Partitioning 2-edge-colored Ore-type graphs by monochromatic cycles

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July 21, 2014

Abstract

Consider a graph G on n vertices satisfying the following Ore-type condition: for any two non-adjacent vertices x and y of G, we have $deg(x)+deg(y) > 3n/2$. We conjecture that if we color the edges of G with 2 colors then the vertex set of G can be partitioned to two vertex disjoint monochromatic cycles of distinct colors. In this paper we prove an asymptotic version of this conjecture.

1 Background, summary of results.

In this paper, we consider the problem of partitioning the vertices of edge-colored graphs into monochromatic cycles. For simplicity, a colored graph means an edge-

*Research supported in part by OTKA Grant No. K104343.

colored graph in this paper. In this context it is conventional to accept *empty graphs and one-vertex graphs* as a cycle (of any color) and also *any edge* as a cycle (in its color). With this convention one can define the *cycle partition number* of any colored graph G as the minimum number of vertex disjoint monochromatic cycles needed to cover the vertex set of G. For complete graphs, [6] posed the following conjecture.

Conjecture 1. *The cycle partition number of any t-colored complete graph K_n is t.*

The $t = 2$ case of this conjecture was stated earlier by Lehel in a stronger form, requiring that the colors of the two cycles must be different. After some initial results [2, 8], Łuczak, Rödl and Szemerédi [19] proved Lehel's conjecture for large enough n, which can be considered as a birth of certain advanced applications of the Regularity Lemma. A more elementary proof, still for large enough n, was obtained by Allen [1]. Finally, Bessy and Thomassé [5] found a completely elementary inductive proof for every n.

The $t = 3$ case of Conjecture 1 was solved asymptotically in [13]. Pokrovskiy [21] showed recently (with a nice elementary proof) that the path partition number of any 3-colored K_n is at most three (for any $n \geq 1$). Later Pokrovskiy [22] surprisingly found a counterexample to Conjecture 1 for all $t \geq 3$. However, in the counterexample all but one vertex can be covered by t vertex disjoint monochromatic cycles, so perhaps the following weaker statement holds.

Conjecture 2. *For every integer $t \geq 2$ there exists a constant $c = c(t)$ such that for any t-colored graph G there are t vertex disjoint monochromatic cycles of G that cover at least $n - c$ vertices.*

For general t, the best bound for the cycle partition number is $O(t \log t)$, see [9]. Note that it is far from obvious that the cycle partition number of K_n can be bounded by *any* function of t.

In [3] we addressed the extension of the cycle and path partition numbers from complete graphs to arbitrary graphs G.

Recently, Schelp [23] suggested in a posthumous paper to strengthen certain Ramsey problems from complete graphs to graphs of given minimum degree. In particular, he conjectured that with $m = R(P_n, P_n)$, minimum degree $3m/4$ is sufficient to find a monochromatic path P_n in any 2-colored graph of order m.[1] Influenced by this, in [3] we posed the following

Conjecture 3. *If G is an n-vertex graph with $\delta(G) > 3n/4$ then in any 2-edge-coloring of G, there are two vertex disjoint monochromatic cycles of different colors, which together cover $V(G)$.*

[1]Some progress towards this conjecture have been done in [14] and [4].

That is, the above mentioned Bessy-Thomassé result [5] would hold for graphs with minimum degree larger than $3n/4$. Note that the condition $\delta(G) \geq 3n/4$ is sharp (see [3]). Indeed, consider the following n-vertex graph, where $n = 4m$. We partition the vertex set into four parts A_1, A_2, A_3, A_4 with $|A_i| = m$. There are no edges from A_1 to A_2 and from A_3 to A_4. Edges in $[A_1, A_3], [A_2, A_4]$ are red and edges in $[A_1, A_4], [A_2, A_3]$ are blue, inside the classes any coloring is allowed. In such an edge-colored graph, there are no two vertex disjoint monochromatic cycles of *different colors* covering G, while the minimum degree is $3m - 1 = 3n/4 - 1$.

In [3] we proved Conjecture 3 in the following asymptotic sense.

Theorem 1. *For every $\eta > 0$, there is an $n_0(\eta)$ such that the following holds. If G is an n-vertex graph with $n \geq n_0$ and $\delta(G) \geq (\frac{3}{4} + \eta)n$, then every 2-edge-coloring of G admits two vertex disjoint monochromatic cycles of different colors covering at least $(1 - \eta)n$ vertices of G.*

The proof of Theorem 1 followed a method of Łuczak [18]. The crucial idea of this method is that "cycles" or "paths" in a statement to be proved are replaced by "connected matchings". In a *connected matching*, the edges of the matching are in the same component of the graph.[2] We prove first this weaker result, then we apply this to the cluster graph of a regular partition of the target graph. Through several technical details, the regularity of the partition is used to "lift back" the connected matching of the cluster graph to a path or cycle in the original graph.

