

# Note on rings in which every proper left-ideal is cyclic

by

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We shall call an arbitrary ring  $R$  *cyclic* if the additive group  $R^+$  is cyclic. The ring  $J$  of rational integers is obviously cyclic. Starting from the fundamental property of the ring  $J$  we introduce the following

**Definition.** An arbitrary ring  $R$  is called a *ring with property  $P$* , if every proper left-ideal  $L$  of  $R$  is cyclic. For example any cyclic ring and any skew-field have the property  $P$ .

**THEOREM.** An arbitrary ring  $R$  has the property  $P$  if and only if  $R$  is a skew-field, or a cyclic ring, or a zero-ring of type  $p^\infty$  or else an arbitrary ring of order  $p^2$  (where  $p$  is a prime).

**Remark.** A skew-field, as a ring without proper left-ideals, can have an arbitrary infinite cardinal, but the order of a finite ring with property  $P$  is necessarily  $p^2$ . For example  $R(p) = \{x, y\}$  is a non-commutative ring with property  $P$  and of order  $p^2$  where  $p$  is a prime number and  $px = py = x^2 = xy = yx - x = y^2 - y = 0$ . We remark that the theorem is a generalization of Lemma 1 (see [7]). The notions of modern algebra can be found in the books [1], [3], [4] and [5], therefore we omit terminological remarks. Now we verify five Lemmas.

**LEMMA 1.** A ring without proper left-ideals is a skew-field or else a zero-ring of prime-number-order.

**Proof.** If there exists an element  $0 \neq a \in R$  for which  $Ra \neq R$ , then  $Ra = 0$ , and thus the zero-ring  $\{a\} \neq 0$ , being a left-ideal, coincides with  $R$  and  $O(R) = p$ . But if for any  $0 \neq a \in R$  the element  $Ra = R$  holds, then  $R$  has no divisors of zero and by the single equation  $ea = a$  we see that  $e \in R$  is the unity of  $R$ . The solvability of all equations  $yb = e$  trivially implies by the associativity law the skew-field behaviour of  $R$ .

**Remarks.** From this short proof we see that only the rings of order  $p$  are without proper subrings; moreover the solvability of all equations  $yb = a$  in a ring implies the solvability of all equations  $bx = a$  in the same ring ( $b \neq 0$ ); and finally we observe that we can similarly prove that if in a ring  $R$  there exists an element  $a \neq 0$  which is not a right an-

nilhilator of  $R$  and if for this element with any  $0 \neq b \in R$  the element  $Rab = Ra$  holds, then  $R$  is a skew-field (see [6]).

LEMMA 2. A ring  $R$  with mixed group  $R^+$  cannot have the property  $P$ .

Proof. We assume that  $R$  is a ring with property  $P$  and with mixed group  $R^+$ . Let  $T$  be the cyclic torsion ring of order  $n \in J$  in  $R$ . Since  $(nR) \cdot T = T \cdot (nR) = 0$  and  $nR \cap T = 0$ , there exists a non-cyclic two-sided ideal  $D = nR + T$  (as a ring-theoretical direct sum) in  $R$ , which by property  $P$  implies  $R = nR + T$  (without the use of the fundamental theorem of [8]). Then  $n^2R$  is likewise a cyclic ideal in  $R$ , consequently  $n^2R + T = R$ . If  $nR = \{nr\}$ , where naturally  $O(r) = \infty$ , we obtain  $nr = k(n^2r) + t$  ( $k \in J$ ,  $t \in T$ ), i. e.,  $n = kn^2$  and  $n = 1$ ,  $T = 0$ .

LEMMA 3. A ring  $R$  with property  $P$  but without divisors of zero is a skew-field or else an infinite cyclic ring.

Proof. If  $0 \neq a \in R$ , then  $Ra \neq 0$ . If for every  $0 \neq a \in R$  it is  $Ra = R$ , thus, by Lemma 1,  $R$  is a skew-field. In the case  $Ra \neq R$  the ring  $R$  is itself cyclic by the property  $P$  and by  $(Ra)^+ \simeq R^+$ , and obviously  $O(R) = \infty$ .

LEMMA 4. A ring with property  $P$ , containing divisors of zero and being of characteristic 0, cannot have an algebraically closed additive group.

