Existence of periodic solutions in linear higher order system of difference equations

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Abstract

In this paper we investigate the existence of nontrivial periodic solutions of a higher order system of difference equations. Our framework is based on an earlier periodicity condition of us. We also use the theory of circulant matrices combined by a theorem of Silvester on the computation the determinant of block matrices. An illustrative application is given to show the effectiveness of of our framework and to point out the connection between our periodicity results and some known stability conditions.

Keywords: periodic solution, higher order difference equation, companion matrix, circulant matrix, eigenvalue and eigenvector, block matrix

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1 Introduction

Existence of periodic solutions of difference equations was studied e.g. in [1], [2], [4], [5] and [13]. For example, Grove and Ladas [5] present a series of results together with many questions and open problems for higher order difference equations having periodic solutions. Based on these works one may formulate the following question:

Is there any easily verifiable test that we can apply to determine whether or not a class of higher order difference equations has a p-periodic solution?

In our recent paper [7], we presented a linear algebraic framework for determining the existence of periodic solutions of systems of higher order linear difference equations with time dependent and independent coefficients. One of the main tools is a novel transformation, recently introduced in [6]. This formulates a given higher order difference equation, in a special way, as a first order recursion.

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In paper [8] the linear algebraic framework is complemented with some basic facts from the circulant matrix theory. This presents easily verifiable elementary necessary and sufficient conditions for the periodicity of the scalar equation

$$x(n) = \sum_{i=1}^{s} a_i x(n-i), \quad n \geq 0,$$

in terms of the scalars a_i $(0 \leq i \leq s)$ and the order s.

In this paper we extend the results of [8] to the case of systems of higher order linear difference equations. Specifically, we investigate equation

$$x(n) = \sum_{i=1}^{s} A_i x(n-i), \quad n \geq 0, \tag{1}$$

$s \geq 2$ is a given integer, and $A_i \in \mathbb{R}^{d\times d}$ $(1 \leq i \leq s)$ are given matrices.

The base of our investigation is an algebraic condition recently given by the authors (see [7]). This condition is formulated in terms of the eigenvalues of a block matrix, constructed from the coefficients $A_1, \ldots, A_s$. A matrix is called block matrix, if its entries are also matrices. In this paper the main tools, used by us, are the theory of circulant matrices and a theorem of Silvester [12] given for the computation of the determinants of block matrices.

In Section 4 we demonstrate the applicability of our main result on the following special case of (1)

$$x(n) = x(n-1) - Ax(n-s), \quad n \geq 0. \tag{2}$$

This equation is investigated very frequently in the literature. In the papers Levin and May [9] and Matsunaga and Hara [10] gave exact asymptotic stability conditions for equation (2) in scalar and two dimensional cases, respectively (see e.g. [9], [10] [11] and the reference therein). We recall that equation (2) is asymptotically stable if and only if all solutions tends to zero at infinity.

Intervals in $\mathbb{R}\cup\{-\infty,\infty\}$ will be denoted by $]a,b[$ (open intervals), $[a,b]$ (closed intervals), $[a,b[$ (left-closed, right-open intervals) and $]a,b]$ (left-open, right-closed intervals).

Theorem S *Consider equation (2).*

(a) (Levin and May [9]) If equation (2) is scalar, then it is asymptotically stable if and only if

$$A \in I := \left]0, 2\sin\left(\frac{\pi/2}{2s-1}\right)\right[. \tag{3}$$

(b) (Matsunaga and Hara [10]) Suppose that the matrix A is given by

$$A = \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}, \quad a_1, a_2, b \in \mathbb{R}. \tag{4}$$

Then equation (2) is asymptotically stable if and only if

$$(a_1, a_2) \in I \times I.$$

(c) (Matsunaga and Hara [10]) Suppose that the matrix A is given by

$$A = q \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix}, \quad q \in \mathbb{R} \setminus \{0\}, \quad 0 < |\vartheta| \le \frac{\pi}{2}. \tag{5}$$

Then equation (2) is asymptotically stable if and only if

$$(\vartheta, q) \in D,$$

where

$$D := \left\{ (\varphi, u) \in \mathbb{R}^2 \mid 0 < |\varphi| \le \frac{\pi}{2}, \quad 0 < u < 2\sin\left(\frac{\pi/2 - |\varphi|}{2s-1}\right) \right\}.$$

The next periodicity result is complementary to Theorem S. The boundary of a set S is denoted by ∂S.

Theorem 1.1 *Consider equation (2). The interval I is defined in (3).*

(a) Let equation (2) be scalar. Then equation (2) has a nontrivial periodic solution, if

$$A \in \partial I.$$

(b) Let A be given in (4). Then equation (2) has a nontrivial periodic solution, if

$$(a_1, a_2) \in \partial (I \times I).$$

(c) Let A be given in (5). Then equation (2) has a nontrivial periodic solution, if

$$(\vartheta, q) \in \partial D,$$

and

$$|\vartheta| = r\frac{\pi}{2}$$

with a rational number $r \in]0,1]$.

It is worth to note that in the cases (a) and (b), equation (2) has a nontrivial periodic solution in any point of the boundary of the asymptotic stability domain. On the other hand, in the case (c), equation (2) has a nontrivial periodic solution only in an everywhere dense subset of the boundary of the asymptotic stability domain.

Let $p \ge 1$ be an integer. $BV^{p,d}$ will mean the pd-dimensional real vector space of block vectors with entries in $\mathbb{R}^d$. The real vector space of $p \times p$ block matrices with entries in $\mathbb{R}^{d\times d}$ will be denoted by $BM^{p,d}$ ($BM^{p,d}$ and $\mathbb{R}^{pd\times pd}$ can be treated as being identical).