In this paper we go one step further and consider graphs satisfying an Ore-type degree condition instead of a minimum degree condition. Here we call a degree condition Ore-type if it gives a lower bound on the degree sum for any two non-adjacent vertices. There has been a lot of efforts in trying to extend results from minimum degree conditions to Ore-type conditions. The first result of this type was proved by Ore [20]: If for any two non-adjacent vertices x and y of G, we have $deg(x) + deg(y) \geq n$, then G is Hamiltonian. Some other results of this type include for example [7] (Ore-type conditions for k-ordered Hamiltonian graphs), [15] (Ore-type results on equitable colorings) or [16] (Ore-type versions of Brooks' theorem).

Generalizing Conjecture 3 for graphs satisfying an Ore-type condition here we pose

Conjecture 4. *If G is an n-vertex graph such that for any two non-adjacent vertices x and y of G, we have $deg(x) + deg(y) > 3n/2$, then in any 2-edge-coloring of G, there are two vertex disjoint monochromatic cycles of different colors, which together cover $V(G)$.*

[2] When the edges are colored, a connected red matching is a matching in a red component.

Here we prove Conjecture 4 in the following asymptotic sense.

Theorem 2. *For every $\eta > 0$, there is an $n_0(\eta) = n_0$ such that the following holds. If G is an n-vertex graph with $n \geq n_0$ such that for any two non-adjacent vertices x and y of G, we have $deg(x) + deg(y) \geq (\frac{3}{2} + \eta)n$, then every 2-edge-coloring of G admits two vertex disjoint monochromatic cycles of different colors covering at least $(1 - \eta)n$ vertices of G.*

The proof follows the same method as outlined above. The relaxed version of Theorem 2 for connected matchings is stated and proved in Section 2 (Theorem 3).

2 Partitioning into connected matchings

In this section we prove the relaxed version of our theorem for connected matchings instead of cycles.

Theorem 3. *Let G be an n-vertex graph, where n is even and G satisfies the following Ore-type condition: for any two non-adjacent vertices x and y of G, we have $deg(x) + deg(y) \geq 3n/2$. If the edges of G are 2-colored with red and blue, then there exist a red connected matching and a vertex-disjoint blue connected matching, which together form a perfect matching of G.*

Proof: Let C_1 be a largest monochromatic component, say red. Let $D = V \setminus V(C_1)$.

Case 1: Assume $|V(C_1)| < n$, i.e. $D \neq \emptyset$. Let A be those vertices in C_1 that are adjacent to D by a blue edge. We claim that $A \cup D$ is a connected blue component. Assume to the contrary that there is a cut in blue $(A_1 \cup D_1, A_2 \cup D_2)$, where $|A_1 \cup D_2| \geq |A_2 \cup D_1|$. Now there is no edge between $V(C_1) \setminus A_2$ and D_2. There is no red edge by the definition of D and no blue edge by the assumption on the cut. Therefore if u is a vertex in $V(C_1) \setminus A_2$ and v is a vertex in D_2 (clearly both sets are non-empty), then $deg(u) + deg(v) \geq 3n/2$. On the other hand, v is non-adjacent to all vertices of $V(C_1) \setminus A_2$ and u is non-adjacent to D_2. Therefore $deg(u) + deg(v) \leq n - 1 - |D_2| + n - 1 - |V(C_1) \setminus A_2| \leq 2n - 2 - |D_2| - |V(C_1) \setminus A| - |A_1| < 2n - n/2 = 3n/2$, a contradiction (here we used the assumption on the size of $A_1 \cup D_2$).

Let C_2 be this blue component covering $D = V \setminus V(C_1)$. Let u be a vertex of $C_1 \setminus C_2$ and v be a vertex of $C_2 \setminus C_1$. Let $|V(C_1) \setminus V(C_2)| = p$ and $|V(C_2) \setminus V(C_1)| = q$, where $p \geq q > 0$ by the choice of C_1. There is no edge between u and v, in fact between $C_1 \setminus C_2$ and $C_2 \setminus C_1$. Therefore $deg(u) + deg(v) \geq 3n/2$. On the other hand $n - 1 - q \geq deg(u)$ and $n - 1 - p \geq deg(v)$. It yields $2n - 2 - (p + q) \geq deg(u) + deg(v) \geq 3n/2$. Therefore $p + q < n/2$ and $|V(C_1) \cap V(C_2)| > n/2$.

