97. An Observation on the Brown-McCoy Radical

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We wish to characterize in this note the Brown-McCoy radical $G(A)$ of an associative ring $A$, as a radical $(1, 1, 1, 1)(A)$, $(1, 1, 1, 0)(A)$, $(1, 1, 0, 1)(A)$ and $(1, 2, 1, 1)(A)$, respectively, where $(k, l, m, n)(A)$ is a well-defined special $F$-radical of the ring $A$ in the sense of Brown-McCoy [3] for arbitrary nonnegative integers $k$, $l$, $m$ and $n$. The concept of a $(k, l, m, n)$-radical ring $A$ can be illustrated by the following elementary remarks. If the elements of $A$ form on the operation $a \cdot b = a + b - ab$ $(a, b \in A)$ a Neumann-regular semigroup (for instance in the case of a Jacobson-radicalring $A$, when $(A, 0)$ is a group), then $A$ is a $(k, 0, 1, 1)$-radicalring and a $(0, l, 1, 1)$-radicalring at the same time for any integers $k, l \geq 0$. Furthermore any $(k, l, m, n)$-semisimple ring $A$ with minimum condition on two-sided principal ideals is, as an $(A, A)$-doublemodule, completely reducible in a weak meaning, which generalizes the classical Wedderburn-Artin structure theorem also. (For the details of radicals, see [1], [2], [3].)

In this note the knowing of the results of Brown-McCoy [3] will be assumed for the reader. We denote the sum of all two-sided principal ideals $(a^{i+1} = a \cdot a^{i} = k \cdot a^{i+1})$ by $(k, l, m, n)(a)$, where $a$ is a fixed element, $X$ a varying element of $A$, $a \cdot b = a + b - ab$, $a^{i} = 0$, $a^{i+1} = a$, $a^{i+1} = a^{i+1} \cdot a$ and $k, l, m, n$ are nonnegative integers. An element $a \in A$ is called $(k, l, m, n)$-regular, if $a \in (k, l, m, n)(a)$. We call an element $a \in A$ strictly $(k, l, m, n)$-regular, if any element $b$ of the two-sided principal ideal $(a)$ generated by $a$ is $(k, l, m, n)$-regular. The set $(k, l, m, n)(A)$ of all strictly $(k, l, m, n)$-regular elements of $A$ is called the $(k, l, m, n)$-radical of $A$. This is evidently a special $F$-radical of $A$ [3]. The rings with $(k, l, m, n)$-radical $(0)$ are called $(k, l, m, n)$-semisimple. We call a subdirectly irreducible $(k, l, m, n)$-semisimple ring $A$ shortly $(k, l, m, n)$-primitive. An element $a \neq 0$ with the condition $(k, l, m, n)(a) = 0$ is called here a $(k, l, m, n)$-distinguished element of $A$. By [3] the $(k, l, m, n)$-radical of $A$ is the intersection of such ideals $\mathfrak{I}$ $(\tau \in \Gamma)$ of $A$, that the factorrings $A/\mathfrak{I}$, $(\tau \in \Gamma)$ are $(k, l, m, n)$-primitive. $A/(k, l, m, n)(A)$ is $(k, l, m, n)$-semisimple, and a subdirect sum of $(k, l, m, n)$-primitive rings. By [3] a subdirectly irreducible ring $A$ is $(k, l, m, n)$-primitive if and only if the minimal ideal $\mathfrak{M}$ of $A$ contains a $(k, l, m, n)$-distinguished element $d \neq 0$ playing the role of unity element in the case of radical
\[(1, 1, 1, 0)(A) = G(A) \text{ of } A.\]

Then holds the following

**Theorem.** An arbitrary \((k, l, m, n)\)-primitive ring \(P\) has no proper twosided ideals, and we have \((1 - d^{\alpha})P(1 - d^{\alpha}) = 0, d = kd \cdot d^{\alpha}\), \(kd^{\alpha} = d^{\alpha+\alpha}\) for a \((k, l, m, n)\)-distinguished element \(d \neq 0\) of \(P\). Furthermore \(G(A) = (1, 1, 1, 0)(A) = (1, 1, 0)(A) = (1, 1, 1)(A) = (1, 2, 1, 1)(A)\) are valid for the Brown-McCoy radical \(G(A)\) of an arbitrary (associative) ring \(A\).