It is clear that the solutions of (1) are uniquely determined by their initial values

$$x(n) = \varphi(n), \quad -s \le n \le -1, \tag{6}$$

where $\varphi(n) \in \mathbb{R}^d$. The unique solution of (1) and (6) is denoted by $x(\varphi) = (x(\varphi)(n))_{n\ge -s}$, where the block vector $\varphi := (\varphi(-s), \ldots, \varphi(-1))^T \in BV^{s,d}$.

Definition 1.1 *(a) The zero matrix and the identity matrix in* $\mathbb{R}^{d\times d}$ *are denoted by* O *and* I*, respectively.*

(b) $\mathcal{O}$ *and* $\mathcal{I}$ *mean the zero matrix and the identity matrix in* $BM^{p,d}$*, respectively.*

We give some basic definitions about periodicity.

Definition 1.2 *Consider equation (1). Let* $\varphi = (\varphi(-s), \dots, \varphi(-1))^T \in BV^{p,d}$ *be a given initial vector.*

(a) The solution $x(\varphi) = (x(\varphi)(n))_{n\geq -s}$ *of (1) and (6) is called periodic if there exists a positive integer* p *such that* $x(\varphi)(n+p) = x(\varphi)(n)$ *for all* $n \geq -s$*. In this case we say that* $x(\varphi)$ *is p-periodic.*

(b) φ *is said to be a p-periodic initial vector of (1) if the solution* $x(\varphi)$ *of (1) and (6) is p-periodic.*

(c) We say that p is the prime period of the solution $x(\varphi)$ *of (1) and (6) if it is p-periodic and p is the smallest positive integer having this property.*

(d) A periodic solution of (1) is called nontrivial if it is different from the zero solution.

Our paper is essentially subdivided into five parts. The main result of this paper is stated in Section 3. It's proof is based on some earlier periodicity results of the authors, and a combination on some fundamental facts on circulant matrices, and a Theorem of Silvester [12] on the computation the determinant of a block matrix. These preliminary results can be found in Section 2. In Section 4 an illustrative application is given to show the effectiveness of our framework for higher order difference systems.

The proofs of our results can be found in Sectin 5.

2 Preliminary results

Let $n \geq 2$ be an integer, R is a ring, and $v := (v_0, v_1, \dots, v_{n-1}) \in R^n$. The circulant matrix associated to v is the $n \times n$ matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right. It is denoted by

$$\operatorname{circ}(v_0, v_1, \dots, v_{n-1}) := \begin{pmatrix} v_0 & v_1 & \dots & v_{n-2} & v_{n-1} \\ v_{n-1} & v_0 & \dots & v_{n-3} & v_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_1 & v_2 & \dots & v_{n-1} & v_0 \end{pmatrix}.$$

For a positive integer n denote $\rho_{n,0}, \dots, \rho_{n,n-1}$ be the nth roots of 1, that is

$$\rho_{n,l} := \exp\left(\frac{2\pi l}{n} i\right) = \cos\left(\frac{2\pi l}{n}\right) + i \sin\left(\frac{2\pi l}{n}\right), \quad 0 \leq l \leq n-1.$$

The following result for the existence of a nontrivial periodic solution of equation (1) can be found in [7].

Theorem A *(see [7]) Consider equation (1). Let $p \geq s$ be an integer. If $p = s$, let $B_i := A_i$ $(1 \leq i \leq s)$, while if $p > s$, let*

$$B_i := \begin{cases} A_i, & 1 \leq i \leq s \\ O, & s+1 \leq i \leq p \end{cases}.$$

(a) Equation (1) has a nontrivial p-periodic solution if and only if 1 is an eigenvalue of the block circulant matrix

$$\operatorname{circ}(B_p, B_{p-1}, \ldots, B_1) = \begin{pmatrix} B_p & B_{p-1} & \ldots & B_2 & B_1 \\ B_1 & B_p & \ldots & B_3 & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{p-1} & B_{p-2} & \ldots & B_1 & B_p \end{pmatrix}. \tag{7}$$

(b) If $\varphi = (\varphi(-s), \ldots, \varphi(-1))^T \in BV^{s,d}$ is a p-periodic initial vector of (1), and

$$\psi := (x(\varphi)(0), \ldots, x(\varphi)(p-s-1), \varphi(-s), \ldots, \varphi(-1))^T \in BV^{p,d},$$

then ψ is an eigenvector of (7) corresponding to the eigenvalue 1.

(c) Conversely, if $\psi = (\psi(-p), \ldots, \psi(-1))^T \in BV^{p,d}$ is an eigenvector of (7) corresponding to the eigenvalue 1, then $\varphi = (\psi(-s), \ldots, \psi(-1))^T \in BV^{s,d}$ is a p-periodic initial vector of (1).

We shall use the following three known results about matrices. The first one deals with determinants of block matrices.

Theorem B *(see [12]) Let R be a commutative subring of $\mathbb{R}^{d\times d}$, and let $M \in R^{p\times p}$ $\left(\subset BM^{p,d}\right)$. Since R is commutative there is a determinant for M, which is denoted by $\det_R(M)$ ($\det_R(M)$ is actually a matrix from $\mathbb{R}^{d\times d}$). Then*

$$\det_{\mathbb{R}}(M) = \det_{\mathbb{R}}(\det_R(M)).$$

Remark 2.1 *For the convenience of the reader, let us consider the block matrix*

$$M := \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where $M_{ij} \in \mathbb{R}^{d\times d}$ $(1 \leq i,j \leq 2)$ and $M_{ij}M_{kl} = M_{kl}M_{ij}$ $(1 \leq i,j,k,l \leq 2)$. Then by Theorem B

$$\det(M) = \det_{\mathbb{R}}(M) = \det_{\mathbb{R}}(\det_R(M)) = \det_{\mathbb{R}}(M_{11}M_{22} - M_{12}M_{21}).$$

Next, the determinant of some circular matrices are calculated.