If $|V(C_1)| = n$, then define C_2 as a largest blue component in G. Now $p = |V(C_1) \setminus V(C_2)|$, $q = |V(C_2) \setminus V(C_1)| = 0$.

Case 2: $|V(C_1)| = n$ and $p \leq n/2$. Now $|V(C_1) \cap V(C_2)| \geq n/2$, just as above.

Therefore in what follows, we unify the proof for the two cases we described so far. Let G_1 be the graph, which we obtain from G by deleting the blue edges induced by $V(C_1) \setminus V(C_2)$ and the red edges induced by $V(C_2) \setminus V(C_1)$ (if these exist).

We claim there is a perfect matching in G_1. Assume the contrary. By Tutte's theorem there exists a set X of vertices in G_1 such that the number of odd components in $G_1 \setminus X$ is larger than $|X|$, which implies that $|X| < n/2$. Let all the components of $G_1 \setminus X$ (not just the odd ones) be $D_1, D_2, \ldots, D_\ell$ in increasing order of their size, $\ell \geq |X| + 1$. Note that $\ell \geq 2$ always holds, even for $X = \emptyset$, as n is even. Let $d_i = |V(D_i)|$ for $i = 1, \ldots, \ell$ and $x = |X|$.

We claim that $(V(C_1) \cap V(C_2)) \cap \left(\cup_{i=1}^{\ell} D_i\right) = \emptyset$. Assume to the contrary that $u \in C_1 \cap C_2$ and $u \in D_i$. Let v be a vertex in a different D_j (using $\ell \geq 2$). Clearly u and v are non-adjacent in G_1, but also in G since we have not deleted any edge adjacent to u. Therefore $deg_G(u) + deg_G(v) \geq 3n/2$. Notice $deg_{G_1}(u) = deg_G(u)$. Now subtract the number of deleted edges adjacent to v, which is at most p or q depending on the position of v. We get $deg_{G_1}(u) + deg_{G_1}(v) \geq n$, since both p and q are at most $n/2$.

On the other hand $deg_{G_1}(u) \leq d_i - 1 + x$ and $deg_{G_1}(v) \leq d_j - 1 + x$. Therefore $deg_{G_1}(u) + deg_{G_1}(v) \leq d_i + d_j + 2x - 2 \leq n - 1$, since $d_i + d_j + 2x - 1$ is at most the number of vertices. This contradiction implies that $(V(C_1) \cap V(C_2)) \subseteq X$. However, this is impossible since $|V(C_1) \cap V(C_2)| \geq n/2$ and $x < n/2$. Therefore G_1 contains a perfect matching.

Case 3: $|V(C_1)| = n$ and $p > n/2$, so the largest blue component has size at most $n/2$. Again we get G_1 from G by deleting the blue edges induced by $V(C_1) \setminus V(C_2)$. We claim again that there is a perfect matching in G_1 and use the same set-up as above. First we show that $V(C_2) \subseteq X$. As before, we select a hypothetical vertex u in $C_2 \cap D_i$ and a vertex v in a different component D_j. Clearly u and v are non-adjacent in G_1, but also in G since we have not deleted any edge adjacent to u. If there were at least $n/2$ blue edges adjacent to v, then we would find a blue component larger than C_2. Therefore $deg_G(v) - deg_{G_1}(v) < n/2$ and $deg_G(u) + deg_G(v) \geq 3n/2$ implies $deg_{G_1}(u) + deg_{G_1}(v) \geq n$. On the other hand, this is impossible since $deg_{G_1}(u) + deg_{G_1}(v) \leq d_i + d_j + 2x - 2 \leq n - 1$ as in the argument above. Therefore $V(C_2) \subseteq X$. This implies that there is no blue component larger than x.

Notice that any potential edge in G between two components of $G_1 \setminus X$ is a blue edge inside $C_1 \setminus C_2$ that was deleted. Let H be the graph formed by the vertices in $V \setminus X$, and these crossing blue edges in $C_1 \setminus C_2$. Since $x < n/2$, we have $|V(H)| > n/2$.

Case 3.a: Assume $x \leq n/4$.

We claim that H is connected in the blue graph. Otherwise there exists a blue cut (A, B) of H, where $A \cap D_1$ is non-empty as well as $B \cap \left(\cup_{i=2}^{\ell} D_i\right)$. Indeed, let us take a blue cut (A, B) of H, where $A \cap D_1$ is non-empty. If $B \cap \left(\cup_{i=2}^{\ell} D_i\right) = \emptyset$, then $B \subseteq D_1$ and we reverse the roles of A and B. Let u be a vertex in $A \cap D_1$ and v be a vertex in $B \cap D_i$ for some $i > 1$. Now u and v are non-adjacent vertices in G. Therefore $deg_G(u) + deg_G(v) \geq 3n/2$. Since the largest blue component has size at most x, there are at most $x - 1$ deleted blue edges at u or v. Therefore $3n/2 - 2x + 2 \leq deg_{G_1}(u) + deg_{G_1}(v) \leq d_1 - 1 + d_i - 1 + 2x \leq n - 1$. This contradicts $x \leq n/4$, so H is indeed connected in blue. But then this is a larger blue component than x, a contradiction, G_1 does have a perfect matching.