**Proof.** If \(P\) is \((k, l, m, n)\)-primitive, then there exists \([8]\) a \((k, l, m, n)\)-distinguished element \(d \neq 0\) in the minimal ideal \(\mathcal{D} \neq 0\) of \(P\). We have from \((k, l, m, n)(d) = 0\) evidently \(d^{\alpha} = x \cdot d^{\alpha} = k \cdot d^{\alpha+\alpha}\) for any \(x \in P\). In the special case \(x = 0\) follows \(d^{\alpha+\alpha} = kd^{\alpha}\) and thus in the case of arbitrary \(x \in P\) is \(x = d^{\alpha} \cdot x + xd^{\alpha} = d^{\alpha} \cdot d^{\alpha} \in \mathcal{D}\) valid. Therefore one has \(P = \mathcal{D}\) for the \((k, l, m, n)\)-primitive rings \(P\), and thus \(P\) cannot have proper twosided ideals. Obviously follows also \((1 - d^{\alpha})P(1 - d^{\alpha}) = 0, d = kd \cdot d^{\alpha+\alpha}\) and \(d = kd \cdot d^{\alpha+\alpha}\) respectively. Let \(A\) be now an arbitrary associative ring. Then \((1, 1, 1, 1)(A) = G(A)\) will be proved by showing, that any \((1, 1, 1, 1)\)-primitive ring \(P\) is a simple ring with unity element, and a similar fact holds for other special \(k, l, m, n\) mentioned in the above theorem. In the four cases \(k, l, m, n\) mentioned above, \(k = 1\), hence \(d = d \cdot d^{\alpha}\) and \(d^{\alpha+\alpha} = d^{\alpha+\alpha+\alpha}\). If \(l = m = n = 1\), then one has \(d^{\alpha} = d\) for the \((k, l, m, n)\)-distinguished element \(d \neq 0\) of the \((k, l, m, n)\)-primitive ring \(P\). By \((1 - d)P(1 - d) = 0\) follows \(C = (1 - d)P + P(1 - d)P = 0\), since \(P\) is by \(d^{\alpha} = d \neq 0\) semi-simple in the sense of Jacobson, and the ideal \(C\) is nilpotent. Thus \((1 - d)P = 0, P = dP = d^{\alpha} = d\) and similarly \(P = Pd\) too. Therefore one has \((1, 1, 1, 1)(A) = G(A)\). If \(k = l = m = 1\) and \(n = 0\), immediately follows

\[(1, 1, 0)(A) = \sum_{a \in A} (a \cdot x \cdot a^{\alpha} - a) = \sum_{x \in A} (X - ax) = (1 - a)A + A(1 - a)A,\]

and thus \((1, 1, 1, 0)(A) = G(A)\) by the definition of the Brown-McCoy radical \(G(A)\) of \(A\) \([3]\). The case \(k = l = n = 1\) and \(m = 0\) is totally similar to the previous case. If \(k = m = n = 1\) and \(l = 2\), then one has \(d = d \cdot d^{\alpha}\) and thus \(d - 2d^{2} = 0\). Then by \(d = 2d^{2} - d^{3}\) \(P_{2} = 0\) is surely \(P_{1} \neq 0\), i.e. \(P\) is semisimple in the sense of Jacobson by the want of proper ideals. By \((1 - d)P(1 - d) = 0\) and \(P_{2} = 0\) follows \(C = (1 - d)P + P(1 - d)P = 0\), since \(C\) is a nilpotent twosided ideal of \(P\). This means \((1 - d)P = 0 = P = dP\). From \((d - d^{\alpha})P = (1 - d)dP = 0\) follows by \(P_{2} = 0\) evidently \(d^{2} = d\), for a Jacobson-semisimple ring we have no annihilator \(d\). Therefore \(d\) is a left unity element of \(P(= dP)\), and similarly one has \(P = Pd\) also, which proves the theorem.

**Remarks.** 1) Any \((k, l, m, n)\)-semisimple ring with minimum condition on twosided principal ideals is the discrete direct sum of \((k, l, m, n)\)-primitive rings (see for these rings the above theorem), and conversely.
2) If the elements of $A$ form with the operation $a \cdot b = a + b - ab$ a Neumann-regular semigroup, then $A$ is a $(k, 0, 1, 1)$-radicalring and a $(0, l, 1, 1)$-radicalring too.

3) It can be proved $A = (0, 0, 0, 0)(A) = (k, 0, 0, 1)(A) = (0, 0, 0, 1)(A) = (k, 0, 1, 0)(A) = (0, l, 1, 0)(A) = (2, 1, 1, 0)(A) = (2, 1, 0, 1)(A) = (2, 1, 1, 1)(A)$.

For instance, if $P$ is a $(2, 1, 1, 1)$-primitive ring, then holds $d^2 = 2d^1$ and $(1 - d) P(1 - d) = 0$, consequently $2d - d^2 = 2d$, $d^2 = 0$ and $0 \frac{d}{d} = d - 2d + d^2 = (1 - d)(1 - d)(1 - d)P(1 - d) = 0$, which is a contradiction. Therefore $P = 0$ and $(0, l, 1, 1)(A) = A$.

4) Any $(k, 0, 1, 1)$-primitive ring $P$ and any $(0, l, 1, 1)$-primitive ring $P$ are simple rings with unity element and with the condition $2P = P \setminus 0$.

