = \begin{cases} b_i & \text{for } i < \kappa, \\ b_\kappa + b_{\kappa+1} & \text{for } i = \kappa, \\ b_{i+1} & \text{for } i > \kappa. \end{cases} \tag{3.20}$$

We therefore obtain that

$$-Y^{(1)}(\tau_1) \sim \text{Exp}\left(\sum_{i=1}^{n-1} b'_i\right)$$

$$Y^{(k)}(\tau_1) - Y^{(k+1)}(\tau_1) \sim \text{Exp}\left(\sum_{i=k+1}^{n-1} b'_i\right) \text{ for } k = 1, \ldots, n-2,$$

and all these gaps are independent. Thus we have shown that

$$\left(Y^{(1)}(\tau_1), \ldots, Y^{(n-1)}(\tau_1)\right) \sim \text{Exp}^{\prec}(\underline{b}').$$

So we can repeat the argument above starting from time τ_1, until the time τ_2 of the next jump, obtaining that $\mathbf{b}_t$ continues to evolve as an ICLD(λ) process. Continuing recursively until the time of the nth jump (when the last death occurs and $\mathbf{b}_t$ becomes identically zero), we obtain the desired result. The proof of Proposition 3.12 is complete. □

We will now prove a corollary of the results and methods developed in Sections 3.2 and 3.3. This corollary will only be used in Section 6.

We consider the particle system introduced in Definition 3.11. Let $\mathbf{b}_t \in \ell_0$ be the vector of sizes of the time-t blocks of the particle system, in decreasing order of their height and let $\mathbf{m}_t = R^{\downarrow}(\mathbf{b}_t)$. Let $\tau_0 = 0$ and denote by $\tau_k, k \geq 1$ the kth jump time of the process $(\mathbf{b}_t)$. Recall the notion of the point measure μ_t from (3.10) (or from (3.15)) and notion of $\mathrm{Exp}(\underline{m})$ from Definition 2.4.

Corollary 3.15. *(i) For any $k \geq 0$, the conditional distribution of μ_{τ_k} given $(\mathbf{m}_{\tau_i}, 0 \leq i \leq k)$ is $\mathrm{Exp}(\mathbf{m}_{\tau_k})$.*
(ii) For any $t \geq 0$, the conditional distribution of μ_t given $(\mathbf{m}_s, 0 \leq s \leq t)$ is $\mathrm{Exp}(\mathbf{m}_t)$.

Before we prove Corollary (3.15), let us state the key claim needed for the proof. The proof of this claim follows from the memoryless property and we omit it.

Claim 3.16. *Let $\underline{m} = (m_1, \dots, m_n) \in \ell_0^{\downarrow}$ and let $\mathbf{b}$ be a size-biased reordering of $\underline{m}$. Conditioned on $\mathbf{b} = \underline{b}$, let $(Y^{(1)}, \dots, Y^{(n)}) \sim \mathrm{Exp}^{\prec}(\underline{b})$ (c.f. Definition 3.14). Then the law of $\mu = \sum_{i=1}^{n} (\mathbf{b})_i \delta_{Y^{(i)}}$ is $\mathrm{Exp}(\underline{m})$ (c.f. Definition 2.4).*

Proof of Corollary 3.15. Given $\mathbf{m}_{\tau_1}, \dots, \mathbf{m}_{\tau_k}$, the distribution of $\mathbf{b}_{\tau_k}$ is that of a random size-biased reordering of $\mathbf{m}_{\tau_k}$ (c.f. P2(k) in the proof of Theorem 3.7 and the proof of Corollary 3.13).

In the proof of Proposition 3.12 we have seen that given $\mathbf{b}_{\tau_1}, \dots, \mathbf{b}_{\tau_k}$, the conditional joint distribution of the heights is

$$\left(Y^{(1)}(\tau_k), \dots, Y^{(n(\tau_k))}(\tau_k)\right) \sim \mathrm{Exp}^{\prec}(\mathbf{b}_{\tau_k}), \qquad n(\tau_k) = n - k.$$

Combining the above observations with Claim 3.16 we obtain that the statement of Corollary 3.15(i) indeed holds.

Now Corollary 3.15(ii) follows from the combination of part (i), the observation that between the jumps of the process $(\mathbf{b}_t)$ the height gaps between particles decrease at constant speed and the memoryless property of the height gap distribution (c.f. Definition 3.14). □

3.4. **Tilt-and-shift representation.** In this section we connect the particle system introduced in Definition 3.11 to the "tilt-and-shift" dynamics presented in Definition 2.9 and prove Proposition 2.10.

Definition 3.17. Assume $g : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ has only finitely many excursions. Denote by

$$\mathcal{E}(g) \in \ell_0$$

the sequence of the lengths of the excursions of g, in order of appearance.

Let $\underline{m} = (m_1, \dots, m_n) \in \ell_0^{\downarrow}$. Let $E_i \sim \mathrm{Exp}(m_i)$ independently for $i = 1, \dots, n$.

Consider the particle system of Section 3.3 in which a particle of size m_i starts at height $Y_i(0) = -E_i$ for each $i = 1, \dots, n$.

Define the function $g_0 \equiv f_0$ as at (2.6) or (2.7), and then $g_t, t > 0$ using the tilt-and-shift procedure of Definition 2.9.

Proposition 3.18. *Let $\mathbf{b}_t \in \ell_0$ be the vector of sizes of the time-t blocks of the particle system in decreasing order of height. Then*

$$(\mathbf{b}_t, t \geq 0) = (\mathcal{E}(g_t), t \geq 0) \text{ with probability } 1. \tag{3.21}$$

Then from Proposition 3.12, Corollary 3.13 and Theorem 3.7 we can immediately deduce the following result:

Corollary 3.19. *$(\mathcal{E}(g_t), t \geq 0)$ has the law of ICLD(λ) started from an initial state which is a size-biased reordering of $\underline{m}$. Hence $\left(\mathcal{E}^{\downarrow}(g_t), t \geq 0\right)$ has the law of MCLD(λ) started from $\underline{m}$ and thus Proposition 2.10 holds.*

Proof of Proposition 3.18. We will show how to find a copy of the particle system embedded within the tilt-and-shift process. Namely, the excursions of g_t correspond to the time-t blocks in the particle system. Specifically, an excursion $[x, x')$ of g_t corresponds to a time-t block whose size is the length $x' - x$ of the excursion, and whose height is the level $g_t(x)$ of the excursion.

The idea is that the "tilt" part of the construction produces an upward drift of the excursion levels which corresponds to the upward movement of the blocks in the particle system; this drift causes neighbouring excursions to merge, corresponding to the merging of blocks in the particle system when they reach the same height. Meanwhile the "shift" mechanism which removes the leftmost excursion when its level reaches 0 corresponds to the death of a block when it reaches height 0 in the particle system.

Recall that $Y^{(1)}(t), \ldots, Y^{(n(t))}(t)$ are the heights of the time-t blocks of the particle system in decreasing order, and $\mathbf{b}_t$ is the vector of the sizes of those blocks (in decreasing order of height).

Let us denote $\mathbf{b}_0 = (b_1, \ldots, b_n)$. Write also

$$x_0 = 0, \quad x_1 = b_1, \quad x_2 = b_1 + b_2, \quad \ldots, \quad x_n = b_1 + \cdots + b_n. \tag{3.22}$$

Then the excursions of g_0 are the intervals

$$[x_0, x_1), \quad [x_1, x_2), \quad \ldots, \quad [x_{n-1}, x_n) \tag{3.23}$$

and the levels of these excursions are

$$g_0(x_0) = Y^{(1)}(0), \quad \ldots, \quad g_0(x_{n-1}) = Y^{(n)}(0) \tag{3.24}$$

(with $g_0(x_n) = -\infty$).

Recall the time τ_1, the time of the first merging or deletion in the particle system, defined in the proof of Proposition 3.12, which we can also write as

$$\tau_1 = \inf\left\{t : Y^{(1)}(t-) = 0 \text{ or } Y^{(k)}(t-) = Y^{(k+1)}(t-) \text{ for some } 1 \leq k < n\right\}. \tag{3.25}$$

Since each block moves upwards at a rate equal to the sum of the sizes of blocks above it plus λ, we have that for $1 \leq k \leq n$ and $t \in [0, \tau_1)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Y^{(k)}(t) &= b_1 + \cdots + b_{k-1} + \lambda \\ &= x_{k-1} + \lambda. \end{aligned} \tag{3.26}$$

But also if we define

$$\tilde{\tau}_1 = \inf\left\{t : g_{t-}(0) = 0 \text{ or } g_{t-}(x_{k-1}) = g_{t-}(x_k) \text{ for some } 1 \leq k < n\right\}, \tag{3.27}$$

then by the definition of the tilt mechanism in Definition 2.9, we have that for $1 \leq k \leq n$ and $t \in [0, \tilde{\tau}_1)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(x_{k-1}) = x_{k-1} + \lambda. \tag{3.28}$$

Since the derivatives on the right-hand sides of (3.26) and (3.28) are the same, and the values at $t = 0$ are also the same by (3.24), we have that $\tau_1 = \tilde{\tau}_1$ and that

$$Y^{(k)}(t) = g_t(x_{k-1}), \quad k = 1, \ldots, n, \quad t \in [0, \tau_1). \tag{3.29}$$

By (3.27), also $g_t(x_{k-1}) > g_t(x_k)$ for all such k and t, and so for all $t \in [0, \tau_1)$, the excursions of g_t are again given by (3.23). So indeed we find that throughtout $[0, \tau_1)$ the block heights and the excursion levels continue to correspond, and the block sizes and excursion lengths do not change.

Now we look at what happens at time τ_1, which is the first time that the particle system has a merge or deletion event. Recall from the proof of Proposition 3.12 the value κ which describes which kind of event happens at time τ_1.

- If $1 \le \kappa \le n-1$ then the event is a merge of the blocks with sizes b_κ and $b_{\kappa+1}$. In that case we have that $Y^\kappa(\tau_1-) = Y^{\kappa+1}(\tau_1-)$ and so also (by (3.29)) we have $g_{\tau_1-}(x_{k-1}) = g_{\tau_1-}(x_k)$. Hence at time τ_1 also the excursions $[x_{\kappa-1}, x_\kappa)$ and $[x_\kappa, x_{\kappa+1})$ merge into a single excursion which is $[x_{\kappa-1}, x_{\kappa+1})$.
- If instead $\kappa = 0$ then $Y^{(1)}(\tau_1-) = 0$ and $g_{\tau_1-}(0) = 0$. Then time τ_1 is the death time of the block of size b_1; also, following (2.9), at time τ_1 there is a shift of size $x^*(\tau_1-) = x_1 = b_1$. Then we obtain $g_{\tau_1}(x) = g_{\tau_1-}(x + b_1)$.

In both of these cases the heights and masses of the particle system at time τ_1 are given by (3.19) and (3.20). If we now define

$$x'_0 = 0, \quad x'_1 = b'_1, \quad x'_2 = b'_1 + b'_2, \quad \dots \quad x'_{n-1} = b'_1 + \dots + b'_{n-1}, \tag{3.30}$$

then we obtain that the excursions of g_{τ_1} are the intervals

$$[x'_0, x'_1), \quad [x'_1, x'_2), \quad \dots, \quad [x'_{n-2}, x'_{n-1}) \tag{3.31}$$

and the levels of these excursions are

$$g_{\tau_1}(x'_0) = Y^{(1)}(\tau_1), \quad \dots, \quad g_{\tau_1}(x'_{n-2}) = Y^{(n-1)}(\tau_1). \tag{3.32}$$

From here, proceeding from (3.31) and (3.32) just as we did from (3.23) and (3.24), we can repeat the argument above from time τ_1 until the time τ_2 of the next jump of $\mathcal{E}(g_t)$. Continuing recursively until the time of the nth jump (when the last death occurs), we obtain that the excursion lengths and block sizes continue to correspond, as required for Proposition 3.18. □

Remark 3.20. Observe that written symbolically, we have shown that

$$\bar{g}_t(\cdot) \equiv f_{\mu_t}(\cdot), \qquad t \ge 0, \tag{3.33}$$

where $\bar{g}_t$ is the minimum process of g_t as defined at (2.1); from the expression (3.15) for μ_t, and the definition of f_μ in Definition 2.3, we have that f_{μ_t} is the non-increasing piecewise constant càdlàg function taking value $Y^{(i)}(t)$ on an interval of length $(\mathbf{b}_t)_i$ for $i = 1, \dots, n(t)$, and otherwise takes the value $-\infty$.

In this formulation the statement of Proposition 3.18 can be seen to follow immediately, since $\mathbf{b}_t = \mathcal{E}(f_{\mu_t})$ and $\mathcal{E}(\bar{g}_t) = \mathcal{E}(g_t)$.

Corollary 3.21. *Recalling the definition of ν from (2.10) and t_i from (3.12), we have*

$$\nu = \sum_{i=1}^{n} m_i \cdot \delta_{t_i}. \tag{3.34}$$

Remark 3.22. We observed in Remark 3.8 that the ICLD(λ) process naturally generalizes to any initial condition $\underline{m} \in \ell_1^\downarrow$. Similarly, the definitions and results of Sections 3.3 and 3.4 also extend to $\underline{m} \in \ell_1^\downarrow$. As a corollary, Proposition 2.10 generalizes to $\underline{m} \in \ell_1^\downarrow$. For initial conditions with infinite total mass we cannot naively extend Definition 2.9, as explained in Remark 2.12. The extension of the tilt-and-shift representation to $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$ will be carried out in Section 5.

4. Preparatory results about μ_0 and excursions

In this section we collect preliminary results that we will later use in Section 5 when we extend the rigid representation result of Section 3 from $\ell_0^\downarrow$ to $\ell_2^\downarrow$.

In Section 4.1 we state and prove some of the analytic properties of the random point measure $\mu \sim \text{Exp}(\underline{m})$ (c.f. Definition 2.4) given some $\underline{m} \in \ell_2^\downarrow$.

In Section 4.2 we state and prove results related to excursions and the $\mathcal{E}^{\downarrow}$ functional (c.f. Definitions 2.1, 2.2).

4.1. Some facts about random point measures.

Lemma 4.1. *Let $\underline{m} \in \ell_2^{\downarrow}$ and $\mu \sim \mathrm{Exp}(\underline{m})$. With probability 1,*
(i) $\mu(A) < \infty$ for every bounded set $A \subseteq (-\infty, 0]$.
(ii) $\mu[y, y+1] \to 0$, as $y \to -\infty$.