Case 3.b: Assume $n/4 < x \leq n/2$.

Here $d_1 \leq 2$, otherwise there would be too many vertices, since $n \geq x + d_1\ell \geq n/4 + 3(n/4 + 1) > n$, a contradiction.

We claim that H is connected in the blue graph. This again leads to a contradiction, since we have a larger blue component than x. Assume the contrary and let A be a blue component in H that intersects D_1 and $H|_{V(H) \setminus V(A)} \cap \left(\cup_{i=2}^{\ell} D_i\right)$ is non-empty. Again, if $(V(H) \setminus V(A) \subseteq D_1$, then we take a blue component in $V(H) \setminus V(A)$ and that will play the role of A. Let $u \in A \cap D_1$. Let $B = H|_{V(H) \setminus V(A)}$. Now (A, B) is a cut of H. Let $v \in B \cap D_i$, where $i > 1$ and i is as small as possible. Now u and v are non-adjacent in G and therefore $deg_G(u) + deg_G(v) \geq 3n/2$. On the other hand using the cut (A, B), we get: $deg_G(u) \leq n - |B| + d_1 - 1$ and $deg_G(v) \leq n - |A| + d_i - 1$. It implies $deg_G(u) + deg_G(v) \leq 2n - (|A| + |B|) + d_1 - 1 + d_i - 1 = n + x + d_1 - 1 + d_i - 1$. However this leads to a contradiction if $x + d_1 - 1 + d_i - 1 < n/2$. In what follows we prove this last inequality.

If $d_1 = 2$, then the inequality simplifies to $x + d_i < n/2$. Notice that $d_i \leq d_\ell$ and $d_\ell \leq n - 3x$, since $\ell \geq x + 1$ and $2 = d_1 \leq d_j$ for any j. Using this and $n/4 < x$ we get $x + d_i \leq x + d_\ell \leq n - 2x < n/2$.

The other possibility is $d_1 = 1$. We have to show $x + d_i - 1 < n/2$ or $x + d_i \leq n/2$. Actually note that in the above the inequality $x + |D_i \cap A| < n/2$ already leads to a contradiction, so it is sufficient to prove this. Firstly if $i = \ell$, then $A \supseteq \cup_{j=1}^{\ell-1} D_j$ by the choice of i. This is a contradiction if $|A| > x$, since now A is larger than a largest blue component C_2. The only exception is $d_j = 1$ for $1 \leq j \leq \ell - 1$ and $A \cap D_\ell = \emptyset$. However in that case $x + |D_i \cap A| = x + 0 < n/2$ holds. Secondly $i < \ell$. Now $\sum_{j \neq i, j \neq \ell} d_j \geq x - 1$ and $d_i + d_\ell \geq 2d_i$. If strict inequality holds in one of these, then we get the following: $n \geq x + \sum_{j \neq i, j \neq \ell} d_j + d_i + d_\ell > 2x - 1 + 2d_i$, which implies $n/2 \geq x + d_i$ as claimed. The only case left is $i = \ell - 1$, $1 = d_1, \ldots, d_{i-1}$ and $d_i = d_\ell$. However this is impossible since now the number of vertices is $2x - 1 + 2d_i$, but we started with an even n.

All these contradictions prove the existence of a perfect matching in G_1. Since the red and blue halves are both connected, we proved our theorem. $\square$

3 Applying the Regularity lemma.

As in many applications of the Regularity Lemma, one has to handle irregular pairs, that translates to exceptional edges in the reduced graph. A graph G on n vertices is *ε-perturbed* if at most $\varepsilon\binom{n}{2}$ of its edges are marked as exceptional (or perturbed). For a perturbed graph G, let G^- denote the graph obtained by removing all perturbed edges. First we need a perturbed version of Theorem 3. These perturbation arguments are fairly standard modifications of the original argument (see e.g. [14]). We give all details to be self-contained.