Theorem C *(see [3] or [14]) Let $v_0, \ldots, v_{n-1} \in \mathbb{R}$, where $n \geq 1$ is a fixed integer. Then*

$$\det(\mathit{circ}(v_0, v_1, \ldots v_{n-1})) = \prod_{l=0}^{n-1}\left(v_0 + \rho_{n,l}v_1 + \rho_{n,l}^2 v_2 \ldots + \rho_{n,l}^{n-1}v_{n-1}\right).$$

Corollary 2.1 *Let $n \geq 1$ be a fixed integer. If $A_0, \dots, A_{n-1} \in \mathbb{R}^{d\times d}$ such that $A_iA_j = A_jA_i$ $(0 \leq i, j \leq s)$, then*

$$\det\left(circ\left(A_0, A_1, \dots, A_{n-1}\right)\right)$$

$$= \prod_{l=0}^{n-1} \det\left(A_0 + \rho_{n,l}A_1 + \rho_{n,l}^2 A_2 \dots + \rho_{n,l}^{n-1}A_{n-1}\right).$$

The third result deals with the eigenvectors of block circulant matrices.

Theorem D *(see [3] or [14]) Let $A_0, \dots, A_{n-1} \in \mathbb{R}^{d\times d}$, where $n \geq 1$ is a fixed integer. Define for some $l \in \{0, \dots, n-1\}$ the matrix*

$$H_l := \sum_{i=0}^{n-1} \rho_{n,l}^i A_i \in \mathbb{R}^{d\times d}.$$

If λ is an eigenvalue of H_l and v is an eigenvector corresponding to λ, then λ is also an eigenvalue of $circ(A_0, A_1, \dots, A_{n-1})$ with eigenvector

$$\left(v, \rho_{n,l}v, \dots, \rho_{n,l}^{n-1}v\right)^T \in BV^{n,d}.$$

3 Main result

In the paper [8] we have proved easily verifiable elementary necessary and sufficient conditions for the periodicity of the solutions of higher order scalar difference equations. In this paper we demonstrate that some of the results formulated for scalar equations can be generalized for systems assuming the commutativity of the coefficient matrices.

Now we state our main theorem which is proved in Section 5.

Theorem 3.1 *Consider equation (1). Assume*

$$A_iA_j = A_jA_i, \quad 1 \leq i, j \leq s.$$

Let $p \geq s$ be an integer. Equation (1) has a nontrivial p-periodic solution if and only if there exists an integer $l \in \{0, 1, \dots, p-1\}$ such that

(a) in case $p = s$

$$\det\left(A_s - I + \sum_{j=1}^{s-1} \rho_{s,l}^j A_{s-j}\right) = \det\left(A_s - I + \sum_{k=1}^{s-1} \rho_{s,l}^{-k} A_k\right) = 0,$$

(b) in case $p > s$

$$\det\left(-I + \sum_{j=p-s}^{p-1} \rho_{p,l}^j A_{p-j}\right) = \det\left(-I + \sum_{k=1}^{s} \rho_{p,l}^{-k} A_k\right) = 0.$$

4 Applications

We illustrate our main result by applying it to the special case of (1)

$$x(n) = x(n-1) - Ax(n-s), \quad n \geq 0, \tag{8}$$

where $s \geq 2$ is an integer and $A \in \mathbb{R}^{d\times d}$.

Consider equation (8). The matrices O, I, $A \in \mathbb{R}^{d\times d}$ commute, and therefore the following result is an immediate consequence of Theorem 3.1.

Corollary 4.1 *Let $p \geq s$ be an integer. Equation (8) has a nontrivial p-periodic solution if and only if there exists an integer $l \in \{0, 1, \ldots, p-1\}$ such that*

$$\det\left(A - \left(\rho_{p,l}^{s-1} - \rho_{p,l}^{s}\right) I\right) = 0.$$

Remark 4.1 *Since $\rho_{p,0} = 1$ $(p \geq 1)$, the previous result shows that if* $\det(A) = 0$*, then equation (8) has a nontrivial p-periodic solution for every $p \geq s$. Really,* $\det(A) = 0$ *implies that equation (8) has a constant solution $x(n) = c$ $(n \geq -s)$ for some $c \neq 0$.*

We can reword the previous corollary by using the eigenvalues of A.

Corollary 4.2 *Let $p \geq s$ be an integer. Equation (8) has a nontrivial p-periodic solution if and only if A has an eigenvalue λ (real or complex) such that*

$$\lambda = \rho_{p,l}^{s-1} - \rho_{p,l}^{s}$$

for some $l \in \{0, 1, \ldots, p-1\}$.

Now we give an explicit and exact condition to the existence of a nontrivial periodic solution of equation (8). $\mathbb{N}_+$ denotes the set of positive integers.