Proof. For any $0 \le a < b$,

$$\mathbb{E}\left(\mu[-b,-a]\right) \overset{(2.5)}{=} \sum_i m_i \mathbb{P}(E_i \in [a,b]) \overset{(2.4)}{=}$$

$$\sum_i m_i e^{-am_i}\left(1 - e^{-(b-a)m_i}\right) < \sum_i m_i^2 (b-a) < \infty$$

since $\underline{m} \in \ell_2^{\downarrow}$, and this is already enough to give (i).

For (ii), we have

$$\mathbb{E}\left(\mu[-k-1,-k]\right) = \sum_i m_i e^{-km_i}\left(1 - e^{-m_i}\right) \le$$

$$\sum_i m_i^2 e^{-km_i} \to 0 \quad \text{as} \quad k \to \infty,$$

and also

$$\mathrm{Var}\left(\mu[-k-1,-k]\right) = \sum_i m_i^2 \, \mathrm{Var}\left(\mathbb{1}[k \le E_i \le k+1]\right) \le$$

$$\sum_i m_i^2 \mathbb{P}\left(k \le E_i \le k+1\right).$$

Thus $\sum_k \mathrm{Var}\left(\mu[-k,-k-1]\right) \le \sum_i m_i^2 < \infty$.

Then let k be large enough such that $\mathbb{E}\left(\mu[-k-1,-k]\right) \le \delta/2$. Then by Chebyshev's inequality,

$$\mathbb{P}(\mu[-k,-k-1] > \delta) \le \frac{\mathrm{Var}\left(\mu[-k,-k-1]\right)}{\left(\delta - \delta/2\right)^2}.$$

Hence $\sum_k \mathbb{P}(\mu[-k,-k-1] > \delta) < \infty$ and the result in (ii) follows from Borel-Cantelli. □

Lemma 4.2. *If $\underline{m} \in \ell_2^{\downarrow} \backslash \ell_1^{\downarrow}$ and $\mu \sim \mathrm{Exp}(\underline{m})$, then with probability 1 we have $\mu[-b,-a] > 0$ for any $0 < a < b$.*

Proof. It is enough to prove that for all pairs of rational numbers $0 < a < b$ we have $\mu[-b,-a] > 0$ with probability 1. This follows from the second Borel-Cantelli lemma and the fact that

$$\sum_{i=1}^{\infty} \mathbb{P}(\, a \le E_i \le b\,) \overset{(2.4)}{=} \sum_{i=1}^{\infty} e^{-am_i}\left(1 - e^{-(b-a)m_i}\right) \overset{(*)}{=} +\infty \quad \text{if} \quad \underline{m} \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow},$$

where $(*)$ follows from $e^{-am}\left(1 - e^{-(b-a)m}\right) \approx (b-a)m$ as $m \to 0$. □

4.2. Good functions.

We define a set $\mathcal{G}$ of "good" functions. Recall the notion of excursions from Definition 2.1.

Definition 4.3. If g is a function from $[0,\infty)$ to $\mathbb{R} \cup \{-\infty\}$, we say $g \in \mathcal{G}$ if:
(i) g is lower semi-continuous and càdlàg.
(ii) If $g(x) = -\infty$ then $g(x') = -\infty$ for all $x' > x$.

(iii) For any $\varepsilon > 0$, g has only finitely many excursions above its minimum with length greater than or equal to ε.
(iv) Let $x_{\max} = \sup\{ x : x > -\infty \} \leq \infty$. The Lebesgue measure of the set of points in $(0, x_{\max})$ which are not contained in some excursion above the minimum is 0.

If $g \in \mathcal{G}$, then $\mathcal{E}^{\downarrow}(g)$ (see Definition 2.2) is well-defined.

Lemma 4.4. *Suppose f_0 is defined from $\underline{m} \in \ell_2^{\downarrow}$ by Definition 2.5. Define f_t by (2.8). Then with probability 1, we have $f_t \in \mathcal{G}$ for all $t \geq 0$.*

Proof. Properties (i), (ii) in Definition 4.3 can be deduced for f_t directly from the definitions (2.6) and (2.8). Property (iv) for f_t follows from the fact that (iv) holds for f_0 (see Remark 2.6(i)) and the observation that every excursion of f_0 is contained in an excursion of f_t.

It remains to justify property (iii). The function f_0 is non-increasing, and Lemma 4.1(ii) tells us that the length of the interval on which f_0 takes values in $[y, y+1]$ tends to 0 as $y \to -\infty$. Hence for every ε there exists a K_ε such that $f_0(x) - f_0(x-\varepsilon) < -1$ for all $x \geq K_\varepsilon$. As a result, $f_t(x) - f_t(x-\varepsilon) < -1 + t\varepsilon$. If $\varepsilon < 1/t$, we find that $f_t(x) < f_t(x-\varepsilon)$, so all excursions intersecting (K_ε, ∞) must have length less than ε, as desired. □

Lemma 4.5. *Given some $f_0 \in \mathcal{G}$ define f_t by (2.8) and assume that $f_t \in \mathcal{G}$ for all $t \geq 0$. The function $\mathcal{E}^{\downarrow}(f_t)$ from $[0, \infty)$ to $\ell_\infty^{\downarrow}$ is càdlàg.*

Proof. Let us write $\underline{m}(t) = (m_1(t), m_2(t), \dots) = \mathcal{E}^{\downarrow}(f_t)$. Since we use the topology of coordinatewise convergence on $\ell_\infty^{\downarrow}$, it is enough to show that the function $t \mapsto m_i(t)$ is càdlàg for all i.

Consider $0 \leq t' < t$. Since f_t is obtained from $f_{t'}$ by adding on an increasing function, any minimum of f_t is also a minimum of $f_{t'}$, and any excursion of $f_{t'}$ is a sub-interval of an excursion of f_t.

Fix t and suppose $[l, r)$ is an excursion of f_t. Take ε with $0 < \varepsilon < 2l$. Recalling the notion of $\bar{f}$ from (2.1), we have $\bar{f}_t(l - \varepsilon/2) > f_t(l)$; hence if δ is sufficiently small, then

$$\bar{f}_{t+\delta}(l - \varepsilon/2) > f_t(l) + \delta l = f_{t+\delta}(l),$$

and so $f_{t+\delta}$ has a minimum in $[l - \varepsilon/2, l]$.

Also, there is some $x \in (r, r + \varepsilon/2)$ with $f_t(x) < f_t(l)$. Hence if δ is sufficiently small, then $f_{t+\delta}$ has a minimum in $[r, r + \varepsilon/2]$.

So for any ε, we can find δ such that the length of the excursion of $f_{t+\delta}$ which includes (l, r) is at most $r - l + \varepsilon$.

Now we will argue that for any $\varepsilon > 0$ there exists small enough δ such that the length $m_1(t+\delta)$ is at most $m_1(t) + \varepsilon$.

Fix any $T > t$ and consider $\delta \in (0, T - t)$. Since the excursions of $f_{t+\delta}$ are contained in the excursions of f_T, any excursion of $f_{t+\delta}$ of length more than $m_1(t) + \varepsilon$ must be contained in an excursion of f_T whose length also exceeds that. There are only finitely many such excursions of f_T since $f_T \in \mathcal{G}$. Let $\mathcal{U}$ be the union of those excursions, which has finite total length, say L.

Now let us look at all the excursions of f_t contained in $\mathcal{U}$. There are at most countably many. We can take a finite number of them whose total length is at least $L - \varepsilon$. Each of them has length no more than $m_1(t)$. From the property above, if we choose δ small enough, then at time $t + \delta$, none of them is contained in an excursion of length more than $m_1(t) + \varepsilon$. But also, since the remaining length of $\mathcal{U}$ outside this set is only ε, then also no other point in $\mathcal{U}$ is contained in an excursion of length more than $m_1(t) + \varepsilon$.

It follows that $m_1(t + \delta) \leq \varepsilon + m_1(t)$ as desired.

In similar fashion we can also obtain that $\sum_{i=1}^k m_i(t+\delta) \le \varepsilon + \sum_{i=1}^k m_i(t)$ for sufficiently small δ, for any k. But note that $\sum_{i=1}^k m_i(t)$ is non-decreasing in t. So for each k, $\sum_{i=1}^k m_i(t)$ is right-continuous with left limits, and hence the same is true for $m_i(t)$ for each i. □

Corollary 4.6. *Given some $f_0 \in \mathcal{G}$ define f_t by (2.8) and assume that $f_t \in \mathcal{G}$ for all $t \ge 0$. Then the set of times t such that f_t has a non-strict excursion (c.f. Definition 2.1) is countable.*

Proof. By Lemma 4.5 the function $t \mapsto \mathcal{E}^{\downarrow}(f_t)$ is càdlàg, therefore it has countably many jumps, c.f. [17, Section 12, Lemma 1]. Thus we only need to show that if f_{t_0} has a non-strict excursion for some $t_0 > 0$ then $t \mapsto \mathcal{E}^{\downarrow}(f_t)$ jumps at t_0. If $[l, r)$ is a non-strict excursion of f_{t_0}, then $f(l) = f(x)$ for some $x \in (l, r)$. Now if $t < t_0$ and $[l', r')$ is an excursion of f_t, then x is not in the interior of $[l', r')$. This implies that for any $t < t_0$ the function f_t has at least two disjoint excursions contained in $[l, r)$ that are separated by x, and these excursions merge at time t_0, thus if k_0 is the smallest index k for which $m_k(t_0) < r - l$ then the non-decreasing function $t \mapsto \sum_{i=1}^{k_0-1} m_i(t)$ jumps at time t_0. □

Definition 4.7. A family of good functions $f^{(i)} \in \mathcal{G}, i \in \mathcal{I}$ is said to be *uniformly good* if for any ε there exists $K_\varepsilon \in \mathbb{R}$ such that for any $i \in \mathcal{I}$ the excursions of $f^{(i)}$ intersecting $[K_\varepsilon, \infty)$ are all shorter than ε.

Lemma 4.8. *Let $f \in \mathcal{G}$ be continuous and assume that all of the excursions of f are strict (c.f. Definition 2.1). Let $f^{(n)} \in \mathcal{G}$, $n \in \mathbb{N}$ be a sequence of (not necessarily continuous) functions that converge to f uniformly on bounded intervals. Let us also assume that the family consisting of f and $f^{(n)}, n \in \mathbb{N}$ is uniformly good. Then $\mathcal{E}^{\downarrow}(f^{(n)}) \to \mathcal{E}^{\downarrow}(f)$ as $n \to \infty$ in the product topology on $\ell_\infty^{\downarrow}$.*

Proof. Suppose (l, r) is an excursion of f. For any given $\gamma > 0$ (with $\gamma < l$), there is a $\delta > 0$ such that the following properties hold:

(i) $f(x) \ge f(l) + \delta$ for all $x \in [0, l - \gamma]$;
(ii) $f(x) \ge f(l) + \delta$ for all $x \in [l + \gamma, r - \gamma]$;
(iii) $f(x) \le f(l) - \delta$ for some $x \in [r, r + \gamma]$.

Here (i) holds since f has a minimum at l and, being continuous, must achieve its bounds on $[0, l-\gamma]$; (ii) holds since the excursion is strict, and (iii) holds since by the definition of excursions, there must be points arbitrarily close to the right of r which take value lower than $f(l)$.

Now suppose n is large enough that $|f^{(n)}(x) - f(x)| < \delta/2$ for all $x \in [0, r + \gamma]$. Then we obtain the following properties:

(i) $f^{(n)}(x) \ge f(l) + \delta/2$ for all $x \in [0, l - \gamma]$;
(ii) $f^{(n)}(x) \ge f(l) + \delta/2$ for all $x \in [l + \gamma, r - \gamma]$;
(iii) $g^{(n)}(x) \le f(l) - \delta/2$ for some $x \in [r, r + \gamma]$;
(iv) $g^{(n)}(l) \in (f(l) - \delta/2, f(l) + \delta/2)$.

Then $f^{(n)}$ must have an excursion which starts somewhere in $[l - \gamma, l + \gamma]$ and ends somewhere in $[r - \gamma, r + \gamma]$.

Now let $\varepsilon > 0$ and choose K_ε such that the excursions of f and $f^{(n)}, n \in \mathbb{N}$ intersecting $[K_\varepsilon, \infty)$ are all shorter than ε.

Now by Definition 4.3(iv) there exists a finite collection of excursions (l_j, r_j) of f, whose union covers all of $[0, K_\varepsilon + \varepsilon]$ except for a set of total length less than $\varepsilon/2$. Let k be the total number of these excursions. Apply the above argument to all of these excursions with $\gamma = \varepsilon/4k$. Then if n is sufficiently large, we have that for each of these excursions of f, there is a corresponding excursion of $f^{(n)}$ whose length is within $\varepsilon/2k$; the remaining length in $[0, K_\varepsilon + \varepsilon]$ amounts to no more than ε; and we know that outside $[0, K_\varepsilon + \varepsilon]$, all excursions (either of $f^{(n)}$ or f) have length less than or equal to ε.

It follows that for any $i > 0$, the ith largest excursion of $f^{(n)}$ and the ith largest excursion of f differ by at most ε. Hence indeed $\mathcal{E}^\downarrow(f^{(n)})$ converges componentwise to $\mathcal{E}^\downarrow(f)$, as desired.

□

5. Extension of rigid representation to $\ell^\downarrow_2$

In this section we will extend the rigid representation results of Section 3 to any initial condition $\underline{m} \in \ell^\downarrow_2$. As we have discussed in Remark 3.22, extension from $\ell^\downarrow_0$ to $\ell^\downarrow_1$ is automatic, so in this section we will assume that $\underline{m} \in \ell^\downarrow_2 \setminus \ell^\downarrow_1$.

In Section 5.1 we prove that the MC admits a tilt representation, i.e., Theorem 2.8.

In Section 5.2 we prove that the MCLD admits a tilt-and-shift representation, i.e., Theorem 2.13.

5.1. Extension of MC tilt representation to $\ell^\downarrow_2$.

The aim of this section is to prove Theorem 2.8.

f_0 is defined from $\underline{m} \in \ell^\downarrow_2$ by Definition 2.5. f_t is defined by (2.8).