Theorem 4. *For every $\eta > 0$, there exist $n_0 = n_0(\eta)$ and $\varepsilon_0 = \varepsilon_0(\eta)(\ll \eta)$ such that the following holds. Suppose that $\varepsilon \leq \varepsilon_0$ and G is a 2-edge-colored ε-perturbed graph on $n \geq n_0$ vertices and G satisfies the following Ore-type condition: for any two non-adjacent vertices x and y of G, we have $deg(x) + deg(y) \geq (3/2 + \eta)n$. All but at most $6\sqrt{\varepsilon}n$ vertices of G can be covered by the vertices of a red connected matching and a vertex-disjoint blue connected matching in G^-.*

Proof: We may assume that n is sufficiently large and $\varepsilon \ll \eta$. Let us start by "trimming" the graph, i.e. by deleting those vertices of G that are adjacent to at least $\sqrt{\varepsilon}n$ exceptional edges. There are less than $\sqrt{\varepsilon}n$ such vertices. We may remove one more arbitrary vertex to guarantee that the number of remaining vertices is even. This way we get a slightly smaller graph G_ε on n' vertices, where n' is even. Secondly we delete the remaining exceptional edges to form the graph G_ε^-. In what follows we mimic the proof of Theorem 3 replacing G by G_ε^-.

Let C_1 be a largest monochromatic component in G_ε^-, say red. We have $n' > (1 - \sqrt{\varepsilon})n$. Let $D = V(G_\varepsilon^-) \setminus V(C_1)$.

Case 1: Assume $|V(C_1)| < n'$, i.e. $D \neq \emptyset$. Let A be those vertices in C_1 that are adjacent to D by a blue edge. We claim that $A \cup D$ is a connected blue component. Assume to the contrary that there is a cut $(A_1 \cup D_1, A_2 \cup D_2)$, where $|A_1 \cup D_2| \geq |A_2 \cup D_1|$. Now again there is no edge in G_ε^- between $V(C_1) \setminus A_2$ and D_2 as before, but now there might be some exceptional edges in G. However if either $|A_1| \geq \sqrt{\varepsilon}n$ or $|D_2| \geq \sqrt{\varepsilon}n$, then we certainly find a pair u, v that are non-adjacent in G as well (so (u, v) cannot be an exceptional edge) and $u \in A_1$ and $v \in D_2$. In the remaining case we have $|A_2 \cup D_1| \leq |A_1 \cup D_2| < 2\sqrt{\varepsilon}$. But then clearly $V(C_1) \setminus A_2 > \sqrt{\varepsilon}n$ and therefore we find a non-adjacent pair u, v in G such that u is a vertex in $V(C_1) \setminus A_2$ and v is a vertex in D_2. Now for this appropriate pair of vertices $deg_G(u) + deg_G(v) \geq (3/2 + \eta)n$. On the other hand, v is non-adjacent to all vertices of $V(C_1) \setminus A_2$ and u is non-adjacent to D_2 in the graph G_ε^-. Therefore $deg_G(u) + deg_G(v) < n - 1 - |D_2| + n - 1 - |V(C_1) \setminus A_2| + 2\sqrt{\varepsilon}n \leq 2n - 2 - |D_2| - |V(C_1) \setminus A| - |A_1| + 2\sqrt{\varepsilon}n < 2n - n/2 + 2\sqrt{\varepsilon}n = 3n/2 + 2\sqrt{\varepsilon}n$, a contradiction using $2\sqrt{\varepsilon} \ll \eta$ (here we used the assumption on the size of $A_1 \cup D_2$).

Let C_2 be this blue component of G_ε^- covering $V(G_\varepsilon^-) \setminus V(C_1)$. Let $|V(C_1) \setminus V(C_2)| = p$ and $|V(C_2) \setminus V(C_1)| = q$, where $p \geq q > 0$ by the choice of C_1. We claim that $p + q < (1/2 - \eta/2)n'$. This clearly holds if $p, q < \sqrt{\varepsilon}n$. Otherwise $p \geq \sqrt{\varepsilon}n$ or $q \geq \sqrt{\varepsilon}n$. Therefore, we find a pair of vertices u and v such that $u \in V(C_1) \setminus V(C_2)$, $v \in V(C_2) \setminus V(C_1)$ and u and v are non-adjacent in G. Thus by the Ore-type condition we have $deg_G(u) + deg_G(v) \geq (3/2 + \eta)n$. On the other hand $n' - 1 - q \geq deg_{G_\varepsilon^-}(u)$ and $n' - 1 - p \geq deg_{G_\varepsilon^-}(v)$. We can also use that $deg_{G_\varepsilon^-}(u) + \sqrt{\varepsilon}n \geq deg_G(u)$ and $deg_{G_\varepsilon^-}(v) + \sqrt{\varepsilon}n \geq deg_G(v)$. Now $(3/2 + \eta)n \leq deg_G(u) + deg_G(v) \leq deg_{G_\varepsilon^-}(u) + deg_{G_\varepsilon^-}(v) + 2\sqrt{\varepsilon}n \leq 2n' - 2 - (p + q) + 2\sqrt{\varepsilon}n \leq 3n/2 + n'/2 - 2 - (p + q) + 2\sqrt{\varepsilon}n$. It yields $n'/2 + 2\sqrt{\varepsilon}n - \eta n > p + q$. Therefore $p + q < (1/2 - \eta/2)n'$ since $\varepsilon \ll \eta$. This implies $|V(C_1) \cap V(C_2)| > (1/2 + \eta/2)n'$.