5) Any $(3, 1, 1, 1)$-primitive ring, any $(3, 1, 1, 0)$-primitive ring and any $(3, 1, 0, 1)$-primitive ring $P$ are simple rings with unity element and with the condition $2P = 0$. Therefore for example a $(3, 1, 1, 1)$-primitive ring $P \setminus 0$ cannot be for instance a $(0, l, 1, 1)$-primitive ring.

6) We have seen $(1, 2, 1, 1)(A) = G(A)$. Then holds $(1, 2, 1, 1)(a) = ((1 - a)A(1 - a)) = (1 - a)A(1 - a) + A(1 - a)A(1 - a) + (1 - a)A(1 - a)A + A(1 - a)A(1 - a)A \supseteq W(a) = A(1 - a)A(1 - a)A$. The following $W$-regularity: $b \in W(b)$ determines a special $P$-radical $W(A)$ of $A$. If $P$ is a $W$-primitive ring $i.e.$ a $W$-semisimple and subdirectly irreducible ring, and if $P^2 = 0$, then $P$ is a simple ring with unity element. If $P$ is a $W$-primitive ring and if $P^2 = 0$, then the additive group $P$ is isomorphic to a group $C(p^r)$, where $1 \leq k \leq \infty$. If finally $P^2 \setminus 0$ but $P^3 = 0$, and $P$ is a $W$-primitive ring, then we have $P \supseteq P = 0$ for the minimal ideal $\mathfrak{I}$ of $P$ and $(P^2) \cong C(p^r)$ holds $(1 \leq k \leq \infty)$. For example $A = \{a, a_2, \ldots; b_i, b_i, \ldots\}$ with $a_i^2 = b_i = p = b_i^2 = b_i^2 = a_i = a_i$ $= 0$ is a $W$-primitive ring with $A^2 = 0$ and $A^2 \setminus 0$, $(A^2) \cong C(p^r)(i \setminus j)$.

7) Let $A$ be an associative ring, $M$ a right $A$-module and $\aleph$ an arbitrary cardinal number. An $A$-submodule $K$ of $M$ is called $\aleph$-homoperfect, if the following conditions are satisfied:

I) $MA + K = M$;
II) $M/K$ is a completely reducible $A$-module of dimension $< \aleph$;
III) $M/K$ has no proper $A$-submodule, which is invariant for all $A$-endomorphism of $M/K$;
IV) if $\varphi$ is an $A$-homomorphism of $M/L$ onto $M/K$ for an $A$-submodule $L$ with the conditions I), II) and III), then $\varphi$ is an isomorphism.

Let $\aleph(A)$ be now itself $M$, if $M$ has no proper $\aleph$-homoperfect submodules. If there exist in $M$ proper $\aleph$-homoperfect submodules $K_i(i \in I)$, then we define $\aleph(A) = \bigcap K_i$. In the case of $1 \in A$, a unitary
$A$-module $M$ and $\mathfrak{M}=2$; $\mathfrak{M}_n(M)$ is the Bourbaki-radical of $M$ [2], and in the case $\mathfrak{M}=2$ and arbitrary $A$ we obtain the Kertész-radical of $M$ [5]. We have proved solving in [6] a problem of Dr. A. Kertész [5] that the Jacobson-radical $\mathfrak{H}(A)$ of $A$ must not coincide with the radical $\mathfrak{M}_n(A)$ of the right $A$-module $A$, if the power $|A|$ of $A$ is no quadratfree finite cardinal number. We have generally only $\mathfrak{M}_n(A) \subseteq \mathfrak{H}(A)$. If in the ring $A$ with left unity element holds the minimum condition on principal right ideals [7] and $\mathfrak{M}=\mathfrak{Z}_n$, then one has evidently $\mathfrak{M}_n(A) \subseteq \mathfrak{H}(A)$ for the above radical $\mathfrak{M}_n(A)$ of the right $A$-module $A$ and the Brown-McCoy radical $G(A)$ of $A$. Now we arise the following

*Problem.* What is a necessary and sufficient condition concerning $A$ for the validity of $\mathfrak{M}_n(A)=G(A)$? (Solve a similar problem of A. Kertész on $\mathfrak{M}_n(A)$ and $\mathfrak{H}(A)$ too?)

References


*It may be remarked that the theory of $F$-radicals can be formulated for $A$-modules too, where $F$ is a well-defined mapping of any $A$-module $M$ onto a set of submodules $F(m)$ of $M \ (m \in M, F(m) \subseteq M)$ with the condition $F(m\gamma)=F(m\eta)$ for any $A$-homomorphism $\gamma$ of $M$. Then $m \in M$ is $F$-regular in the case $m \in F(m)$. Then the $F$-radical of $M$ is the set $\{m; m \in M, n \in F(n), n \in \{m\}\}$.}