Definition 5.1. By Lemma 4.4 and Corollary 4.6 the (random) set of times t such that f_t has a non-strict excursion (c.f. Definition 2.1) is countable. Hence for all but countably many t, the probability that all excursions of f_t are strict is equal to 1. Let $\mathcal{T}$ denote this (deterministic) set of "good" times t.

From Lemma 4.4 and Lemma 4.5 it follows that $t \mapsto \mathcal{E}^\downarrow(f_t)$ is a càdlàg process with respect to the product topology on $\ell^\downarrow_\infty$. The graphical representation of the multiplicative coalescent $\mathbf{m}_t$ constructed in [2, Section 1.5] is also a càdlàg process with respect to the topology of the $\mathrm{d}(\cdot,\cdot)$-metric defined in (1.2) (see [30, Lemma 2.8]), thus it is càdlàg with respect to the weaker product topology on $\ell^\downarrow_\infty$.

Hence, since $\mathcal{T}$ is dense, if we can show that for any finite collection $t_1, \ldots, t_r \in \mathcal{T}$, we have

$$(\mathcal{E}^\downarrow(f_{t_i}), 1 \le i \le r) \stackrel{d}{=} (\mathbf{m}_{t_i}, 1 \le i \le r), \tag{5.1}$$

then indeed the law of $\mathcal{E}^\downarrow(f_t)$ is that of the MC.

For each n, let $\underline{m}^{(n)}$ be given by

$$m^{(n)}_i = \begin{cases} m_i, & i \le n \\ 0, & i > n \end{cases}. \tag{5.2}$$

For each n, $\underline{m}^{(n)} \in \ell^\downarrow_0$, and $\underline{m}^{(n)} \to \underline{m}$ in $\ell^\downarrow_2$ as $n \to \infty$.

We couple the processes starting from $\underline{m}^{(n)}$, $n \ge 1$, by using the same height variables $Y_i = -E_i$ throughout. If we define

$$f^{(n)}_t(x) = f^{(n)}_0(x) + tx,$$

then by the $\lambda = 0$ case of Proposition 2.10 we have

$$(\mathcal{E}^\downarrow(f^{(n)}_{t_i}), 1 \le i \le r) \stackrel{d}{=} (\mathbf{m}^{(n)}_{t_i}, 1 \le i \le r), \tag{5.3}$$

where $\mathbf{m}^{(n)}_t, t \ge 0$ is the MC started from $\underline{m}^{(n)}$.

By the Feller property of the MC (see [2, Proposition 5]) we have

$$(\mathbf{m}^{(n)}_{t_i}, 1 \le i \le r) \stackrel{d}{\to} (\mathbf{m}_{t_i}, 1 \le i \le r), \qquad n \to \infty \tag{5.4}$$

(with respect to the topology of $\ell^\downarrow_2$ and hence also coordinatewise).

We will let $n \to \infty$, and show that

$$\begin{gathered} \mathcal{E}^{\downarrow}(f_t^{(n)}) \to \mathcal{E}^{\downarrow}(f_t) \quad \text{for all} \quad t \in \mathcal{T} \\ \text{coordinate-wise with probability 1.} \end{gathered} \tag{5.5}$$

Putting together (5.3),(5.5) and (5.4) we obtain (5.1) as required.

It remains to show (5.5). We will achieve this by checking that the conditions of Lemma 4.8 almost surely hold if $t \in \mathcal{T}$. We may assume that $\underline{m} \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$, as discussed in the first paragraph of Section 5.

Lemma 5.2. *Fix $t > 0$. With probability 1, the family of functions that consists of $f_t^{(n)}$, $n \geq 1$ and f_t is uniformly good (c.f. Definition 4.7).*

Proof. Recalling Definitions 2.3 and 2.5 we see that

$$f_0^{(n)} = f_{\mu_0^{(n)}}, \qquad \text{where} \qquad \mu_0^{(n)} = \sum_{i=1}^{n} m_i \cdot \delta_{Y_i}.$$

Now $\mu_0 - \mu_0^{(n)} = \sum_{n<i} m_i \cdot \delta_{Y_i}$ is a non-negative measure for each $n \in \mathbb{N}$, thus μ_0 dominates $\mu_0^{(n)}$ for each n and we obtain the proof of Lemma 5.2 by repeating the argument of proof Lemma 4.4, uniformly in n. □

Lemma 5.3. *If $\underline{m} \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$ then for any $t \geq 0$ the function $f_t(\cdot)$ is continuous and $f_t^{(n)} \to f_t$ uniformly on bounded intervals.*

Proof. Since $f_t^{(n)}(x) = f_0^{(n)}(x) + tx$, and $f_t(x) = f_0(x) + tx$, it is enough to show the statements of the lemma for $t = 0$.

The function f_0 is non-increasing, moreover by Lemma 4.2 the values $E_i, i \geq 0$ are dense in $[0, \infty)$, thus by Remark 2.6(i) the values f_0 takes are dense in $(-\infty, 0)$. Hence f_0 is continuous.

Since f_0 is a continuous function on $[0, \infty)$, it is uniformly continuous on any bounded sub-interval.

Fix any $U < \infty$. Let us define $n_0 = \min\{ n : \sum_{i=1}^{n} m_i > U \}$. For all $n \geq n_0$ we have $f_0^{(n)}(U) \geq f_0^{(n_0)}(U) =: -S$.

Consider $x \leq U$, $n \geq n_0$. By (2.7) have $f_0^{(n)}(x) = Y_k$, where k is such that

$$x \in \left[\sum_{j \leq n \,:\, E_j < E_k} m_j \,,\; m_k + \sum_{j \leq n \,:\, E_j < E_k} m_j \right).$$

By considering the interval on which the function f_0 takes the same value Y_k, we have

$$f_0^{(n)}(x) = f_0(x + \delta) \qquad \text{where} \qquad \delta = \sum_{j > n \,:\, E_j < E_k} m_j.$$

For all $n \geq n_0$ we have $\delta \leq \sum_{j>n \,:\, E_j < S} m_j$. This goes to 0 as $n \to \infty$ by Lemma 4.1 and dominated convergence.

Then by the uniform continuity of f_0 on bounded intervals, we have that $f_0^{(n)} \to f_0$ uniformly on $[0, U]$, as desired. □

Finally, we can insert the properties derived in Definition 5.1 and Lemmas 5.2, 5.3 into Lemma 4.8 to obtain (5.5). This completes the proof of Theorem 2.8.

5.2. **Extension of MCLD tilt-and-shift representation to $\ell_2^\downarrow$.** The aim of this section is to prove Theorem 2.13.

In Section 5.2.1 we give quantitative bounds on the effect of the insertion of a new particle in a finite particle system on the death times of other particles.

In Section 5.2.2 we construct the death times of each particle in the infinite particle system associated to the initial state $\underline{m} \in \ell_2^\downarrow$ by inserting particles one by one and showing that death times converge.

In Section 5.2.3 we collect some technical lemmas that allow us to deduce the convergence of the ordered sequence of excursion lengths from the approximation results of Section 5.2.2.

In Section 5.2.4 we extend the MCLD tilt-and-shift representation from $\ell_1^\downarrow$ to $\ell_2^\downarrow$, i.e., we prove Theorem 2.13.

5.2.1. *Perturbation of the particle system.* Recall the particle system introduced in Section 3.3. The main result of Section 5.2.1 is Lemma 5.4, which quantifies the effect of the insertion of a new particle on the death times of other particles. This perturbation result will play an important role when we extend our rigid representation of MCLD(λ) from $\ell_0^\downarrow$ to $\ell_2^\downarrow$ in Section 5.2.2 using truncation and approximation. Note that Lemma 5.4 is a *deterministic* result, i.e., it holds for any initial configuration of particles.

Lemma 5.4. *For any $\underline{m} = (m_1, \ldots, m_n) \in \ell_0^\downarrow$ and $Y_1(0), \ldots, Y_n(0)$, let us define $Y_1(t), \ldots, Y_n(t)$, μ_t and $t_1, \ldots, t_n$ as in Definition 3.11.*

Let us initialize a new particle system $\widetilde{Y}_0(t), \ldots, \widetilde{Y}_n(t)$ by letting $\widetilde{Y}_i(0) = Y_i(0)$ for any $i = 1, \ldots, n$ and by adding a new particle with initial height $\widetilde{Y}_0(0)$ and mass $\widetilde{m}$.

Let us then define $\widetilde{Y}_0(t), \ldots, \widetilde{Y}_n(t)$, $\widetilde{\mu}_t$ and $\widetilde{t}_0, \ldots, \widetilde{t}_n$ analogously to Definition 3.11. Then we have

$$|\widetilde{t}_i - t_i| \le \mathbb{1}[\widetilde{Y}_0(0) > Y_i(0)] \frac{\widetilde{m}|Y_i(0)|}{\lambda^2} \exp\left(\frac{\mu_0(Y_i(0), 0)}{\lambda}\right), \quad i = 1, \ldots, n. \tag{5.6}$$

The rest of Section 5.2.1 is devoted to the proof of Lemma 5.4.

Without loss of generality we may assume $Y_1(0)$, $Y_2(0)$, $\ldots Y_n(0)$ are all distinct, because if $Y_i(0) = Y_j(0)$ for some $i \neq j$ then we can replace our particle system by another one with fewer particles in which these two particles are merged.

With a slight abuse of notation, for the rest of Section 5.2.1 we will relabel our particles so that we have $Y_1(0) > Y_2(0) > \cdots > Y_n(0)$ and denote by m_i the weight and t_i the death time of the particle with initial location $Y_i(0)$. Thus we have

$$t_1 \le t_2 \le \cdots \le t_n.$$

The next lemma gives a recursive formula for $t_1, \ldots, t_n$.

Lemma 5.5. *Let us formally define $t_0 = 0$. We have*

$$t_i = t_{i-1} \vee \frac{|Y_i(0)| - \sum_{j=1}^{i-1} m_j t_j}{\lambda}, \qquad 1 \le i \le n. \tag{5.7}$$

Proof. First observe that we have

$$|Y_i(0)| \overset{(3.12)}{=} Y_i(t_i) - Y_i(0) = \int_0^{t_i} \dot{Y}_i(t)\,\mathrm{d}t \overset{(3.11)}{=} \int_0^{t_i} (\lambda + \mu_t(Y_i(t), 0))\,\mathrm{d}t \overset{(3.10)}{=}$$
$$\int_0^{t_i} \left(\lambda + \sum_{j=1}^{i-1} m_j \mathbb{1}[\,Y_i(t) < Y_j(t) < 0\,]\right) \mathrm{d}t. \tag{5.8}$$

We will prove (5.7) by considering two cases separately.

First case: If $t_i > t_{i-1}$ then

$$|Y_i(0)| \stackrel{(*)}{=} \int_0^{t_i} \left(\lambda + \sum_{j=1}^{i-1} m_j \mathbb{1}[t < t_j] \right) \mathrm{d}t = \lambda t_i + \sum_{j=1}^{i-1} m_j t_j, \tag{5.9}$$

where in $(*)$ we used (5.8) and the fact that $t_i > t_{i-1}$ implies $Y_j(t) > Y_i(t)$ for any $t < t_i$ and $j \leq i-1$. Rearranging (5.9) we obtain that if $t_i > t_{i-1}$ then we have $t_i = t_i^*$, where

$$t_i^* := \frac{|Y_i(0)| - \sum_{j=1}^{i-1} m_j t_j}{\lambda},$$

therefore (5.7) holds.

Second case: Now we assume $t_i = t_{i-1}$. First note that

$$|Y_i(0)| \stackrel{(5.8)}{\leq} \int_0^{t_i} \left(\lambda + \sum_{j=1}^{i-1} m_j \mathbb{1}[t < t_i] \right) \mathrm{d}t = \lambda t_i + \sum_{j=1}^{i-1} m_j t_j. \tag{5.10}$$

Rearranging (5.10) we obtain $t_i^* \leq t_i$. Now if $t_i = t_{i-1}$, then $t_i^* \leq t_{i-1}$, and therefore (5.7) holds. □

Recall the notion introduced in the statement of Lemma 5.4. Denote by

$$i^* = \inf\{\, i \ : \ Y_i(0) < \widetilde{Y}_0(0) \,\}. \tag{5.11}$$

In particular, we define $i^* = \infty$ if $Y_i(0) \geq \widetilde{Y}_0(0)$ for all $i \in [1, n]$.

By (3.11) the speed of $Y_i(t)$ only depends on the locations of particles strictly above it, so we have $\widetilde{Y}_i(t) \equiv Y_i(t)$ for any $1 \leq i < i^*$, thus we have $t_i = \widetilde{t}_i$ for any $1 \leq i < i^*$, and (5.6) trivially follows for these particles.

For any $i^* \leq i \leq n$ we will think about $\widetilde{t}_i = \widetilde{t}_i(\widetilde{m})$ as a function of the variable $\widetilde{m} \geq 0$, which represents the weight of the inserted particle. With this definition we have $\widetilde{t}_i(0) = t_i$. Let us define the Lipschitz constant L_i by

$$L_i = \sup_{0 \leq \widetilde{m} < \widetilde{m}'} \frac{|\widetilde{t}_i(\widetilde{m}') - \widetilde{t}_i(\widetilde{m})|}{\widetilde{m}' - \widetilde{m}}.$$

In order to prove (5.6), we only need to show

$$L_i \leq \frac{|Y_i(0)|}{\lambda^2} \exp\left(\frac{\mu_0(Y_i(0), 0)}{\lambda} \right), \quad i = i^*, \ldots, n. \tag{5.12}$$

We begin with the following claim. The proof is trivial and we omit it.

Claim 5.6. *If F and G are both Lipschitz-continuous functions of the variable $\widetilde{m}$ with Lipschitz-constants L_F and and L_G, and $H(\widetilde{m}) := F(\widetilde{m}) \vee G(\widetilde{m})$, then the Lipschitz-constant L_H satisfies $L_H \leq L_G \vee L_H$.*

Denote by $\widetilde{t}_0(\widetilde{m})$ the death time of particle $\widetilde{Y}_0(t)$ in the particle system $\widetilde{Y}_0(t), \widetilde{Y}_1(t), \ldots, \widetilde{Y}_n(t)$. Note that we have $\widetilde{t}_0(\widetilde{m}) = \widetilde{t}_0(0)$, i.e., the death time of the inserted particle does not depend on its weight, so we will omit dependence of $\widetilde{t}_0$ on $\widetilde{m}$.