If $|V(C_1)| = n'$, then define C_2 as a largest blue component in G_ε^-. Now $p = |V(C_1) \setminus V(C_2)|$, $q = |V(C_2) \setminus V(C_1)| = 0$.

Case 2: $|V(C_1)| = n'$ and $p \leq (1/2 - \eta/2)n'$. Then again $|V(C_1) \cap V(C_2)| \geq (1/2 + \eta/2)n'$, just as above.

Therefore in what follows, we unify the proof for the two cases we described so far. Let G_1 be the graph, which we obtain from G_ε^- by deleting the blue edges induced by $V(C_1) \setminus V(C_2)$ and the red edges induced by $V(C_2) \setminus V(C_1)$ (if these exist).

We claim there is a perfect matching in G_1. Assume the contrary. By Tutte's theorem there exists a set X of vertices in G_1 such that the number of odd components in $G_1 \setminus X$ is larger than $|X|$, which implies that $|X| < n/2$. Let all the components of $G_1 \setminus X$ (not just the odd ones) be $D_1, D_2, \ldots, D_\ell$ in increasing order of their size, $\ell \geq |X| + 1$. Note that $\ell \geq 2$ always holds, even for $X = \emptyset$, as n' is even. Let $d_i = |V(D_i)|$ for $i = 1, \ldots, \ell$ and $x = |X|$.

We claim $|(V(C_1) \cap V(C_2)) \cap \cup_{i=1}^{\ell} D_i| \leq 2\sqrt{\varepsilon}n$. Assume the contrary. Now we want to copy the corresponding part of the proof of Theorem 3. Although some non-adjacent u, v pairs in G_1 might be connected by an exceptional edge in G, the size of $|(V(C_1) \cap V(C_2)) \cap \cup_{i=1}^{\ell} D_i|$ now assures that we find a non-adjacent pair as follows. We can find an index j such that $D_j \cap (V(C_1) \cap V(C_2)) = U$ is non-empty. We can think of U as a collection of potential u's. Let $\bar{D}_j = \cup_{i=1, i \neq j}^{\ell} D_i$. If $|\bar{D}_j| \geq \sqrt{\varepsilon}n$, then pick any vertex $u \in U$. There are less than $\sqrt{\varepsilon}n$ exceptional edges adjacent to u. Therefore, we find a vertex $v \in \bar{D}_j$ that is non-adjacent to u in G. If $|\bar{D}_j| < \sqrt{\varepsilon}n$, then $|U| \geq \sqrt{\varepsilon}n$. Now pick a vertex $v \in \bar{D}_j$ (using $\ell \geq 2$). There are less than $\sqrt{\varepsilon}n$ exceptional edges adjacent to v. Therefore, we find a vertex $u \in U$ that is non-adjacent to v in G.

Now we may use the Ore-type condition $deg_G(u) + deg_G(v) \geq (3/2 + \eta)n$. Notice $deg_{G_1}(u) \geq deg_G(u) - \sqrt{\varepsilon}n$. Let us subtract the number of deleted non-exceptional edges adjacent to v, which is at most p or q depending on the position of v. We get $deg_{G_1}(u) + deg_{G_1}(v) \geq deg_G(u) - \sqrt{\varepsilon}n + deg_G(v) - p - \sqrt{\varepsilon}n \geq (3/2 + \eta)n - 2\sqrt{\varepsilon}n - p \geq n$, since both p and q are less than $n/2$ and $\varepsilon \ll \eta$. On the other hand

$deg_{G_1}(u) \leq d_i - 1 + x$ and $deg_{G_1}(v) \leq d_j - 1 + x$. Therefore $deg_{G_1}(u) + deg_{G_1}(v) \leq d_i + d_j + 2x - 1 - 1 \leq n' - 1 \leq n - 1$, since $d_i + d_j + 2x - 1$ is at most the number of vertices in G_1. This contradiction implies $|(V(C_1) \cap V(C_2)) \cap \cup_{i=1}^{\ell} D_i| \leq 2\sqrt{\varepsilon}n$. Using this we get $n/2 + 2\sqrt{\varepsilon}n > x + 2\sqrt{\varepsilon}n \geq |V(C_1) \cap V(C_2)| \geq (1/2 + \eta/2)n' > (1/2 + \eta/2)(1 - \sqrt{\varepsilon})n$. However, this is a contradiction since $\varepsilon \ll \eta$. Therefore G_1 contains a perfect matching.