Theorem 4.1 *Equation (8) has a nontrivial periodic solution if and only if at least one of the following conditions holds:*

(a) A has a real eigenvalue λ such that

$$\lambda \in \left\{2(-1)^k \sin\left(\frac{\pi(2k+1)}{2(2s-1)}\right) \mid k = 0, \ldots, s-1\right\} \bigcup \{0\}. \tag{9}$$

(b) A has a complex eigenvalue $\lambda = q\exp(\vartheta i)$ such that either

$$|\vartheta| = \frac{\pi}{2}$$

and

$$q \in \left\{2(-1)^k \sin\left(\frac{k}{2s-1}\pi\right) \mid k = 1, \ldots, 2s-2\right\}, \tag{10}$$

or for some positive integers $\alpha < \beta$

$$|\vartheta| = \frac{\alpha}{\beta}\frac{\pi}{2}$$

and

$$q \in \left\{ 2\left(-1\right)^k \sin\left(\left(\frac{\beta-\alpha}{\beta\left(4s-2\right)} + \frac{k}{2s-1} \right)\pi \right) \mid k = 0, 1, \ldots, 2s-2 \right\}. \tag{11}$$

The notation $(a, b) = 1$ is used if $a, b \in \mathbb{N}_+$ are relative prime.

The next three results give further information on the periodic solutions of equation (8). The first concerns the real eigenvalues.

Theorem 4.2 *Consider equation (8).*

(a) If 0 *is an eigenvalue of* A*, then there is a nonzero constant solution.*

(b) Suppose that for some $k \in \{0, \ldots, s-1\}$ *the number*

$$\lambda = 2\left(-1\right)^k \sin\left(\frac{\pi\left(2k+1\right)}{2\left(2s-1\right)} \right)$$

is an eigenvalue of A*, and let* v *be an eigenvector corresponding to* λ*. Let*

$$\frac{\left(2k+1\right)}{2\left(2s-1\right)} = \frac{l}{p},$$

where l*,* $p \in \mathbb{N}_+$*,* $l < p$ *and* $(l, p) = 1$*. Define the block vectors*

$$\varphi_c := \left(\cos\left(\frac{2\pi l}{p}\left(s-1\right) \right) v, \ldots, \cos\left(\frac{2\pi l}{p} \right) v, v \right)^T \in BV^{s,d}$$

and

$$\varphi_s := \left(\sin\left(\frac{2\pi l}{p}\left(s-1\right) \right) v, \ldots, \sin\left(\frac{2\pi l}{p} \right) v, 0 \right)^T \in BV^{s,d}.$$

(b_1) If $k = s-1$*, then* $p = 2$*, while if* $k \neq s-1$*, then* p *is an even divisor of* $4s-2$ *greater or equal than* 6*.*

(b_2) If $k = s-1$*, then* $x(\varphi_c)$ *is a periodic solution with prime period* 2*. Moreover* $\varphi_s = 0$*.*

(b_3) If $k \neq s-1$*, then* $x(\varphi_c)$ *and* $x(\varphi_s)$ *are both periodic solutions with prime period* p*.*

(b_4) If $k \neq s-1$ *and* $2s-1$ *is prime, then the prime period of* $x(\varphi_c)$ *and* $x(\varphi_s)$ *is* $4s-2$*.*

Next, we deal with the case of complex eigenvalues in two separate theorems.

Theorem 4.3 *Consider equation (8). Suppose that for some* $k \in \{1, \ldots, 2s-2\}$ *the number*

$$\lambda = 2\left(-1\right)^k \sin\left(\frac{k}{2s-1}\pi \right) \exp\left(\frac{\pi}{2} i \right)$$

is an eigenvalue of A*, and let* $v = v_1 + v_2 i$ *be an eigenvector corresponding to* λ*. Let*

$$\frac{k}{2s-1} = \frac{l}{p},$$

where $l,\ p \in \mathbb{N}_+$, $l < p$ *and* $(l,p) = 1$. *Define the block vectors*

$$\varphi_1 := \left(\cos\left(\frac{2\pi l}{p}(s-1)\right)v_1 + \sin\left(\frac{2\pi l}{p}(s-1)\right)v_2, \ldots,\right. \tag{12}$$

$$\left.\cos\left(\frac{2\pi l}{p}\right)v_1 + \sin\left(\frac{2\pi l}{p}\right)v_2, v_1\right)^T \in BV^{s,d}$$

and

$$\varphi_2 := \left(-\sin\left(\frac{2\pi l}{p}(s-1)\right)v_1 + \cos\left(\frac{2\pi l}{p}(s-1)\right)v_2, \ldots,\right. \tag{13}$$

$$\left.-\sin\left(\frac{2\pi l}{p}\right)v_1 + \cos\left(\frac{2\pi l}{p}\right)v_2, v_2\right)^T \in BV^{s,d}.$$

(a) p *is a (odd) divisor of* $2s-1$ *greater or equal than* 3.

(b) $x(\varphi_1)$ *and* $x(\varphi_2)$ *are both periodic solutions with prime period* p.

Theorem 4.4 *Consider equation (8). Suppose that for some* $k \in \{0, 1, \ldots, 2s-2\}$ *and* $\alpha, \beta \in \mathbb{N}_+$ $(\alpha < \beta,\ (\alpha,\beta) = 1)$ *the number*

$$\lambda = 2(-1)^k \sin\left(\left(\frac{\beta-\alpha}{\beta(4s-2)} + \frac{k}{2s-1}\right)\pi\right)\exp\left(\frac{\alpha}{\beta}\frac{\pi}{2}i\right)$$

is an eigenvalue of A, *and let* $v = v_1 + v_2 i$ *be an eigenvector corresponding to* λ. *Let*

$$\frac{\beta-\alpha}{\beta(4s-2)} + \frac{k}{2s-1} = \frac{l}{p}, \tag{14}$$

where $l,\ p \in \mathbb{N}_+$, $l < p$ *and* $(l,p) = 1$. *Consider the block vectors (12) and (13) with* p, l *defined in (14).*

(a) p *is greater or equal than* 3, *and* p *has the form* $p = t\beta$, *where* t *is a positive divisor of* $4s-2$.

(b) $x(\varphi_1)$ *and* $x(\varphi_2)$ *are both periodic solutions with prime period* p.