Lemma 5.7. *For any $i^* \leq i \leq n$ we have*

$$L_i \leq \frac{1}{\lambda} \left(\widetilde{t}_0 + \sum_{j=i^*}^{i-1} m_j \cdot L_j \right). \tag{5.13}$$

Proof. For $i^* \le i \le n$, the recursion (5.7) for the new particle system reads

$$\widetilde{t}_i(\widetilde{m}) = \widetilde{t}_{i-1}(\widetilde{m}) \vee \frac{|Y_i(0)| - \sum_{j=1}^{i^*-1} m_j t_j - \widetilde{m}\widetilde{t}_0 - \sum_{j=i^*}^{i-1} m_j \widetilde{t}_j(\widetilde{m})}{\lambda}. \tag{5.14}$$

We will prove (5.13) by induction on i. We begin with $i = i^*$. Since $\widetilde{t}_{i^*-1}(\widetilde{m}) = t_{i^*-1}$ and thus $L_{i^*-1} = 0$, we obtain from Claim 5.6 and (5.14) that $L_{i^*} \le \widetilde{t}_0/\lambda$, thus (5.13) holds if $i = i^*$.

As for the induction step, we obtain from Claim 5.6 and (5.14) that

$$L_i \le L_{i-1} \vee \frac{1}{\lambda}\left(\widetilde{t}_0 + \sum_{j=i^*}^{i-1} m_j \cdot L_j\right). \tag{5.15}$$

Now (5.13) holds for i by (5.15) and the induction hypothesis (i.e., the assumption that (5.13) holds for $i-1$).

□

From the recursive inequalities (5.13) one readily deduces by induction on i the following explicit bound:

$$L_i \le \frac{\widetilde{t}_0}{\lambda} \cdot \prod_{j=i^*}^{i-1}\left(1 + \frac{m_j}{\lambda}\right), \qquad i^* \le i \le n. \tag{5.16}$$

Now we are ready to prove (5.12) for any $i^* \le i \le n$:

$$L_i \overset{(5.16)}{\le} \frac{t_i}{\lambda} \cdot \prod_{j=1}^{i-1}\left(1 + \frac{m_j}{\lambda}\right) \le \frac{t_i}{\lambda} \cdot \exp\left(\sum_{j=1}^{i-1} \frac{m_j}{\lambda}\right) \overset{(3.11),(3.12)}{\le}$$
$$\frac{|Y_i(0)|}{\lambda^2} \exp\left(\sum_{j=1}^{i-1} \frac{m_j}{\lambda}\right) = \frac{|Y_i(0)|}{\lambda^2} \exp\left(\frac{\mu_0(Y_i(0), 0)}{\lambda}\right).$$

The proof of (5.12) and Lemma 5.4 is complete.

5.2.2. *Truncation and approximation.* The main result of Section 5.2.2 is Lemma 5.8 which extends the "shift" operator from $\ell_0^\downarrow$ to $\ell_2^\downarrow$ using truncations and approximation, c.f. Remarks 2.12 and 2.14(iii).

Given some $\underline{m} \in \ell_2^\downarrow$, let us generate $E_1, E_2, \dots$ as in (2.4). Define the truncation $\underline{m}^{(n)}$ by (5.2). We define $g_0^{(n)}(\cdot) \equiv f_0^{(n)}(\cdot)$ using $E_1, E_2, \dots, E_n$ by (2.6). Note that we still denote by $f_t(\cdot)$ the function constructed from the un-truncated $\underline{m}$ by (2.6) and (2.8), and also that we use the same sequence of random variables $E_1, E_2, \dots$ to obtain a coupling of $g_0^{(n)}(\cdot), n \in \mathbb{N}$. We define $g_t^{(n)}(\cdot)$ from $g_0^{(n)}(\cdot)$ using Definition 2.9. This gives rise to the measure $\nu^{(n)}$ by (2.10) and the function $\Phi^{(n)}$ by (2.11). By (2.12) we have

$$g_t^{(n)}(x) = g_0^{(n)}(x + \Phi^{(n)}(t)) + \lambda t + \int_0^t \left(x + \Phi^{(n)}(t) - \Phi^{(n)}(s)\right) \mathrm{d}s. \tag{5.17}$$

Our next result states that $\mathbb{P}$-almost surely $\nu^{(n)}$ vaguely converges to some ν as $n \to \infty$.

Lemma 5.8. *If $\nu^{(n)}$, $n \in \mathbb{N}$ is defined as above then $\mathbb{P}$-almost surely there exists a locally finite measure ν on $[0, \infty)$ such that for any compactly supported continuous function $h : [0, \infty) \to \mathbb{R}$ we have*

$$\lim_{n\to\infty} \int_0^\infty h(t)\,\mathrm{d}\nu^{(n)}(t) = \int_0^\infty h(t)\,\mathrm{d}\nu(t), \qquad \mathbb{P} - a.s. \tag{5.18}$$

Corollary 5.9. *By the portemanteau theorem* (5.18) *implies*

$$\lim_{n\to\infty} \Phi^{(n)}(t) = \Phi(t) \quad \text{if} \quad \nu(\{t\}) = 0, \quad \text{where} \quad \Phi(t) := \nu([0,t]) \tag{5.19}$$

Proof of Lemma 5.8. We will use the particle representation (see Section 3.3)

$$Y_1^{(n)}(t), \dots, Y_n^{(n)}(t)$$

of $g_t^{(n)}(\cdot)$. We can then write $\nu^{(n)} = \sum_{i=1}^n m_i \cdot \delta_{t_i^{(n)}}$, see (3.34). Note that

$$Y_i^{(n)}(0) = Y_i(0) \qquad \text{for any} \qquad 1 \le i \le n.$$

Let us assume that the function $h : [0,\infty) \to \mathbb{R}$ for which we want to show (5.18) is supported on $[0,T]$.

Recalling the definition $\mu_0 = \sum_{i=1}^\infty m_i \cdot \delta_{Y_i(0)}$, we analogously define $\mu_0^{(n)} = \sum_{i=1}^n m_i \cdot \delta_{Y_i(0)}$. Then by Lemma 4.1 there exists a $\mathbb{P}$-almost surely finite random variable K_0 such that

$$\sup_{n\ge 0} \frac{\mu_0^{(n)}[-K,0]}{K} = \frac{\mu_0[-K,0]}{K} \le \frac{1}{2T}, \qquad \text{for any} \quad K \ge K_0. \tag{5.20}$$

If $|Y_i(0)| = E_i > K_0$ then for any $t \ge 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_i^{(n)}(t) \overset{(3.11)}{\le} \lambda + \mu_t^{(n)}(Y_i^{(n)}(t), 0) \le \lambda + \mu_0^{(n)}(Y_i(0), 0) \overset{(5.20)}{\le} \lambda + \frac{|Y_i(0)|}{2T}. \tag{5.21}$$

This implies that if $Y_i(0) < Y := -(K_0 \vee 2\lambda T)$, then

$$Y_i^{(n)}(T) \overset{(5.21)}{\le} Y_i(0) + \left(\lambda + \frac{|Y_i(0)|}{2T}\right) \cdot T < 0,$$

which implies that the time of death $t_i^{(n)}$ of particle i (see (3.34)) satisfies

$$h(t_i^{(n)}) = 0 \quad \text{if} \quad Y_i(0) < Y. \tag{5.22}$$

Our aim is to show that the sequence $\int_0^\infty h(t)\,\mathrm{d}\nu^{(n)}(t), n \in \mathbb{N}$ is Cauchy. In order to show this we let $n \le m$ and bound

$$\left| \int_0^\infty h(t)\,\mathrm{d}\nu^{(m)}(t) - \int_0^\infty h(t)\,\mathrm{d}\nu^{(n)}(t) \right| \overset{(3.34),(5.22)}{\le}$$
$$\sum_{i=1}^n m_i \cdot \left| h(t_i^{(m)}) - h(t_i^{(n)}) \right| \cdot \mathbb{1}[Y_i(0) \ge Y] + \|h\|_\infty \cdot \sum_{i=n+1}^m m_i \cdot \mathbb{1}[Y_i(0) \ge Y]. \tag{5.23}$$

In order to bound the first term on the right-hand side of (5.23) we observe that if $1 \le i \le n$ and $Y_i(0) \ge Y$ then

$$\left| t_i^{(m)} - t_i^{(n)} \right| \le \sum_{k=n}^{m-1} \left| t_i^{(k+1)} - t_i^{(k)} \right| \overset{(5.6)}{\le}$$
$$\sum_{k=n}^{m-1} \mathbb{1}[Y_{k+1}(0) \ge Y] \frac{m_{k+1}|Y_i(0)|}{\lambda^2} \exp\left(\frac{\mu_0^{(k)}(Y_i(0),0)}{\lambda} \right) \le$$
$$\frac{|Y|}{\lambda^2} \exp\left(\frac{\mu_0(Y,0)}{\lambda} \right) \sum_{k=n}^\infty m_{k+1} \cdot \mathbb{1}[Y_{k+1}(0) \ge Y]. \tag{5.24}$$

Note that Lemma 4.1 implies that with probability 1 we have

$$\sum_{i=n}^\infty m_i \cdot \mathbb{1}[Y_i(0) \ge Y] \to 0 \quad \text{as} \quad n \to \infty.$$

If we combine this with (5.23), (5.24) and the fact that $h(\cdot)$ is uniformly continuous, we can conclude that $\int_0^\infty h(t)\,\mathrm{d}\nu^{(n)}(t)$, $n \in \mathbb{N}$ is a Cauchy sequence for any $h \in \mathcal{C}_0(\mathbb{R})$, from which it follows that the exists ν for which (5.18) holds. □

Lemma 5.10. *If ν is the random measure obtained in Lemma 5.8, then for every $t \geq 0$ there exists $y \in (-\infty, 0]$ such that*

$$\nu[0,t] = \mu_0[y,0]. \tag{5.25}$$

Proof. Note that $\nu^{(n)} = \sum_{i=1}^n m_i \cdot \delta_{t_i^{(n)}}$ for every $n \in \mathbb{N}$, where $Y_i \geq Y_j$ implies $t_i^{(n)} \leq t_j^{(n)}$ for every $1 \leq i, j \leq n$. Since $\nu^{(n)}$ is an atomic measure with masses $(m_i)_{i=1}^n$ located at $\left(t_i^{(n)}\right)_{i=1}^n$ and $\nu^{(n)} \to \nu$ vaguely, we can conclude that ν is also an atomic measure with masses $(m_i)_{i=1}^\infty$ located at $(t_i)_{i=1}^\infty$ where $\lim_{n\to\infty} t_i^{(n)} = t_i$, thus $Y_i \geq Y_j$ implies $t_i \leq t_j$ for every $i, j \in \mathbb{N}$. From this (5.25) readily follows. □

5.2.3. *Tilt-and-shift continuity lemmas.* We will prove Theorem 2.13(i) in Section 5.2.4 by replicating the argument given in Section 5.1. In Section 5.2.3 we collect some ingredients of this proof.

Given some $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$ we defined $g_0(\cdot) = f_0(\cdot)$ by (2.6), ν by Lemma 5.8, $\Phi(\cdot)$ by (5.19) and $g_t(\cdot)$ by (2.12). The next lemma is the tilt-and-shift version of Lemma 4.5.

Lemma 5.11. *If $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$ then the function $t \mapsto \mathcal{E}^\downarrow(g_t)$ is càdlàg with respect to the product topology on $\ell_\infty^\downarrow$.*

Proof. Let us fix $t \geq 0$ and define the auxiliary functions

$$g^*_{t+\Delta t}(x) = g_t(x + \Phi(t+\Delta t) - \Phi(t)), \qquad g^{**}_{t+\Delta t}(x) = g_t(x) + \Delta t \cdot x.$$

From (2.12) we obtain

$$g_{t+\Delta t}(x) = g^*_{t+\Delta t}(x) + \lambda \Delta t + \int_t^{t+\Delta t} (x + \Phi(t+\Delta t) - \Phi(s))\,\mathrm{d}s, \tag{5.26}$$

$$g_{t+\Delta t}(x) = g^{**}_{t+\Delta t}(x + \Phi(t+\Delta t) - \Phi(t)) + \lambda \Delta t - \int_t^{t+\Delta t} \Phi(s)\,\mathrm{d}s. \tag{5.27}$$

If $\underline{m}, \underline{m}' \in \ell_\infty^\downarrow$, we say that $\underline{m} \preceq \underline{m}'$ if $\sum_{j=1}^i m_j \leq \sum_{j=1}^i m'_j$ for any $i \in \mathbb{N}$.

We are going to show

$$\mathcal{E}^\downarrow(g^*_{t+\Delta t}) \preceq \mathcal{E}^\downarrow(g_{t+\Delta t}), \tag{5.28}$$

$$\mathcal{E}^\downarrow(g_{t+\Delta t}) \preceq \mathcal{E}^\downarrow(g^{**}_{t+\Delta t}), \tag{5.29}$$

$$\mathcal{E}^\downarrow(g^*_{t+\Delta t}) \to \mathcal{E}^\downarrow(g_t) \quad \text{in} \quad \ell_\infty^\downarrow \quad \text{as} \quad \Delta t \searrow 0, \tag{5.30}$$

$$\mathcal{E}^\downarrow(g^{**}_{t+\Delta t}) \to \mathcal{E}^\downarrow(g_t) \quad \text{in} \quad \ell_\infty^\downarrow \quad \text{as} \quad \Delta t \searrow 0. \tag{5.31}$$

As soon as we show (5.28)–(5.31), we immediately obtain

$$\mathcal{E}^\downarrow(g_{t+\Delta t}) \to \mathcal{E}^\downarrow(g_t) \quad \text{in} \quad \ell_\infty^\downarrow \quad \text{as} \quad \Delta t \searrow 0,$$

i.e., the right-continuity of $t \mapsto \mathcal{E}^\downarrow(g_t)$ with respect to the $\ell_\infty^\downarrow$ topology. The proof of the existence of left limits is similar and we omit it.

(5.28) follows from the fact that $g_{t+\Delta t}$ is obtained from $g^*_{t+\Delta t}$ by adding an increasing function (see (5.26)), thus the collection of excursions of $g_{t+\Delta t}$ are obtained by merging some excursions of $g^*_{t+\Delta t}$.

(5.29) follows from the fact that $g_{t+\Delta t}$ is obtained from $g^{**}_{t+\Delta t}$ by a shift to the left plus an addition of a constant (see (5.27)), thus the excursions of $g_{t+\Delta t}$ are obtained by deleting/splitting some excursions of $g^*_{t+\Delta t}$.