Case 3: $|V(C_1)| = n'$ and $p > (1/2 - \eta/2)n'$, so the largest blue component of G_ε^- has size at most $(1/2 + \eta/2)n'$. We claim in this case that there is a matching in G_1 covering all but at most $5\sqrt{\varepsilon}n$ vertices of G_1. We use the same set-up and notation as previously. Thus by Tutte's theorem now we have slightly more components than before: $\ell \geq |X| + 5\sqrt{\varepsilon}n$. This implies $x = |X| < n'/2 - 2\sqrt{\varepsilon}n$.

We can show $|V(C_2) \cap \cup_{i=1}^{\ell} D_i| \leq 2\sqrt{\varepsilon}n$ as before. Assume the contrary and as above select a vertex u in $C_2 \cap D_i$ and a vertex v in a different component D_j. Clearly u and v are non-adjacent in G_1, but also in G since we have not deleted any edge adjacent to u. If there were at least $(1/2+\eta/2)n$ ($\geq (1/2+\eta/2)n'$) deleted non-exceptional blue edges adjacent to v, then we would find a blue component larger than C_2. Therefore $deg_G(v) - deg_{G_1}(v) < (1/2 + \eta/2)n + \sqrt{\varepsilon}n$, $deg_{G_1}(u) \geq deg_G(u) - \sqrt{\varepsilon}n$ and $deg_G(u) + deg_G(v) \geq (3/2+\eta)n$ imply $deg_{G_1}(u) + deg_{G_1}(v) \geq (1+\eta/2)n - 2\sqrt{\varepsilon}n > n$. On the other hand, this is impossible since $deg_{G_1}(u) + deg_{G_1}(v) \leq d_i + d_j + 2x - 1 - 1 \leq n' - 1 < n$ as in the previous argument. Therefore $|V(C_2) \cap \cup_{i=1}^{\ell} D_i| \leq 2\sqrt{\varepsilon}n$. This implies that there is no blue component larger than $x + 2\sqrt{\varepsilon}n$.

Notice that any non-exceptional edge in G between two components of $G_1 \setminus X$ is a blue edge inside $C_1 \setminus C_2$ that was deleted. Let H be the graph formed by the vertices in $V(G_1) \setminus X$, and these crossing non-exceptional blue edges in $C_1 \setminus C_2$. Now we have $x = |X| < n'/2 - 2\sqrt{\varepsilon}n$ and therefore $|V(H)| > n'/2 + 2\sqrt{\varepsilon}n$.

Case 3.a: Assume $x \leq (1 + \eta)n/4$.
We claim that H is connected in the blue graph except for possibly $2\sqrt{\varepsilon}n$ vertices. This leads to the final contradiction, since $|V(H)| - 2\sqrt{\varepsilon}n > n'/2 > x + 2\sqrt{\varepsilon}n$. (We found a blue connected component larger than the size of a largest.)

Assume the contrary. Then there exists a blue cut (A, B) of H where we have $|A|, |B| > 2\sqrt{\varepsilon}n$, $A \cap D_1$ is non-empty and $|B \cap \cup_{i=2}^{\ell} D_i| \geq \sqrt{\varepsilon}n$. Indeed, let us take a blue cut (A, B) of H, where $|A|, |B| > 2\sqrt{\varepsilon}n$ and $A \cap D_1$ is non-empty. If $|B \cap \cup_{i=2}^{\ell} D_i| < \sqrt{\varepsilon}n$, then $|B \cap D_1| \geq \sqrt{\varepsilon}n$ and we reverse the roles of A and B. Let u be a vertex in $A \cap D_1$. Since $|B \cap \cup_{i=2}^{\ell} D_i| \geq \sqrt{\varepsilon}n$, we find a vertex v in $B \cap \bar{D}_1$ such that u and v are non-adjacent in G. Therefore $deg_G(u) + deg_G(v) \geq (3/2 + \eta)n$. Since the largest blue component has size at most $x + 2\sqrt{\varepsilon}n$, there are at most $x + 2\sqrt{\varepsilon}n - 1$ deleted non-exceptional blue edges at u or v. Therefore $(3/2+\eta)n - 2x - 4\sqrt{\varepsilon}n + 2 - 2\sqrt{\varepsilon}n \leq deg_{G_1}(u) + deg_{G_1}(v) \leq d_1 - 1 + d_i - 1 + 2x \leq n - 1$. This yields $n/2 + \eta n - 6\sqrt{\varepsilon}n \leq 2x$. This contradicts $x \leq (1 + \eta)n/4$, since $\varepsilon \ll \eta$.