Remark 4.2 *For every odd integer* $p \geq 3$ *there is an equation of the form (8) which has a* p*-periodic solution with prime period* p. *Really, since* p *is odd, either* $p+1$ *or* $p+4$ *can be divided by* 4.

Suppose 4 *divides* $p+1$, *and let* $s := \frac{3(p+1)}{4}$, $k := 2s-3$, $\beta := p$ *and* $\alpha := 1$ *in (14). Then*

$$\frac{\beta-\alpha}{\beta(4s-2)} + \frac{k}{2s-1} = \frac{p-1}{p(3p+1)} + \frac{3(p+1)-6}{3p+1} = \frac{p-1}{p}.$$

Suppose 4 *divides* $p+3$, *and let* $s := \frac{5p+3}{4}$, $k := 2s-4$, $\beta := p$ *and* $\alpha := 1$ *in (14). Then*

$$\frac{\beta-\alpha}{\beta(4s-2)} + \frac{k}{2s-1} = \frac{p-1}{p(5p+1)} + \frac{5p+3-8}{5p+1} = \frac{p-1}{p}.$$

Corollary 4.3 *Consider equation (8).*

(a) The minimal positive number in the set (9) is

$$2 \sin \left(\frac{\pi/2}{2s-1} \right).$$

(b) The minimal positive q either in the set (10) or in the set (11) belonging to a fixed $\vartheta = r\frac{\pi}{2}$ ($r \in]0,1]$) is

$$2 \sin \left(\frac{\pi/2 - |\vartheta|}{2s-1} \right).$$

5 Proofs

Proof of Theorem 3.1. By Theorem A, equation (1) has a nontrivial p-periodic solution if and only if 1 is an eigenvalue of the block circular matrix (7). This is equivalent with

$$\det \left(\operatorname{circ} \left(B_p - I, B_{p-1}, \ldots, B_1 \right) \right) = 0. \tag{15}$$

It follows from $A_i A_j = A_j A_i$ $(1 \le i, j \le s)$ that the matrices $B_p - I, B_{p-1}, \ldots B_1$ also commute, and therefore Theorem B and Corollary 2.1 show that

$$\det \left(\operatorname{circ} \left(B_p - I, B_{p-1}, \ldots B_1 \right) \right) = \det \left(\prod_{l=0}^{p-1} \left(B_p - I + \sum_{j=1}^{p-1} \rho_{p,l}^{j} B_{p-j} \right) \right)$$

$$= \prod_{l=0}^{p-1} \det \left(B_p - I + \sum_{j=1}^{p-1} \rho_{p,l}^{j} B_{p-j} \right).$$

Hence, (15) holds if and only if there exists an integer $l \in \{0, 1, \ldots, p-1\}$ such that

$$\det \left(B_p - I + \sum_{j=1}^{p-1} \rho_{p,l}^{j} B_{p-j} \right) = 0.$$

By using the forms of the $B_1, \ldots, B_p$ matrices, we have the result.

The proof is complete. ■

We need the integer part and the ceiling functions: Let $x \in \mathbb{R}$. $[x]$ denotes the largest integer not greater than x, while $\lceil x \rceil$ means the smallest integer not less than x. The fractional part, denoted by $\{x\}$, is defined by $\{x\} := x - [x]$.

Proof of Theorem 4.1. If equation (8) has a nontrivial periodic solution, then it has a nontrivial p-periodic solution with some $p \ge s$.

Let $p \ge s$ be an integer. By Corollary 4.2, equation (8) has a nontrivial p-periodic solution if and only if A has an eigenvalue λ (real or complex) such that

$$\lambda = \rho_{p,l}^{s-1} - \rho_{p,l}^{s} \tag{16}$$

for some $l \in \{0, 1, \ldots, p-1\}$.

Suppose λ is real.

We have from (16) that

$$\sin\left(\frac{2\pi l\,(s-1)}{p}\right) - \sin\left(\frac{2\pi ls}{p}\right) = 0,$$

and thus

$$2\cos\left(\frac{\pi l\,(2s-1)}{p}\right)\sin\left(-\frac{\pi l}{p}\right) = 0.$$

This yields by $l \in \{0, 1, \ldots, p-1\}$, that either $l = 0$ or

$$\cos\left(\frac{\pi l\,(2s-1)}{p}\right) = 0. \tag{17}$$

If $l = 0$, then $\lambda = 0$.

(17) means that

$$\frac{\pi l\,(2s-1)}{p} = \frac{\pi}{2} + k\pi$$

for some integer k. Therefore

$$\frac{l}{p} = \frac{2k+1}{2\,(2s-1)}. \tag{18}$$

This and $l \in \{0, 1, \ldots, p-1\}$ together give

$$0 \leq k \leq 2s-2.$$

If (18) holds, then by (16)

$$\lambda = \cos\left(\frac{2\pi l\,(s-1)}{p}\right) - \cos\left(\frac{2\pi ls}{p}\right)$$

$$= -2\sin\left(\frac{\pi l\,(2s-1)}{p}\right)\sin\left(-\frac{\pi l}{p}\right) = 2\,(-1)^k \sin\left(\frac{\pi\,(2k+1)}{2\,(2s-1)}\right).$$

We have different values of λ if $k = 0, \ldots, s-1$, and every such k is possible providing $p = 4s-2$ in (18).

From the above establishments (a) follows.

Suppose λ is complex.

Then $\lambda = q\exp(\vartheta i)$, where $q \in \mathbb{R} \setminus \{0\}$ and $0 < |\vartheta| \leq \frac{\pi}{2}$.