If we apply Lemma 5.10 with μ_t in place of μ_0 then it follows that for every $\Delta t \geq 0$ there exists some $y \in (-\infty, 0]$ such that $\Phi(t+\Delta t) - \Phi(t) = \mu_t[y, 0]$ (see (3.10)), thus the collection of excursions of $g^*_{t+\Delta t}$ is obtained by removing some excursions of g_t whose total length is $\Phi(t+\Delta t) - \Phi(t)$. From this (5.30) follows, since $\Phi(t+\Delta t) - \Phi(t) = \nu(t, t+\Delta t] \to 0$ as $\Delta t \searrow 0$.

From (2.12) and Lemma 4.4 it follows that $g^{**}_{t+\Delta t}(x) \in \mathcal{G}$ for any $\Delta t \geq 0$, thus Lemma 4.5 implies (5.31). The proof of Lemma 5.11 is complete. □

Corollary 5.12. *If $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$ then the set of times t such that g_t has a non-strict excursion (c.f. Definition 2.1) is countable.*

Proof. By Lemma 5.11 the function $t \mapsto \mathcal{E}^\downarrow(g_t)$ is càdlàg, therefore it has countably many jumps, c.f. [17, Section 12, Lemma 1]. Also, the measure ν has countably many atoms.

Thus we only need to show that if g_{t_0} has a non-strict excursion for some $t_0 > 0$ and if $\nu(\{t_0\}) = 0$ then $t \mapsto \mathcal{E}^\downarrow(g_t)$ jumps at t_0.

The rest of the proof is identical to that of Corollary 4.6, with the additional observation that the jump at time t_0 created by the merger of excursions coming from the "tilt" operation cannot be cancelled by the deletion of excursions coming from the "shift" operation, since $t \mapsto \Phi(t)$ is continuous from the left at time t_0.

□

Claim 5.13. *Let $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$ and define $g_t(\cdot)$ using (2.12) and $g_t^{(n)}(\cdot)$ using (5.17). Then $g_t(\cdot)$ is continuous and if $t \in [0,\infty)$ satisfies $\nu(\{t\}) = 0$ then*

$$g_t^{(n)}(\cdot) \to g_t(\cdot) \quad \textit{uniformly on compacts.} \tag{5.32}$$

Proof. (5.32) follows from Corollary 5.9, Lemma 5.3 and (2.12), (5.17). □

Definition 5.14. By Corollary 5.12 the (random) set of times t such that either g_t has a non-strict excursion or $\nu(\{t\}) > 0$ is countable. Hence for all but countably many t, the probability that $\nu(\{t\}) = 0$ and all excursions of g_t are strict is equal to 1. Let $\mathcal{T}^*$ denote this (deterministic) set of "good" times t.

Lemma 5.15. *If $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$, $t \in \mathcal{T}^*$ then almost surely $\mathcal{E}^\downarrow(g_t^{(n)}) \to \mathcal{E}^\downarrow(g_t)$ in the product topology on $\ell_\infty^\downarrow$.*

Proof. First note that the functions $g_t(\cdot)$ and $g_t^{(n)}(\cdot), n \in \mathbb{N}$ are uniformly good (c.f. Definition 4.7): this follows from Lemma 5.2 and the fact that g_t is a "shifted" version of f_t:

$$g_t(x) \overset{(2.8),(2.12)}{=} f_t(x + \Phi(t)) + \lambda t - \int_0^t \Phi(s)\, \mathrm{d}s, \tag{5.33}$$

and similarly, $g_t^{(n)}$ is a shifted version of $f_t^{(n)}$. Now Lemma 5.15 follows from Definition 5.14, Claim 5.13 and Lemma 4.8. □

5.2.4. *Proof of Theorem 2.13.*

Proof of Theorem 2.13(i). Given $\underline{m} \in \ell_2^\downarrow \setminus \ell_1^\downarrow$, we constructed two stochastic processes:

- the graphical construction in [30, Section 3] of the MCLD(λ) process $\mathbf{m}_t, t \geq 0$,
- the process $\mathcal{E}^\downarrow(g_t), t \geq 0$, defined by (2.12), where the initial state $g_0(\cdot) := f_0(\cdot)$ is defined by (2.6) and the control function $\Phi(\cdot)$ is defined by (5.19).

We now want to show that these two $\ell_\infty^\downarrow$-valued processes have the same law. Both processes are càdlàg with respect to the product topology on $\ell_\infty^\downarrow$ by [30, Proposition 1.1] and Lemma 5.11.

Hence, since the set $\mathcal{T}^*$ introduced in Definition 5.14 is dense, if we can show that for any finite collection $t_1, \dots, t_r \in \mathcal{T}^*$, we have

$$(\mathcal{E}^{\downarrow}(g_{t_i}), 1 \le i \le r) \stackrel{d}{=} (\mathbf{m}_{t_i}, 1 \le i \le r), \tag{5.34}$$

then indeed Theorem 2.13(i) will follow.

By Proposition 2.10 we have

$$(\mathcal{E}^{\downarrow}(g^{(n)}_{t_i}, 1 \le i \le r) \stackrel{d}{=} (\mathbf{m}^{(n)}_{t_i}, 1 \le i \le r), \tag{5.35}$$

where $\mathbf{m}^{(n)}_t, t \ge 0$ is the MCLD(λ) process started from $\underline{m}^{(n)}$. By the Feller property of MCLD(λ) (see [30, Theorem 1.2]) we have

$$(\mathbf{m}^{(n)}_{t_i}, 1 \le i \le r) \stackrel{d}{\to} (\mathbf{m}_{t_i}, 1 \le i \le r), \qquad n \to \infty \tag{5.36}$$

(with respect to the topology of $\ell_2^{\downarrow}$ and hence also coordinatewise). Putting together (5.35), Lemma 5.15 and (5.36) we obtain (5.34). The proof of Theorem 2.13(i) is complete. □

Proof of Theorem 2.13(ii). It is enough to show that for any $K > 0$ and $\varepsilon > 0$ we almost surely have $-\varepsilon \le g_t(0) \le 0$ for any $0 \le t \le K$. Let us fix $K, \varepsilon > 0$. Recall from (5.20)-(5.22) that there exists $Y < 0$ such that if $Y_i(0) < Y$ then $t_i^{(n)} > K$ for any $i \le n$. By Lemma 4.2 there exists an almost surely finite n_0 such that $\mu_0^{(n_0)}[y-\varepsilon, y] > 0$ for any $Y \le y \le 0$, thus for any $n \ge n_0$ and any $x \ge 0$ such that $Y \le g_0^{(n)}(x) \le 0$ we have $g_0^{(n)}(x_-) - g_0^{(n)}(x) \le \varepsilon$. In words: the gaps between consecutive particles initially located in $[Y, 0]$ are smaller than or equal to ε. By Definition 3.11, these gaps can only decrease with time, thus for any $n \ge n_0$ and $t \le K$ there is a particle in $[-\varepsilon, 0]$, i.e., we have $-\varepsilon \le g_t^{(n)}(0) \le 0$. Now $g_t^{(n)}(0) \to g_t(0)$ as $n \to \infty$ for all except countably many values of $t \in [0, K]$ by Claim 5.13, moreover $g_t(0)$ is a càdlàg function of t by (2.12), therefore $-\varepsilon \le g_t(0) \le 0$ holds for every $0 \le t \le K$. □

Remark 5.16. Note that Theorem 2.13(ii) also implies that the function $\Phi(\cdot)$ is strictly increasing. Indeed, if we indirectly assume that $\Phi(s) = \Phi(t)$ for all $s \in [t, t + \Delta t]$, where $\Delta t > 0$, then by (2.12) we obtain $g_{t+\Delta t}(0) = g_t(0) + \lambda \Delta t$, which contradicts Theorem 2.13(ii).

Proof of Theorem 2.13(iii). Let us assume that $\underline{m} = (m_1, m_2, \dots) \in \ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$. We recursively define

$$n_1 = 1, \qquad n_k = \min\{\, i \,:\, m_i < m_{n_{k-1}} \,\},\ k \ge 2, \qquad \widetilde{m}_k = m_{n_k}.$$

Thus we have $\{m_1, m_2, \dots\} = \{\widetilde{m}_1, \widetilde{m}_2, \dots\}$ and $\widetilde{m}_1 > \widetilde{m}_2 > \dots$.

Let us fix $x \ge 0$ and let $y = g_0(x)$. Definition 2.5 and Lemma 4.2 imply

$$\mu_0[y, 0) \ge x, \qquad \mu_0[y + \varepsilon, 0) < x \qquad \text{for any} \quad \varepsilon > 0. \tag{5.37}$$

The restriction of the measure μ_0 to $(y, 0)$ can be determined by looking at $\mathcal{F}_x := \sigma\left(g_0(x'), 0 \le x' \le x\right)$, moreover for any $k \ge 1$ the restriction of $\mu_0^{(n_k - 1)} = \sum_{i=1}^{n_k - 1} m_i \cdot \delta_{Y_i(0)}$ to $(y, 0)$ is also determined by $\mathcal{F}_x$, simply by removing all atoms with a weight strictly smaller than $\widetilde{m}_{k-1}$.

As we have already discussed in the proof of Lemma 5.10, ν is an atomic measure with masses $(m_i)_{i=1}^{\infty}$ located at $(t_i)_{i=1}^{\infty}$ where $\lim_{n\to\infty} t_i^{(n)} = t_i$. Now if $Y_i(0) \in [y, 0)$ for some $i \ge 1$ and $n_k \ge i$ then $t_i^{(n_k - 1)}$ is $\mathcal{F}_x$-measurable, because the value of the death time $t_i^{(n_k-1)}$ can be determined by (3.11) if we know the initial particle configuration strictly above $Y_i(0)$ and the value of $Y_i(0)$ (we do not need to know the mass m_i of particle i).

Therefore $\lim_{k\to\infty} t_i^{(n_k)} = t_i$ is also $\mathcal{G}_x$-measurable. If we define $t_{y'} = \sup\{\, t_i \,:\, Y_i \in [y', 0) \,\}$, then $y' \mapsto t_{y'}$ is continuous, t_y is $\mathcal{F}_x$-measurable; moreover by (5.37) we have

$$\{\Phi(t) \geq x\} = \{t \geq t_y\} \in \mathcal{F}_x. \tag{5.38}$$

Hence $\{\Phi(t) \leq x\} \in \bigcap_{\varepsilon>0} \mathcal{F}_{x+\varepsilon} = \mathcal{F}_x^+$. This completes the proof of Theorem 2.13(iii). □

6. Applications

6.1. **Eternal multiplicative coalescents.** In Section 6.1 we restrict to the case $\lambda = 0$, i.e. the multiplicative coalescent (with no deletion).

We showed in Theorem 2.8 that a multiplicative coalescent $\mathbf{m}_t, t \geq 0$ started from any initial condition $\mathbf{m}_0$ in $\ell_2^{\downarrow}$ has a "tilt" representation, as $\mathbf{m}_t = \mathcal{E}^{\downarrow}(f_t)$, where f_0 is a random function and $f_t(x) = f_0(x) + xt$.

This leaves the question of *eternal* coalescents, i.e. those defined for all times $t \in (-\infty, \infty)$.

Recall from Definition 1.1 the notion of Brownian motion with parabolic drift, or briefly BMPD(u), where $u \in \mathbb{R}$ is the "tilt parameter".

$$\text{If } h \sim \text{BMPD}(u), \text{ denote by } \mathcal{M}(u) \text{ the law of } \mathcal{E}^{\downarrow}(h). \tag{6.1}$$

In [2, Corollary 24] Aldous showed that there exists an eternal version of the MC, the *standard multiplicative coalescent*, such that the marginal distribution of the coalescent at any given time $t \in \mathbb{R}$ is given by $\mathcal{M}(t)$. Armendáriz [8], and then also Broutin-Marckert [18], showed that if $h_0 \sim \text{BMPD}(u)$, and $h_t(x) = h_0(x) + tx$ for all $t \in \mathbb{R}$, then the process $\mathcal{E}^{\downarrow}(h_t), t \in \mathbb{R}$ is in fact the standard MC. We will provide an alternative proof of this result in Corollary 6.6 below.

Aldous and Limic [5] described the set of all eternal multiplicative coalescents. They showed that the marginal distributions of any of these coalescents can be given by the set of excursion lengths of a suitable stochastic process. To state their results we need to recall some more notation from [5].

Definition 6.1. Given $\kappa \geq 0$, $\tau \in \mathbb{R}$ and $\underline{c} \in \ell_3^{\downarrow}$, define

$$W^{\kappa,\tau,\underline{c}}(x) = \kappa^{1/2} W(x) - \frac{1}{2}\kappa x^2 + \sum_{i=1}^{\infty} \left(c_i \mathbb{1}[E_i \leq x] - c_i^2 x \right) + \tau x \tag{6.2}$$

for $x \geq 0$, where W is a standard Brownian motion, and where, for each i, $E_i \sim \text{Exp}(c_i)$, independently of each other and of W.

If $\underline{c} = \underline{0}$ and $\kappa = 1$, then $W^{\kappa,\tau,\underline{c}} \sim \text{BMPD}(\tau)$. For general $\underline{c}$, the processes defined in (6.2) have been called *Lévy processes without replacement*; each jump of size c_i occurs at rate c_i, but once such a jump has occurred it does not happen again. Define also the parameter space

$$\mathcal{I} = \Big[(0,\infty) \times (-\infty,\infty) \times \ell_3^{\downarrow} \Big] \cup \Big[\{0\} \times (-\infty,\infty) \times \left(\ell_3^{\downarrow} \setminus \ell_2^{\downarrow} \right) \Big]. \tag{6.3}$$

We can phrase the results of [5, Theorems 2 and 3] as follows:

Theorem 6.2. (i) *For each $(\kappa, \tau, \underline{c}) \in \mathcal{I}$, there exists an eternal multiplicative coalescent $\mathbf{m}_t, t \in \mathbb{R}$ such that for each t, $\mathbf{m}_t \overset{d}{=} \mathcal{E}^{\downarrow}(W^{\kappa,\tau+t,\underline{c}})$.*

(ii) *Let $\mu(\kappa, \tau, \underline{c})$ be the distribution of the MC in (i). Then the extreme points of the set of eternal MC distributions are $\mu(\kappa, \tau, \underline{c})$ for $(\kappa, \tau, \underline{c}) \in \mathcal{I}$, together with the distributions of "constant" processes (such that $\mathbf{m}_t = (y, 0, 0, 0, \dots)$ for all t, for some $y \geq 0$).*

Our results can be applied to show that all these eternal coalescents also have a "tilt" representation. First, we make a definition:

Definition 6.3. Let $W(x), x \geq 0$ be a random function. We say that W has the *exponential excursion levels* property if the following holds: conditional on the sequence of excursion lengths $\mathcal{E}^{\downarrow}(W) = (m_1, m_2, \dots)$, the levels of the excursions of W are independent, with the excursion of length m_i occurring at level $-E_i$ where $E_i \sim \text{Exp}(m_i)$.