Proof. We suppose that  $R$ , being a ring with divisors of zero and having an algebraically closed additive group  $R^+$ , is of characteristic 0. Then  $(Rb)^+$  cannot be simultaneously cyclic and algebraically closed, and therefore  $Rb = 0$  or else  $Rb = R$ . By Lemma 1 and by our hypothesis there exists a  $z \neq 0$  right-annihilator of  $R$ , i. e.,  $Rz = 0$ . Then the set  $Z \neq 0$  of all right-annihilators of  $R$  is a two-sided ideal in  $R$ , whose additive group  $Z^+$  is a serving subgroup in  $R^+$ . The ideal  $Z$  cannot be cyclic, since  $Z^+$  is likewise algebraically closed, therefore  $R = Z$  and  $R$  is a zero-ring. But in an algebraically closed group there exists a subgroup which is not cyclic, and this contradiction proves our Lemma.

LEMMA 5. Let  $F$  be a (finite or infinite) elementary  $p$ -ring for which  $F^2 \neq 0$  and  $O(F) \geq p^2$ , and let moreover  $\mathcal{I}$  be a two-sided ideal of order  $p$  in a ring  $R$ . If  $R/\mathcal{I} \simeq F$ , then  $R$  is without property  $P$ .

Proof. We shall assume that  $R$  has the property  $P$  and we shall show a contradiction. The complete endomorphism ring of  $\mathcal{I}^+$  has the order  $p$ , and by the endomorphism  $j \mapsto j e_r = j r$  ( $j \in \mathcal{I}$ ,  $r \in R$ ) of  $\mathcal{I}^+$  we have a ring-theoretical homomorphism  $r \mapsto e_r$  of  $R$  into the complete endomorphism ring of  $\mathcal{I}^+$ . The kernel of this mapping  $r \mapsto e_r$  is an ideal  $N$ , for which  $RN = 0$  and  $O(R/N) \leq p$  holds. Consequently by  $O(R) \geq p^2$  and by property  $P$  obviously  $R = N$ ; therefore  $R$  is a zero-ring. But then likewise  $R/\mathcal{I} \simeq F$  is a zero ring, which contradicts our hypothesis.

Now we give an elementary

**Proof of Theorem.** Let  $R$  be a ring with property  $P$ . By Lemma 3 we can suppose the existence of divisors of zero. If  $R$  contains an element of infinite order, then by Lemma 2 and 4 there exists a number  $n \in J$  for which  $0 \subset nRCR$ . But by  $R^+ \cong (nR)^+$  and by property  $P$ ,  $R$  is cyclic.

If  $R^+$  is a torsion group, then a ring theoretical direct decomposition  $R = \sum_p R_p$  holds, where the ideal  $R_p$  is generated by all elements of  $p$  power order of  $R$ . If  $R \neq R_p$ , then  $R$  is a finite cyclic ring. Now let  $R$  be a  $p$ -ring in which  $R'$  is generated by all elements of order  $p$  of  $R$ . If  $R' \neq R$ , then  $R$  is cyclic or else of type  $p^\infty$  because in both cases  $R'$  is cyclic [2]. Finally we assume that  $R' = R$ . By the existence of divisors of zero, by  $pR = 0$ , by Lemma 1 and by property  $P$  the existence of a left-ideal  $L$  of order  $p$  of  $R$  is necessarily ensured. Now we show the impossibility of  $O(R) \geq p^3$ . It is clear that  $Lr$  is a left-ideal in  $R$  ( $r \in R$ ). If there exists an element  $0 \neq r \in R$  for which  $Lr \neq 0$  and  $L \cap Lr = 0$  holds, then for the left-ideal  $D = \{L, Lr\}$  it is  $R = D$ , i. e.,  $O(R) = p^2$ . But if  $Lr \subseteq L$  for all  $r \in R$ , the subring  $L$  is a two-sided ideal in  $R$ . Then  $R/L$  has the property  $P$  and consequently has no proper left-ideals. By  $O(R) \geq p^3$  we can assume that  $R/L$  is a skew-field, and thus not a zero-ring, but has the property  $P$ . By  $O(R/L) \geq p^2$  and by Lemma 5 we have obtained a contradiction, which completes the proof.

### References

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