(16) gives

$$q\exp(\vartheta i) = \rho_{p,l}^{s-1} - \rho_{p,l}^{s},$$

and therefore

$$q = \exp\left(\left(\frac{2\pi l\,(s-1)}{p} - \vartheta\right)i\right) - \exp\left(\left(\frac{2\pi ls}{p} - \vartheta\right)i\right)$$

$$= -2\sin\left(\frac{\pi l\,(2s-1)}{p} - \vartheta\right)\sin\left(-\frac{\pi l}{p}\right)$$
$$+2i\cos\left(\frac{\pi l\,(2s-1)}{p} - \vartheta\right)\sin\left(-\frac{\pi l}{p}\right).$$

Since $q \in \mathbb{R}$
$$\cos\left(\frac{\pi l\,(2s-1)}{p} - \vartheta\right)\sin\left(-\frac{\pi l}{p}\right) = 0.$$
According to $l \in \{0, 1, \ldots, p-1\}$, either $l = 0$ or
$$\cos\left(\frac{\pi l\,(2s-1)}{p} - \vartheta\right) = 0.$$
Because $q \neq 0$, it is enough to consider the second case, which implies that
$$\frac{\pi l\,(2s-1)}{p} - \vartheta = \frac{\pi}{2} + k\pi$$
for some integer k. This insures that
$$\vartheta = \frac{\pi l\,(2s-1)}{p} - \frac{\pi}{2} - k\pi. \tag{19}$$

Taking account of $0 < |\vartheta| \leq \frac{\pi}{2}$ and $l \in \{1, \ldots, p-1\}$, it follows from (19) that either (in case $0 < \vartheta \leq \frac{\pi}{2}$)
$$1 + 2k < \frac{l\,(4s-2)}{p} \leq 2 + 2k, \quad k \in \{0, 1, \ldots, 2s-2\} \tag{20}$$
or (in case $-\frac{\pi}{2} \leq \vartheta < 0$)
$$2k \leq \frac{l\,(4s-2)}{p} < 1 + 2k, \quad k \in \{0, 1, \ldots, 2s-2\}. \tag{21}$$
(20) shows that $\left\lceil \frac{l(4s-2)}{p} \right\rceil$ must be even, and then, by (19)
$$\vartheta = \frac{\pi}{2}\left(\frac{l\,(4s-2)}{p} - \left\lceil \frac{l\,(4s-2)}{p} \right\rceil + 1\right), \quad l \in \{1, \ldots, p-1\}.$$
Similarly, (21) shows that $\left\lfloor \frac{l(4s-2)}{p} \right\rfloor$ must be even, and then
$$\vartheta = \frac{\pi}{2}\left(\frac{l\,(4s-2)}{p} - \left\lfloor \frac{l\,(4s-2)}{p} \right\rfloor - 1\right), \quad l \in \{1, \ldots, p-1\}.$$

Since A is real, $\bar{\lambda} = q\exp(-\vartheta i)$ is also an eigenvalue of A. It now follows that either $\left\lceil \frac{l(4s-2)}{p} \right\rceil$ must be even, and then
$$|\vartheta| = \frac{\pi}{2}\left(\frac{l\,(4s-2)}{p} - \left\lceil \frac{l\,(4s-2)}{p} \right\rceil + 1\right), \quad l \in \{1, \ldots, p-1\},$$

or $\left[\frac{l(4s-2)}{p}\right]$ must be even, and then

$$|\vartheta| = \frac{\pi}{2}\left(\left[\frac{l\,(4s-2)}{p}\right] - \frac{l\,(4s-2)}{p} + 1\right), \quad l \in \{1, \ldots, p-1\},$$

Let $l \in \{1, \ldots, p-1\}$. It is easy to see that $\left\lceil\frac{l(4s-2)}{p}\right\rceil$ is even if and only if $\left[\frac{(p-l)(4s-2)}{p}\right]$ is even, and

$$\frac{\pi}{2}\left(\frac{l\,(4s-2)}{p} - \left\lceil\frac{l\,(4s-2)}{p}\right\rceil + 1\right)$$
$$= \frac{\pi}{2}\left(\left[\frac{(p-l)\,(4s-2)}{p}\right] - \frac{(p-l)\,(4s-2)}{p} + 1\right).$$

This implies that the previous two conditions for ϑ can be written in a common form: $\left[\frac{l(4s-2)}{p}\right]$ must be even, and then

$$|\vartheta| = \frac{\pi}{2}\left(\left[\frac{l\,(4s-2)}{p}\right] - \frac{l\,(4s-2)}{p} + 1\right), \quad l \in \{1, \ldots, p-1\}.$$

For every possible ϑ

$$q = 2\,(-1)^{\frac{1}{2}\left[\frac{l(4s-2)}{p}\right]} \sin\left(\frac{\pi l}{p}\right).$$

Summarized, for a complex $\lambda = q\exp(\vartheta i)$, (8) has a nontrivial periodic solution if and only if there exists a rational number $r \in]0,1[$ such that $[r\,(4s-2)]$ is even,

$$|\vartheta| = \frac{\pi}{2}\,(1 - \{r\,(4s-2)\}) \tag{22}$$

and

$$q = 2\,(-1)^{\frac{1}{2}[r(4s-2)]} \sin(r\pi). \tag{23}$$

Assume $|\vartheta| = \frac{\pi}{2}$. Then (22) holds with $r = \frac{1}{2s-1}$. Moreover, $r \in]0,1[$ and $[r\,(4s-2)] = 0$ is even.