Now we can state the key property that we need:

Claim 6.4. *Let $(\kappa, \tau, \underline{c}) \in \mathcal{I}$. Then $W^{\kappa,\tau,\underline{c}}$ has the exponential excursion levels property.*

Remark 6.5. Equation (72) of [2] states that BMPD(t) has the exponential excursion levels property for any $t \in \mathbb{R}$. From this fact the statement of Claim 6.4 follows for $W^{\kappa,\tau,\underline{0}}$ for any $\kappa > 0$ and $\tau \in \mathbb{R}$ by scaling. For a short sketch proof of this fact, see the $\underline{c} = \underline{0}$ case of Section 6.1.1 below.

Among other things, Claim 6.4 implies that the excursions of $W^{\kappa,\tau,\underline{c}}$ occur in size-biased order, c.f. Claim 3.3. From Claim 6.4, we can deduce the following result:

Corollary 6.6. *Let $(\kappa, \tau, \underline{c}) \in \mathcal{I}$. Then the process $\mathcal{E}^{\downarrow}(W^{\kappa,\tau+t,\underline{c}}), t \in \mathbb{R}$ is an eternal multiplicative coalescent.*

We describe below the straightforward way in which Corollary 6.6 follows from Claim 6.4. We then give a sketch of the proof of Claim 6.4 in Section 6.1.1. The proof is not difficult but a detailed account would occupy many more pages, and require some more technical extensions of earlier results (in particular the material in Section 4.2) to the setting of functions with positive jumps.

Corollary 6.6 is also the subject of current work by Vlada Limic which is available as an interim version at [23]. Limic gives a proof of this result based on the *breadth-first walk* construction used by Aldous [2] and by Aldous and Limic [5], and also using new ideas from the thesis by Uribe Bravo [35]. Even without giving the full proof, we feel that abstracting the property in Claim 6.4 gives valuable insight in complement to the different approach of [23].

Proof of Corollary 6.6 (using Claim 6.4). Fix any $\tau \in \mathbb{R}$. Let $h_t = W^{\kappa,\tau+t,\underline{c}}$ for $t \geq 0$. Note that

$$h_t(x) = h_0(x) + tx$$

for $x \geq 0$, $t \geq 0$.

Condition on $\mathcal{E}^{\downarrow}(W^{\kappa,\tau\underline{c}}) = \underline{m}$. Define $f_0 = \bar{h}_0$. Then Claim 6.4 tells us precisely that the distribution of f_0 is the same as the one defined at (2.6).

Define f_t by $f_t(x) = f_0(x) + tx$ as at (2.8). Then by the $\lambda = 0$ case of Remark 2.14(v) we have $\mathcal{E}^{\downarrow}(h_t) = \mathcal{E}^{\downarrow}(f_t)$ for all $t \geq 0$.

But Theorem 2.8 says that $\mathcal{E}^{\downarrow}(f_t), t \geq 0$ is a MC. Hence the same is true of $\mathcal{E}^{\downarrow}(h_t), t \geq 0$. That is, $\mathcal{E}^{\downarrow}(W^{\kappa,\tau+t,\underline{c}}), t \geq 0$ is a MC for all τ. But by considering values of τ tending to $-\infty$, and applying Kolmogorov extension, the statement of Corollary 6.6 follows. ☐

6.1.1. *Sketch of proof of Claim 6.4.* One necessary tool is an extension of Lemma 4.8, in two directions. First, we need not just that the collection of excursion lengths converges, but also that their levels converge; this is straightforward. Second, we need to cover the case where the limit f is allowed to have positive jumps (since this is true of the functions $W^{\kappa,\tau,\underline{c}}$ whenever $\underline{c}$ is not identically zero); this introduces a few technicalities (but doesn't require new ideas).

Now the idea is to take an appropriate sequence of initial conditions $\underline{m}^{(n)} \in \ell_0^{\downarrow}$, and times t_n. Define $f_0^{(n)}$ based on $\underline{m}^{(n)}$ according to (2.7), and define $f_{t_n}^{(n)}$ as in (2.8). Then we want to show convergence, in an appropriate sense, of $f_{t_n}^{(n)}$ to $W_{\kappa,\tau,\underline{c}}$; then, to use the extension of Lemma 4.8 to deduce that the lengths and levels of the excurions of $f_{t_n}^{(n)}$

converge in distribution to those of $W_{\kappa,\tau,\underline{c}}$. Finally, observe that for any n, the function $f^{(n)}_{t_n}$ satisfies the exponential excursion lengths property (this is a translation of Corollary 3.15(ii) from the particle system context into the tilt representation context using the identity (3.33)). This property is then inherited by the limit $W_{\kappa,\tau,\underline{c}}$ of the sequence $f^{(n)}_{t_n}$ and Claim 6.4 follows.

The main part of the work here is showing the convergence of $f^{(n)}_{t_n}$ to $W_{\kappa,\tau,\underline{c}}$ in such a way that we can deduce the convergence of the lengths and levels of the excursions.

For convenience, take $\tau = 0$, and $\kappa = 0$ or $\kappa = 1$; other cases are almost identical.

Case 1: $\underline{c} = \underline{0}$ and $\kappa = 1$. The distribution of $W^{1,0,\underline{0}}$ is simply BMPD(0). The fact that BMPD(u) satisfies the exponential excursion heights property was already discussed in Remark 6.5. Alternatively, let the initial condition $\underline{m}^{(n)}$ consist of n blocks each of size $n^{-2/3}$, and let $t_n = n^{1/3}$. The function $f^{(n)}_0$ has n excursions each of length $n^{-2/3}$. The gaps between the levels of these excursions are given by independent exponential random variables; for $0 \le k < n$, let F_k be the $(k+1)$st such gap, with rate $n^{1/3}\frac{n-k}{n}$.

Then the increments of $f^{(n)}_{n^{1/3}}$ on the intervals $[(kn^{-2/3}, (k+1)n^{-2/3}]$ are independent over $1 \le k < n$; the kth such increment is given by $n^{-1/3} - F_k$. Let us denote

$$x = n^{-2/3}k, \quad \mathrm{d}x = n^{-2/3}, \quad \mathrm{d}f^{(n)}_{n^{1/3}}(x) = f^{(n)}_{n^{1/3}}(x + \mathrm{d}x) - f^{(n)}_{n^{1/3}}(x).$$

We have

$$\begin{aligned}\mathbb{E}(\mathrm{d}f^{(n)}_{n^{1/3}}(x)) &= -kn^{-4/3}(1 + O(k/n)) \approx -x\mathrm{d}x,\\ \mathrm{Var}(\mathrm{d}f^{(n)}_{n^{1/3}}(x)) &= n^{-2/3}(1 + O(k/n)) \approx \mathrm{d}x.\end{aligned}$$

In this way we obtain a functional limit theorem; the distribution of $f^{(n)}_{n^{1/3}}$ converges (in the sense of uniform convergence on finite intervals) to BMPD(0).

(By this method we get an alternative, self-contained proof of the results of [8, 18] claiming that the tilt procedure applied to the BMPD family gives the (eternal) standard MC.)

Case 2: $\kappa = 0$. Now the process in (6.2) is given just by the compensated jumps according to the vector $\underline{c}$. In keeping with (6.3), we now need $\underline{c} \in \ell^{\downarrow}_3 \setminus \ell^{\downarrow}_2$.

Here, for an appropriate initial condition take $t_n = c_1^2 + \cdots + c_n^2$ and $\underline{m}^{(n)} = t_n^{-1}(c_1, c_2, \ldots, c_n, 0, 0, \ldots)$. We can couple the sequence of initial functions $f^{(n)}_0$, $n \in \mathbb{N}$ with $W^{0,0,\underline{c}}$ in the following way. For $k \le n$, let $f^{(n)}_0$ have an excursion on an interval of length $t_n^{-1}c_k$ at level $-t_nE_k$ where $E_k \sim \mathrm{Exp}(c_k)$ independently for each k.

Consider what happens on this interval in $f^{(n)}_{t_n}$. Because of the tilt, the function increases by $t_nt_n^{-1}c_k = c_k$ over the course of the interval. The length $t_n^{-1}c_k$ goes to 0 as $n \to \infty$.

By Remark 2.6, the horizontal location of the start of the interval is

$$\sum_{j=1}^{n} t_n^{-1}c_j\mathbb{1}(E_j < E_k); \tag{6.4}$$

conditional on E_k, this has mean converging to E_k and variance converging to 0, and (using simple martingale calculations) can be shown to converge almost surely as $n \to \infty$ to E_k. Hence in the limit process this interval produces a jump of size c_k occurring at time E_k, matching the term on the right of (6.2).

The vertical location of the start of the interval is $-t_n E_k$ in the function $f_0^{(n)}$; applying the tilt corresponding to the horizontal distance in (6.4), its vertical location in $f_{t_n}^{(n)}$ is

$$-t_n E_k + t_n \sum_{j=1}^{n} t_n^{-1} c_j \mathbb{1}(E_j < E_k),$$

which, after simplification, is $\sum_{j=1}^{n} \left(c_j \mathbb{1}(E_j < E_k) - c_j^2 E_k \right)$. Comparing with (6.2), this converges to $W^{0,0,\underline{c}}(E_k-)$ as $n \to \infty$, so in the limit process the jump indeed appears at the correct height.

In the limit process, these jumps are dense. In this case we do not have uniform convergence of $f_{t_n}^{(n)}$ to $W^{0,0,\underline{c}}$, but by considering suitable time-changes one obtains that $f_{t_n}^{(n)}$ has the same excursions as a function $h^{(t_n)}$ which converges to $W^{0,0,\underline{c}}$ in the Skorohod topology, and this is enough to give convergence of the lengths and levels of excursions as required.

Case 3: $\underline{c} \neq \underline{0}$ and $\kappa = 1$. Now the process $W^{1,0,\underline{c}}$ has both a Brownian part and positive jumps. Here a suitable sequence of initial conditions is given by taking $\underline{m}^{(n)}$ to consist of blocks of sizes $n^{-1/3}(c_1, c_2, \ldots, c_{k(n)})$ along with n blocks of size $n^{-2/3}$, where $k(n)$ is chosen such that $\sum_{i=1}^{k(n)} c_i^2 \ll n^{1/3}$ as $n \to \infty$. (This is the same regime used in the proof of Lemma 8 of [5]). Similarly to the two previous cases, choose $t_n = \|\underline{m}^{(n)}\|_2^{-1}$. The ideas of the two previous cases can be combined to give the desired result for $W^{1,0,\underline{c}}$ also.

6.2. **Tilt-and-shift of** BMPD**.** Recall the definition of BMPD(u) from Definition 1.1. We show that applying the tilt-and-shift procedure starting from an initial state h_0 which is a BMPD results in a MCLD process. Furthermore, the function h_t remains in the class of BMPD processes (with a random parameter).

Proposition 6.7. *Let $u \in \mathbb{R}$, and let $h_0 \sim$ BMPD(u). Let $g_0 = \bar{h}_0$. Let $\Phi(t), t \geq 0$ and $g_t, t \geq 0$ be given by the tilt-and-shift procedure in Theorem 2.13, and let h_t be given by* (2.13)*.*

(i) The process $\mathcal{E}^{\downarrow}(h_t), t \geq 0$ is a MCLD(λ) *process.*
(ii) Given $\Phi(t)$ and $(h_0(x), x \leq \Phi(t))$, the conditional law of h_t is

$$\text{BMPD}(u + t - \Phi(t)). \tag{6.5}$$

Proof. We first note that by Remark 6.5 the function h_0 (and thus g_0) has the exponential excursion levels property (c.f. Definition 6.3). Together with Definition 2.5 this implies that g_0 is a suitable initial state of the tilt-and-shift representation. Then by Theorem 2.13, $\mathcal{E}^{\downarrow}(g_t)$ is a MCLD(λ) process, and, as observed in Remark 2.14(v), so is $\mathcal{E}^{\downarrow}(h_t)$. This completes the proof of part (i).

For part (ii), since $h_0 \sim$ BMPD(u), we have

$$h_t(x) \overset{(2.13),(1.4)}{=} B(x + \Phi(t)) - \frac{1}{2}(x + \Phi(t))^2 + u \cdot (x + \Phi(t)) + \lambda t + \int_0^t (x + \Phi(t) - \Phi(s))\, \mathrm{d}s. \tag{6.6}$$

Since $\mathcal{E}^{\downarrow}(g_0) = \mathcal{E}^{\downarrow}(h_0)$ is in $\ell_2^{\downarrow} \setminus \ell_1^{\downarrow}$ with probability 1, we have $h_t(0) = g_t(0) = 0$ by Theorem 2.13(ii), and if we combine this with (6.6), we obtain

$$0 = B(\Phi(t)) - \frac{1}{2}(\Phi(t))^2 + u\Phi(t) + \lambda t + \int_0^t (\Phi(t) - \Phi(s))\, \mathrm{d}s. \tag{6.7}$$

Subtracting (6.7) from (6.6) we get

$$h_t(x) = (B(x+\Phi(t)) - B(\Phi(t))) - \frac{1}{2}x^2 + (u+t-\Phi(t))x, \quad x \geq 0. \tag{6.8}$$

Now Theorem 2.13(iii) states that $\Phi(t)$ is a stopping time w.r.t. the filtration $(\mathcal{F}_x^+)$. Then by the strong Markov property for Brownian motion w.r.t. $(\mathcal{F}_x^+)$ (see [28, Theorem 2.14]), we have that $B\big(x+\Phi(t)\big) - B\big(\Phi(t)\big)$ is a standard Brownian motion independent of $\Phi(t)$ and $\big(B(x), 0 \leq x \leq \Phi(t)\big)$. Using (6.8) then gives that the conditional distribution of h_t is (6.5). □

Remark 6.8. Proposition 6.7(ii) gives a MCLD process whose marginal distributions are all given by mixtures of distributions $\mathcal{M}(u)$ (c.f. Definition (6.1)). In [24] we find eternal processes with this property, which may be stationary or non-stationary. We show that such processes arise naturally as scaling limits of discrete models such as frozen percolation processes (see Definition 6.9 below) or forest fire processes (see Section 6.3).

In Section 6.2.1 below we observe a simple case of such a scaling limit.

6.2.1. *Scaling limit of frozen percolation started from a critical Erdős-Rényi graph.* First we recall the notion of *mean-field frozen percolation process* from [31] (using slightly different notation).

Definition 6.9 ($\mathrm{FP}(n, \lambda(n))$)**.** We start with a graph $F_0^{(n)}$ on n vertices. Between each pair of unconnected vertices an edge appears with rate $1/n$; also, every connected component of size k is deleted with rate $\lambda(n) \cdot k$. (When a component is deleted, its vertices as well as its edges are removed from the graph.) Let $F_t^{(n)}$ be the graph at time t. Denote by

$$\mathbf{M}^{(n)}(t) = \left(M_1^{(n)}(t), M_2^{(n)}(t), \dots\right) \in \ell_0^{\downarrow}$$

the sequence of component sizes of $F_t^{(n)}$, arranged in decreasing order.

Then $\mathbf{M}^{(n)}(t), t \geq 0$ is a Markov process – let us call it here the frozen percolation component process on n vertices with lightning rate $\lambda(n)$, or briefly $\mathrm{FP}(n, \lambda(n))$.

In order to achieve self-organized criticality, one chooses $n^{-1} \ll \lambda(n) \ll 1$, c.f. [31, Theorem 1.2]. The case $\lambda(n) \asymp n^{-1/3}$ holds special significance, c.f. [31, Conjecture 1.1].

The next result gives a scaling limit for $\mathrm{FP}(n, \lambda(n))$, in a setting where $\lambda(n) \asymp n^{-1/3}$ and the initial state has the distribution of an Erdős-Rényi random graph at some point within the "critical window".

Proposition 6.10. *Fix $u \in \mathbb{R}$ and let $F_0^{(n)}$ be an Erdős-Rényi graph $\mathcal{G}(n,p)$ with edge probability $p = \frac{1+un^{-1/3}}{n}$. Let $\lambda > 0$ and let $\mathbf{M}^{(n)}(t), t \geq 0$ be the $\mathrm{FP}(n, \lambda n^{-1/3})$ process with initial state $F_0^{(n)}$.*

Define $\mathbf{m}^{(n)}(t), t \geq 0$ by

$$\mathbf{m}^{(n)}(t) := \left(n^{-2/3}M_1^{(n)}(n^{-1/3}t), n^{-2/3}M_2^{(n)}(n^{-1/3}t), \dots\right). \tag{6.9}$$

Then as $n \to \infty$ the sequence of processes $\mathbf{m}^{(n)}(t), t \geq 0$ converge in law to the $\mathrm{MCLD}(\lambda)$ *process $\mathcal{E}^{\downarrow}(h_t), t \geq 0$ given by Proposition 6.7.*

Proof. Corollary 2 of Aldous [2] (or, alternatively, the method sketched in the $\underline{c} = \underline{0}$ case of Section 6.1.1) implies that the sequence $\mathbf{m}^{(n)}(0)$ converges in distribution as $n \to \infty$ in the $(\ell_2^{\downarrow}, \mathrm{d}(\cdot,\cdot))$ space to $\mathcal{E}^{\downarrow}(h_0)$, where $h_0 \sim \mathrm{BMPD}(u)$.

Further, the process $\mathbf{m}^{(n)}(t)$ defined by (6.9) is a $\mathrm{MCLD}(\lambda)$ process (as can be readily seen by comparing the definition (1.3) of the $\mathrm{MCLD}(\lambda)$ process with Definition 6.9).

Then the statement of Proposition 6.10 follows by applying Proposition 6.7 together with the Feller property of $\mathrm{MCLD}(\lambda)$ (see [30, Theorem 1.2]).

□

6.3. **Forest fire model.** This section contains a particle representation of the mean-field forest fire model of [32]; see Section 6.3.1. The representation is an adaptation of the one in Section 3.3, and we will briefly explain in Section 6.3.2 how it sheds some new light on a certain controlled non-linear PDE problem (see (6.17) below) which played a central role in the theory developed in [32] and [19].

In [32] Ráth and Tóth modify the dynamical Erdős-Rényi model to obtain the mean-field forest fire model:

Definition 6.11 ($\mathrm{FF}(n,\lambda(n))$)**.** We start with a graph on n vertices. Between each pair of unconnected vertices an edge appears with rate $1/n$; moreover each connected component of size k "burns" with rate $\lambda(n)\cdot k$, i.e., the edges of the component are deleted. The total number of vertices remains n.

Denote by $\mathcal{C}^n(i,t)$ the connected component of vertex i at time t.

We define the empirical component size densities by

$$\mathbf{v}_k^n(t)=\frac{1}{n}\sum_{i=1}^n \mathbb{1}[|\mathcal{C}^n(i,t)|=k], \qquad \mathbf{v}^n(t)=(\mathbf{v}_k^n(t))_{k=1}^n\,. \tag{6.10}$$

With the above definitions $\mathbf{v}^n(t), t\geq 0$ is a Markov process, let us call it here the forest fire component size density Markov process on n vertices with lightning rate $\lambda(n)$, or briefly $\mathrm{FF}(n,\lambda(n))$.

6.3.1. *Particle representation of the forest fire model.* In Definition 6.14 and Proposition 6.15 below we are going to give a novel particle representation of $\mathrm{FF}(n,\lambda(n))$ by slightly modifying Definition 3.11.

Definition 6.12. If $n\in\mathbb{N}_+$ we let

$$\mathcal{V}^n=\left\{\ \underline{v}^n=(v_k^n)_{k=1}^n\ :\ \sum_{k=1}^n v_k^n=1\ \text{ and }\ \frac{n}{k}v_k^n\in\mathbb{N}\ \text{ for all }\ k\ \right\},$$

$$\mathcal{M}^n=\left\{\ \underline{m}^n=\left(m_j^n\right)_{j=1}^N\in\ell_0^{\downarrow}\ :\ \sum_{j=1}^N m_j^n=n\ \text{ and }\ m_j^n\in\mathbb{N}_+\ \text{ for all }\ j\ \right\}.$$

We say that the component size density vector $\underline{v}^n\in\mathcal{V}^n$ and the ordered list of component sizes $\underline{m}^n\in\mathcal{M}^n$ correspond to each other if

$$v_k^n=\sum_{j=1}^N \frac{k}{n}\mathbb{1}[\,m_j^n=k\,]\quad\text{for all }\ k. \tag{6.11}$$

Note that this correspondence is one-to-one.

In plain words, $\underline{v}^n$ and $\underline{m}^n$ correspond to each other if there is a graph G on n vertices such that $\underline{v}^n$ and $\underline{m}^n$ both arise from G.

Definition 6.13. If $\widetilde{\mu}^n$ is a finite point measure on $\mathbb{R}_-$ such that $\widetilde{\mu}^n(\mathbb{R}_-)=1$ and the masses of the atoms of $n\widetilde{\mu}^n$ are integers then we define $\mathbf{v}(\widetilde{\mu}^n)$ to be the element of $\mathcal{V}^n$ corresponding to the element of $\mathcal{M}^n$ which consists of the ordered list of masses of the atoms of $n\widetilde{\mu}^n$.

Now we define the particle representation of the $\mathrm{FF}(n,\lambda(n))$ model.

Definition 6.14. Given $\underline{v}^n(0)=(v_k^n(0))_{k=1}^n\in\mathcal{V}^n$ and the corresponding $\underline{m}^n=\left(m_j^n\right)_{j=1}^N\in\mathcal{M}^n$, we define the initial heights of the particles $\widetilde{Y}_i(t), 1\leq i\leq n$ by letting $\widetilde{Y}_i(0)=-E_j$,

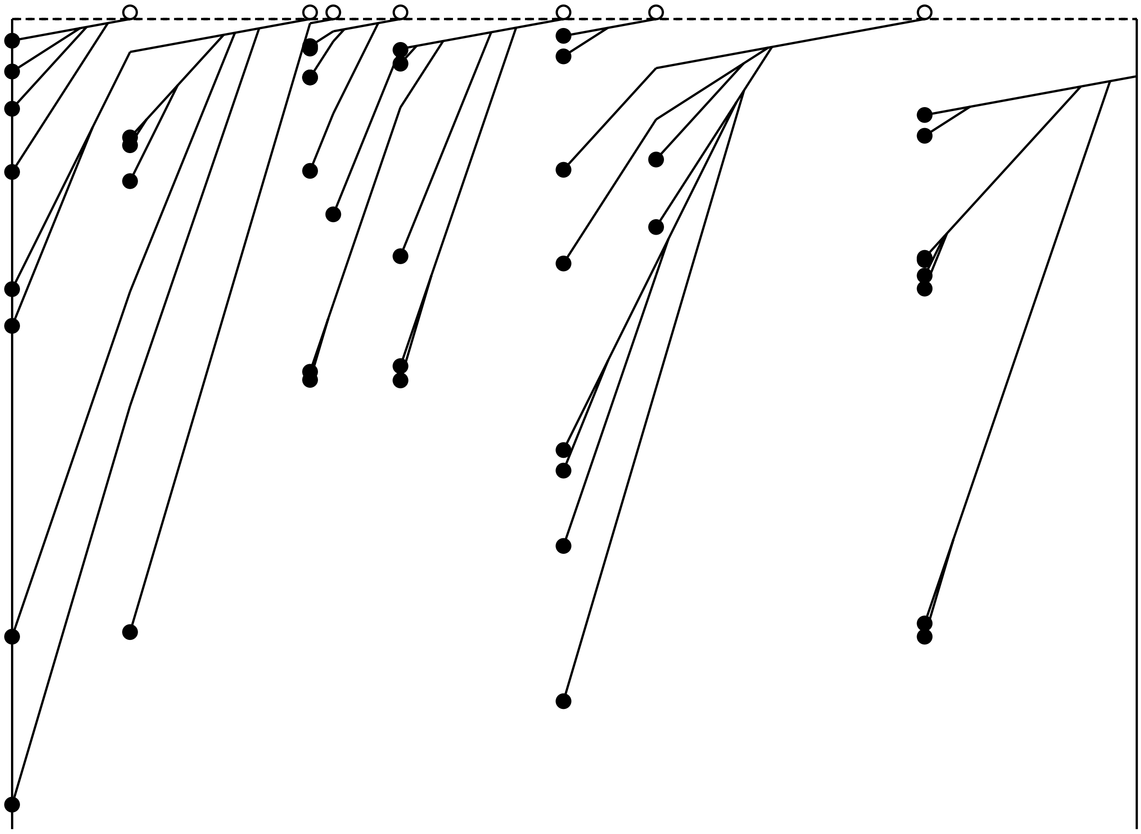

FIGURE 6.1. A simulation of the particle system realising the forest fire model. The system has $n = 8$ particles and $\lambda = 0.4$, and is shown on the time interval $[0, 2]$. Compare to the systems realising the MCLD and the multiplicative coalescent in Figures 1.2 and 1.3. The burning events (which occur when a block reaches 0) here involve blocks of sizes 4, 7, 1, 3, 8, 2, 8 respectively. Note that in this case the process of burnings is a Poisson process of rate $n\lambda$.

where $E_j \sim \text{Exp}(m_j^n)$, $1 \le j \le N$ are independent and vertex i initially belongs to component j in the forest fire model. We define

$$\widetilde{\mu}_t^n = \sum_{i=1}^{n} \frac{1}{n} \delta_{\widetilde{Y}_i(t)} \quad (\text{ Note: } \mathbf{v}(\widetilde{\mu}_0^n) = \underline{v}^n(0) \text{ }). \tag{6.12}$$

If $\widetilde{Y}_i(t_-) < 0$ then we let

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{Y}_i(t) = \lambda(n) + \widetilde{\mu}_t^n(\widetilde{Y}_i(t), 0), \tag{6.13}$$

and if $\widetilde{Y}_i(t_-) = 0$ then we say that vertex i burns and we let $-\widetilde{Y}_i(t)$ have Exp(1) distribution, independently from everything else.

In words, a clustered family of particles with mass $1/n$ start at negative locations, move up and merge with other particle clusters, but if a time-t block of particles with total mass k/n reaches 0, then this block burns and gets replaced by k particles of mass $1/n$ with i.i.d. locations with negative Exp(1) distribution.

Proposition 6.15. *(i) For any $n \in \mathbb{N}_+$ and any initial state $\underline{v}^n(0) \in \mathcal{V}^n$, the process $\mathbf{v}(\widetilde{\mu}_t^n), t \ge 0$ is a $\mathrm{FF}(n, \lambda(n))$ process with initial state $\underline{v}^n(0)$ (see Definitions 6.11, 6.14 and 6.13 for the definitions of $\mathrm{FF}(n, \lambda(n))$, $\widetilde{\mu}_t^n$ and $\mathbf{v}(\widetilde{\mu}^n)$, respectively).*

(ii) *For any $t \geq 0$, the conditional distribution of $n\widetilde{\mu}_t^n$ given the σ-algebra $\sigma(\mathbf{m}_s^n, 0 \leq s \leq t)$ is* $\mathrm{Exp}(\mathbf{m}_t^n)$ *(c.f. Definition 2.4), where $\mathbf{m}_t^n$ is the $\mathcal{M}^n$-valued random variable corresponding to the $\mathcal{V}^n$-valued random variable $\mathbf{v}(\widetilde{\mu}_t^n)$ (c.f. Definition 6.12).*

See Figure 6.1 for a simulation of the particle system realising the forest fire model.

Proof of Proposition 6.15. Recalling Definition 2.4 we observe that by Definition 6.14 we have $n\widetilde{\mu}_0^n \sim \mathrm{Exp}(\underline{m}^n)$, where $\underline{m}^n \in \mathcal{M}^n$ corresponds to $\underline{v}^n(0) \in \mathcal{V}^n$.

Denote by τ the first burning time of the particle system $\widetilde{Y}_i(t), 1 \leq i \leq n$.