Assume $|\vartheta| = \frac{\alpha}{\beta}\frac{\pi}{2}$, where $\alpha, \beta \in \mathbb{N}_+$ and $\alpha < \beta$. It follows from (22) that

$$\{r\,(4s-2)\} = \frac{\beta-\alpha}{\beta}. \tag{24}$$

If $r = \frac{\beta-\alpha}{\beta(4s-2)}$, then $r \in]0,1[$, $[r\,(4s-2)] = 0$ is even and (24) is satisfied.

These two alternatives give that either $|\vartheta| = \frac{\pi}{2}$ or

$$|\vartheta| \in \left\{\frac{\alpha}{\beta}\frac{\pi}{2} \mid \alpha, \beta \in \mathbb{N}_+, \quad \alpha < \beta\right\}.$$

Next, we would like to characterize those rational numbers from $]0,1[$ which define the same $|\vartheta|$. To study this problem, let r_1, $r_2 \in \,]0,1[$ such that $r_1 \neq r_2$, $[r_i\,(4s-2)]$ is even $(i=1,2)$ and

$$\{r_1\,(4s-2)\} = \{r_2\,(4s-2)\}\,.$$

This implies that $|r_1 - r_2|\,(4s-2)$ is an even positive integer. Let $|r_1 - r_2| = \frac{a}{b}$, where $a, b \in \mathbb{N}_+$, $a < b$, and $(a,b) = 1$. It follows that b is a divisor of $4s-2$. If b is even, then a is odd, and therefore $|r_1 - r_2|\,(4s-2) = \frac{a(2s-1)}{b/2}$ is also odd. Thus b is an odd divisor of $4s-2$ greater than 1. Since the odd divisors of $4s-2$ are exactly the divisors of $2s-1$, hence $|r_1 - r_2|$ has the form $\frac{k}{2s-1}$ for some $k = 1, \ldots, 2s-2$. Every $k = 1, \ldots, 2s-2$ is possible because $\frac{k}{2s-1}\,(4s-2) = 2k$ is even.

We have seen that if $|\vartheta| = \frac{\pi}{2}$, then $r = \frac{1}{2s-1}$ defines $|\vartheta|$ in (22), while if $|\vartheta| = \frac{\alpha}{\beta}\frac{\pi}{2}$ $(\alpha, \beta \in \mathbb{N}_+$ and $\alpha < \beta)$, then $|\vartheta|$ belongs to $r = \frac{\beta-\alpha}{\beta(4s-2)}$ in (22). Since $0 < \frac{\beta-\alpha}{\beta(4s-2)} < \frac{1}{4s-2}$, an application of the assumptions in the previous paragraph yields:

(i) If $|\vartheta| = \frac{\pi}{2}$, then $|\vartheta|$ is determined in (22) exactly the numbers

$$\left\{ \frac{k}{2s-1} \mid k = 1, \ldots, 2s-2 \right\}.$$

(ii) If $|\vartheta| = \frac{\alpha}{\beta}\frac{\pi}{2}$ $(\alpha, \beta \in \mathbb{N}_+$ and $\alpha < \beta)$, then $|\vartheta|$ is determined in (22) exactly the numbers

$$\left\{ \frac{\beta-\alpha}{\beta\,(4s-2)} + \frac{k}{2s-1} \mid k = 0, 1, \ldots, 2s-2 \right\}.$$

Finally, we have to give the different q's in (23) for a fixed ϑ. If $|\vartheta| = \frac{\pi}{2}$, and $|\vartheta|$ is determined by $\frac{k}{2s-1}$ for some $k = 1, \ldots, 2s-2$, then by (23)

$$q = 2\,(-1)^k \sin\left(\frac{k}{2s-1}\pi\right).$$

It is easy to see that all these numbers are different.

If $|\vartheta| = \frac{\alpha}{\beta}\frac{\pi}{2}$ $(\alpha, \beta \in \mathbb{N}_+$ and $\alpha < \beta)$, and $|\vartheta|$ is determined by $\frac{\beta-\alpha}{\beta(4s-2)} + \frac{k}{2s-1}$ for some $k = 0, 1, \ldots, 2s-2$, then by (23)

$$q = 2\,(-1)^k \sin\left(\left(\frac{\beta-\alpha}{\beta\,(4s-2)} + \frac{k}{2s-1}\right)\pi\right).$$

It is also not too hard to check that all these numbers are different.

The proof is now complete. ■

Proof of Theorem 4.2. (a) See Remark 4.1.

(b) It follows from the proof of Theorem 4.1 (a) that

$$\lambda = \rho^{s-1}_{4s-2,2k+1} - \rho^{s}_{4s-2,2k+1},$$

and equation (8) has a $(4s-2)$-periodic solution. For simplicity, let $\rho = \rho_{4s-2,2k+1}$.

Since

$$\left(A - \left(\rho^{s-1} - \rho^{s}\right) I\right) v = 0$$

is equivalent with

$$\left(-\rho^{4s-2-s} A + \rho^{4s-2-1} I\right) v = v,$$

we have from the construction of the block circular matrix

$$\operatorname{circ}\left(B_{4s-2}, B_{4s-3}, \ldots, B_{1}\right) \tag{25}$$

in (7) with $p = 4s-2$, and from Theorem D that

$$\left(v, \rho v, \ldots, \rho^{4s-3} v\right)^{T}$$

is a complex eigenvector of (25) corresponding to the eigenvalue 1. The matrix (7) is real, and hence

$$\left(v, \cos\left(\frac{\pi\left(2k+1\right)}{2s-1}\right) v, \ldots, \cos\left(\frac{\pi\left(2k+1\right)}{2s-1}\left(4s-3\right)\right) v\right)^{T} \tag{26}$$

and

$$\left(v, \sin\left(\frac{\pi\left(2k+1\right)}{2s-1}\right) v, \ldots, \sin\left(\frac{\pi\left(2k+1\right)}{2s-1}\left(4s-3\right)\right) v\right)^{T} \tag{27}$$

are real eigenvectors of (25) corresponding to the eigenvalue 1.

With the help of Theorem A (c) this yields that

$$\varphi_{c} = \left(\cos\left(\frac{\pi\left(2k+1\right)}{2s-1}\left(s-1\right)\right) v, \ldots, \cos\left(\frac{\pi\left(2k+1\right)}{2s-1}\right) v, v\right)^{T} \in BV^{s,d}$$

and

$$\varphi_{s} = \left(\sin\left(\frac{\pi\left(2k+1\right)}{2s-1}\left(s-1\right)\right) v, \ldots, \sin\left(\frac{\pi\left(2k+1\right)}{2s-1}\right) v, 0\right)^{T} \in BV^{s,d}$$

are $(4s-2)$-periodic initial vectors.

(b_1) If $k = s-1$, then $\frac{(2k+1)}{2(2s-1)} = \frac{1}{2}$, and thus $p = 2$. Suppose $0 \leq k < s-1$. Since $2k+1$ is odd, p is even. Because $2k+1$ and $2s-1$ are odd, $p \geq 3$. The result now follows.

(b_2) Let $k = s-1$. Then

$$\varphi_{c} = \left(\left(-1\right)^{s-1} v, \left(-1\right)^{s-2} v, \ldots, v\right)^{T}$$

and $\varphi_{s} = 0$, which give the claim.

(b_3) By Theorem A (b), it is enough to prove that (26) and (27) are p-periodic block vectors and p is the smallest period.

Consider the block vector (26). Since

$$\cos\left(\frac{\pi\left(2k+1\right)}{2s-1}t\right)=\cos\left(\frac{2\pi l}{p}t\right),\quad 0\le t\le 4s-3,$$

we have to prove that the function $x\to\cos\left(\frac{2\pi l}{p}x\right)$ $(x\in\mathbb{R})$ is p-periodic with prime period p. This is clear, by $(l,p)=1$.

The other case can be proved similarly.

(b_4) Because $2s-1$ is prime, $p=4s-2$, and therefore (d) can be applied.

The proof is complete. ■

Proof of Theorem 4.3. (a) This is obvious.

(b) By the proof of Theorem 4.1 (b)

$$\lambda=\rho_{2s-1,k}^{s-1}-\rho_{2s-1,k}^{s},$$

and equation (8) has a $(2s-1)$-periodic solution.

Exactly as in the proof of Theorem 4.2, we have that

$$\left(v_1,\cos\left(\frac{2\pi k}{2s-1}\right)v_1-\sin\left(\frac{2\pi k}{2s-1}\right)v_2,\dots,\right.$$
$$\left.\cos\left(\frac{2\pi k}{2s-1}\left(2s-2\right)\right)v_1-\sin\left(\frac{2\pi k}{2s-1}\left(2s-2\right)\right)v_2\right)^T$$

and

$$\left(v_2,\sin\left(\frac{2\pi k}{2s-1}\right)v_1+\cos\left(\frac{2\pi k}{2s-1}\right)v_2,\dots,\right.$$
$$\left.\sin\left(\frac{2\pi k}{2s-1}\left(2s-2\right)\right)v_1+\cos\left(\frac{2\pi k}{2s-1}\left(2s-2\right)\right)v_2\right)^T$$

are real eigenvectors of

$$\operatorname{circ}\left(B_{2s-1},B_{2s-2},\dots,B_1\right)$$

(see (7)) corresponding to the eigenvalue 1. Therefore Theorem A (c) yields that

$$\varphi_1=\left(\cos\left(\frac{2\pi k}{2s-1}\left(s-1\right)\right)v_1+\sin\left(\frac{2\pi k}{2s-1}\left(s-1\right)\right)v_2,\dots,\right.$$
$$\left.\cos\left(\frac{2\pi k}{2s-1}\right)v_1+\sin\left(\frac{2\pi k}{2s-1}\right)v_2,v_1\right)^T\in BV^{s,d}$$

and

$$\varphi_2=\left(-\sin\left(\frac{2\pi k}{2s-1}\left(s-1\right)\right)v_1+\cos\left(\frac{2\pi k}{2s-1}\left(s-1\right)\right)v_2,\dots,\right.$$
$$\left.-\sin\left(\frac{2\pi k}{2s-1}\right)v_1+\cos\left(\frac{2\pi k}{2s-1}\right)v_2,v_2\right)^T\in BV^{s,d}$$

are $(2s-1)$-periodic initial vectors.

We can accomplish the proof as in Theorem 4.2 (b_3).

The proof is complete. ■

Proof of Theorem 4.4. (a) Since $(\beta, \beta-\alpha)=1$, it follows from

$$(\beta-\alpha+2\beta k)\,p=l\beta\,(4s-2) \tag{28}$$

that β is a divisor of p, that is $p=t\beta$ for some positive integer t.

$(l,p)=1$ implies that $(t,l)=1$, and hence by (28), $4s-2$ is divided by t.

Suppose $p=2$. Then $l=1$ and $\beta=2$, and thus (28) has the form

$$\alpha=4+4k-4s$$

which contradicts to $(\alpha,\beta)=1$.

(b) In this case Theorem 4.1 (b) shows that

$$\lambda=\rho^{s-1}_{\beta(4s-2),\beta-\alpha+2\beta k}-\rho^{s}_{\beta(4s-2),\beta-\alpha+2\beta k},$$

and equation (8) has a $(2s-1)$-periodic solution.

Now we can apply the same method as in the proof of Theorem 4.3 (b).

The proof is complete. ■

Proof of Corollary 4.3. It is easy to check them. ■

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