Denote by $Y_i(t) := \widetilde{Y}_i(nt)$ and $\mu_t := n\widetilde{\mu}_{nt}^n = \sum_{i=1}^n \delta_{Y_i(t)}$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_i(t) \overset{(6.13)}{=} n\lambda(n) + \mu_t(Y_i(t), 0), \quad 0 \leq t < \frac{\tau}{n},$$

thus the evolution of the particle system $Y_i(t), 1 \leq i \leq n$ satisfies Definition 3.11 (with $\lambda = n\lambda(n)$) up to time τ/n, including the time-τ/n block that burns.

Likewise, if $\mathbf{v}^n(t), t \geq 0$ is a $\mathrm{FF}(n, \lambda(n))$ process, then the $\mathcal{M}^n$-valued process corresponding to the $\mathcal{V}^n$-valued process $\mathbf{v}^n(nt), 0 \leq t \leq \tau/n$ satisfies the definition of a $\mathrm{MCLD}(n\lambda(n))$ process, including the time of the first deletion event and the component that gets deleted.

Therefore by Corollary 3.13 the statement of Proposition 6.15(i) holds for $\mathbf{v}(\widetilde{\mu}_t^n), 0 \leq t \leq \tau$ (including time τ, since in both Definitions 6.11 and 6.14 we add k singletons of mass $1/n$ if a block of size k is deleted).

Next we observe that the conditional distribution of $n\widetilde{\mu}_\tau^n$ given the σ-algebra $\sigma(\mathbf{m}_s^n, 0 \leq s \leq \tau)$ is $\mathrm{Exp}(\mathbf{m}_\tau^n)$. Indeed, this follows from Corollary 3.15(i) and the fact that we insert k particles of mass $1/n$ with i.i.d. $-\mathrm{Exp}(1)$ distribution if a block of k particles burn at time τ, which is exactly what we have to do to maintain the property required by Definition 2.4.

Therefore we can inductively repeat this argument using Corollary 3.15(i) again and again to show that Proposition 6.15(i) holds for $\mathbf{v}(\widetilde{\mu}_t^n), 0 \leq t \leq \tau_i$, where τ_i is the i'th burning time. The proof of Proposition 6.15(ii) similarly follows from Corollary 3.15(ii). □

Remark 6.16. Let us recall a closely related dynamic random graph model of self-organized criticality, studied by Fournier and Laurencot in [21] and by Merle and Normand in [26]. One starts with a graph on n vertices, the coagulation mechanism is the same as in Definition 6.9 and Definition 6.11 (between each pair of unconnected vertices an edge appears with rate $1/n$), but the deletion mechanism is different: connected components are deleted forever when their size exceeds a threshold $\omega(n)$. In order to achieve self-organized criticality, one chooses $1 \ll \omega(n) \ll n$.

Let us remark that this model also admits a particle representation: the initial weights and heights of particles should be the same as in Definition 6.14, and the particle dynamics between deletion times should be given by (6.13). The deletion mechanism is obvious: if the weight of a time-t block exceeds the threshold $\omega(n)$, we remove that time-t block. The proof of the validity of this particle representation can be carried out analogously to the proof of Proposition 6.15.

This is a "rigid" representation, i.e., all of the randomness is encoded in the initial state. However, in contrast to the case of the frozen percolation model (i.e., MCLD), the above described threshold-deletion model does not admit a clean tilt-and-shift representation, since particles from the middle can be deleted (as opposed to the case of linear deletion, where only the top particle can be deleted).

6.3.2. *The controlled Burgers equation.* In this section we use the particle representation of the $\mathrm{FF}(n, \lambda(n))$ process to give a new interpretation of a certain controlled non-linear

PDE problem (see (6.17) below) which played a central role in the theory developed in [32] and [19]. This new interpretation will be presented in Remark 6.17.

One investigates the FF$(n, \lambda(n))$ when $\frac{1}{n} \ll \lambda(n) \ll 1$ as $n \to \infty$. This is called the *self-organized critical regime* of the lightning rate $\lambda(n)$. We assume that $\mathbf{v}_k^n(0) \to v_k(0)$ for all $k \in \mathbb{N}$ as $n \to \infty$, where $\sum_k k^3 v_k(0) < +\infty$.

Under these assumptions [32, Theorem 2] states that

$$\mathbf{v}_k^n(t) \to v_k(t) \quad \text{in probability as} \quad n \to \infty, \tag{6.14}$$

where $(v_k(t))_{k=1}^{\infty}$ is the unique solution of the following system of ODE's:

$$\forall\, k \geq 2 \quad \frac{\partial}{\partial t} v_k(t) = \frac{k}{2} \sum_{l=1}^{k} v_l(t) v_{k-l}(t) - k v_k(t), \qquad \sum_{k=1}^{\infty} v_k(t) \equiv 1. \tag{6.15}$$

This system of equations is a modification of Smoluchowski's coagulation equations with multiplicative kernel (c.f. [3, Section 2.1]).

In order to prove that (6.15) is well-posed (c.f. [32, Theorem 1]), one looks at the Laplace transform

$$V(t, x) = \sum_{k=1}^{\infty} v_k(t) e^{-kx} - 1 \tag{6.16}$$

which satisfies the following controlled PDE (c.f. [32, (43)]):

$$\frac{\partial}{\partial t} V(t, x) = -V(t, x) \frac{\partial}{\partial x} V(t, x) + \varphi(t) e^{-x}, \qquad V(t, 0) \equiv 0, \tag{6.17}$$

where the control function $\varphi(t)$ measures the intensity of fires at time t:

$$\varphi(t) = \frac{\partial}{\partial t} r(t), \quad r(t) = \lim_{n\to\infty} \mathbf{r}^n(t), \quad \mathbf{r}^n(t) = \frac{1}{n} \sum_{i=1}^{n} B^n(i, t), \tag{6.18}$$

and $B^n(i, t)$ denotes the number of times vertex i has burnt before time t. Note that (6.17) is a controlled variant of the Burgers equation.

Given a solution $V(\cdot, \cdot)$ of (6.17) one defines the corresponding characteristic curves (c.f. [32, (66)]) as the solutions of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}s} \xi(s) = V(s, \xi(s)). \tag{6.19}$$

These curves are useful because by (6.17) they satisfy $\frac{\mathrm{d}^2}{\mathrm{d}s^2} \xi(s) = \varphi(s) e^{-\xi(s)}$, hence given $\varphi(\cdot)$ they can be constructed (c.f. [32, (65)]) without solving (6.17).

Remark 6.17. Let us assume that $\widetilde{\mu}_t^n$ converges weakly in probability to some measure $\widetilde{\mu}_t$ as $n \to \infty$. Denote by

$$\widetilde{V}(t, y) = \widetilde{\mu}_t(y, 0), \qquad y \leq 0.$$

We will non-rigorously derive a PDE for $\widetilde{V}(t, y)$, see (6.20) below.

We have $\widetilde{\mu}_t[y - \mathrm{d}y, y] = -\frac{\partial}{\partial y} \widetilde{V}(t, y) \mathrm{d}y$, moreover by (6.13), each "particle" near the location y moves with speed $\widetilde{V}(t, y)$ (since $\lambda(n) \ll 1$), thus $\widetilde{V}(t, y)$ increases by $\widetilde{\mu}_t[y - \mathrm{d}y, y]$ on the time interval $[t, t + \mathrm{d}t]$, where $\mathrm{d}y = \widetilde{V}(t, y)\mathrm{d}t$. The mass $\widetilde{V}(t, y)$ also decreases by $\varphi(t)\mathrm{d}t$ because of burning (see (6.18)) and increases by $(1 - e^y)\varphi(t)\mathrm{d}t$ because of the re-insertion of burnt mass with distribution $-\mathrm{Exp}(1)$. Putting these effects together we obtain

$$\frac{\partial}{\partial t} \widetilde{V}(t, y) = -\widetilde{V}(t, y) \frac{\partial}{\partial y} \widetilde{V}(t, y) - e^y \varphi(t). \tag{6.20}$$

By comparing (6.17) and (6.20), we observe that $V(t, x)$ solves the same PDE as $-\widetilde{V}(t, -x)$. Indeed, by (6.14) and Proposition 6.15(ii) we have $\widetilde{V}(t, y) = \sum_k v_k(t)(1 - e^{ky})$, which is equal to $-V(t, -y)$ by (6.16).

Moreover, if $1 \ll n$ then $\widetilde{\mu}^n_t(y,0) \approx \widetilde{V}(t,y)$, thus by comparing (6.19) and (6.13) we see that the trajectories $-\widetilde{Y}_i(s), s \geq 0$ of particles can be viewed as discrete approximations of characteristic curves.

Acknowledgments

We thank Christina Goldschmidt for several very helpful discussions at various stages of this work, Vlada Limic for sending us a preliminary version of [23], and Ines Armendáriz for sending us the paper [8].

This work was supported by EPSRC grant EP/J019496/1. JBM was supported by EPSRC Fellowship EP/E060730/1, and BR was supported by OTKA (Hungarian National Research Fund) grant K100473, the Postdoctoral Fellowship of the Hungarian Academy of Sciences and the Bolyai Research Scholarship of the Hungarian Academy of Sciences.

References

[1] L. Addario-Berry, N. Broutin, and C. Goldschmidt. The continuum limit of critical random graphs. *Probab. Theory Related Fields*, 152(3-4):367–406, 2012.

[2] D. Aldous. Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab.*, 25(2):812–854, 1997.

[3] D. Aldous. Deterministic and stochastic models for coalescence (aggregation and coagulation): a review of the mean-field theory for probabilists. *Bernoulli*, 5(1):3–48, 1999.

[4] D. Aldous. The percolation process on a tree where infinite clusters are frozen. *Math. Proc. Cambridge Philos. Soc.*, 128(3):465–477, 2000.

[5] D. Aldous and V. Limic. The entrance boundary of the multiplicative coalescent. *Electron. J. Probab.*, 3:no.3, 59 pp., 1998.

[6] D. Aldous and J. Pitman. The standard additive coalescent. *Ann. Probab.*, 26(4):1703–1726, 1998.

[7] I. Armendáriz. Brownian excursions and coalescing particle systems. PhD thesis, New York University, 2001.

[8] I. Armendáriz. Dual fragmentation and multiplicative coagulation. Unpublished preprint, 2005.

[9] J. Bertoin. A fragmentation process connected to Brownian motion. *Probab. Theory Related Fields*, 117(2):289–301, 2000.

[10] S. Bhamidi, N. Broutin, S. Sen, and X. Wang. Scaling limits of random graph models at criticality: Universality and the basin of attraction of the Erdős-Rényi random graph. `arXiv:1411.3417`.

[11] S. Bhamidi, A. Budhiraja, and X. Wang. The augmented multiplicative coalescent, bounded size rules and critical dynamics of random graphs. *Probab. Theory Related Fields*, 160(3-4):733–796, 2014.

[12] S. Bhamidi and S. Sen. Geometry of the vacant set left by random walk on random graphs, Wright's constants, and critical random graphs with prescribed degrees. `arXiv:1608.07153`.

[13] S. Bhamidi, S. Sen, and X. Wang. Continuum limit of critical inhomogeneous random graphs. *Probab. Theory Related Fields*. To appear. `arXiv:1411.3417`.

[14] S. Bhamidi, R. van der Hofstad, and S. Sen. The multiplicative coalescent, inhomogeneous continuum random trees, and new universality classes for critical random graphs. `arXiv:1508.04645`.

[15] S. Bhamidi, R. van der Hofstad, and J. van Leeuwaarden. Scaling limits for critical inhomogeneous random graphs with finite third moments. *Electron. J. Probab.*, 15:no. 54, 1682–1703, 2010.

[16] S. Bhamidi, R. van der Hofstad, and J. van Leeuwaarden. Novel scaling limits for critical inhomogeneous random graphs. *Ann. Probab.*, 40(6):2299–2361, 2012.

[17] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999.

[18] N. Broutin and J.-F. Marckert. A new encoding of coalescent processes: applications to the additive and multiplicative cases. *Probab. Theory Related Fields*, 166(1-2):515–552, 2016.

[19] E. Crane, N. Freeman, and B. Tóth. Cluster growth in the dynamical Erdős-Rényi process with forest fires. *Electron. J. Probab.*, 20:no. 101, 33 pp., 2015.

[20] A. Dembo, A. Levit, and S. Vadlamani. Brownian excursions and critical quantum random graphs. `arXiv:1404.5705`.

[21] N. Fournier and P. Laurençot. Marcus-Lushnikov processes, Smoluchowski's and Flory's models. *Stochastic Process. Appl.*, 119(1):167–189, 2009.

[22] A. Joseph. The component sizes of a critical random graph with given degree sequence. *Ann. Appl. Probab.*, 24(6):2560–2594, 2014.

[23] V. Limic. Extreme eternal multiplicative coalescent is encoded by its Lévy-type processes. Interim version available at `arxiv:1601.01325v2`.

[24] J. Martin and B. Ráth. Scaling limits of mean-field frozen percolation and forest fire processes. In preparation.

[25] J. Martin and D. Yeo. A scaling limit for the component sizes of critical random forests. In preparation.

[26] M. Merle and R. Normand. Self-organized criticality in a discrete model for Smoluchowski's equation. arXiv:1410.8338.

[27] M. Merle and R. Normand. Self-organized criticality in a discrete model for Smoluchowski's equation with limited aggregations. arXiv:1509.00934.

[28] P. Mörters and Y. Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.

[29] A. Nachmias and Y. Peres. Critical percolation on random regular graphs. *Random Structures Algorithms*, 36(2):111–148, 2010.

[30] B. Ráth. Feller property of the multiplicative coalescent with linear deletion. arXiv:1610.00021.

[31] B. Ráth. Mean field frozen percolation. *J. Stat. Phys.*, 137(3):459–499, 2009.

[32] B. Ráth and B. Tóth. Erdős-Rényi random graphs + forest fires = self-organized criticality. *Electron. J. Probab.*, 14:no. 45, 1290–1327, 2009.

[33] O. Riordan. The phase transition in the configuration model. *Combin. Probab. Comput.*, 21(1-2):265–299, 2012.

[34] G. Uribe Bravo. Coding multiplicative coalescence by an inhomogeneous random walk. Work in progress.

[35] G. Uribe Bravo. Markovian bridges, Brownian excursions, and stochastic fragmentation and coalescence. PhD thesis, UNAM, 2007.