Case 3.b: Assume $(1 + \eta)n/4 < x < n'/2 - 2\sqrt{\varepsilon}n$.

Here $d_1 \leq 2$, otherwise there would be too many vertices, since $n \geq x + d_1\ell > x + 3x = 4x > (1+\eta)n$, a contradiction.

We claim again that H is connected in the blue graph except for possibly $2\sqrt{\varepsilon}n$ vertices, a contradiction again, since $|V(H)| - 2\sqrt{\varepsilon}n > n'/2 > x + 2\sqrt{\varepsilon}n$. (We found a blue connected component larger than the size of a largest.) Assume the contrary and let A be a blue component in H that intersects D_1. Let $u \in A \cap D_1$. Let $B = H|_{V(H)\setminus V(A)}$. Now (A, B) is a cut of H. We may assume $|B| \geq 2\sqrt{\varepsilon}n$ (since otherwise we are done) and thus $\left|B \cap \left(\cup_{i=2}^{\ell} D_i\right)\right| \geq \sqrt{\varepsilon}n$ (using $d_1 \leq 2$). Let $v \in B \cap D_i$ such that u and v are non-adjacent in G and $i > 1$ is as small as possible. Now u and v are non-adjacent in G and therefore $deg_G(u) + deg_G(v) \geq (3/2 + \eta)n$. On the other hand using the cut (A, B), we get: $deg_G(u) \leq n - |B| + d_1 - 1 + \sqrt{\varepsilon}n$ and $deg_G(v) \leq n - |A| + d_i - 1 + \sqrt{\varepsilon}n$. It implies $(3/2+\eta)n \leq deg_G(u) + deg_G(v) \leq 2n - (|A| + |B|) + d_1 - 1 + d_i - 1 + 2\sqrt{\varepsilon}n = 2n - n' + x + d_1 - 1 + d_i - 1 + 2\sqrt{\varepsilon}n = n + x + d_1 + d_i - 2 + 3\sqrt{\varepsilon}n$. However this is a contradiction if $x + d_1 + d_i - 2 + 3\sqrt{\varepsilon}n < n/2 + \eta n$. Using $d_1 \leq 2$, it suffices to prove $x + d_i + 3\sqrt{\varepsilon}n < n/2 + \eta n$, or $x + d_i < (1+\eta)n/2$. In what follows we prove this last inequality.

Let $d_1 = 2$. Notice that $d_i \leq d_\ell$ and $d_\ell \leq n - 3x$, since $\ell > x$ and $2 = d_1 \leq d_j$ for any j. Using this and $(1+\eta)n/4 < x$ we get $x + d_i \leq x + d_\ell \leq n - 2x < n - (1+\eta)n/2 < n/2 < (1+\eta)n/2$, as desired.

The other possibility is $d_1 = 1$. Firstly if $i = \ell$, then $A \supseteq \cup_{j=1}^{\ell-1} D_j$. This is a contradiction since $|A| > x + 2\sqrt{\varepsilon}n$, we have a blue component that is larger than a largest blue component C_2. Secondly $i < \ell$. Now $\sum_{j \neq i, j \neq \ell} d_j \geq x$ and $d_i + d_\ell \geq 2d_i$. Thus we get the following: $n \geq x + \sum_{j \neq i, j \neq \ell} d_j + d_i + d_\ell > 2x + 2d_i$, which implies $x + d_i \leq n/2 < (1+\eta)n/2$, as claimed.

All these contradictions prove the existence of a matching in G_1 of the desired size (covering all but at most $\sqrt{\varepsilon}n + 5\sqrt{\varepsilon}n = 6\sqrt{\varepsilon}n$ vertices of G). Since the red and blue halves are both connected, we proved our theorem. $\square$

4 Building cycles from connected matchings.

Next we show how to prove Theorem 2 from Theorem 4 and the Szemerédi Regularity Lemma [24]. The material of this section is fairly standard by now (see [3, 9, 10, 11, 12, 13] so we omit some of the details.

We need a 2-edge-colored version of the Szemerédi Regularity Lemma.[3]

Lemma 1. *For every integer m_0 and positive ε, there is an $M_0 = M_0(\varepsilon, m_0)$ such that for $n \geq M_0$ the following holds. For any n-vertex graph G, where $G = G_1 \cup G_2$*

[3]For background, this variant and other variants of the Regularity Lemma see [17].

with $V(G_1) = V(G_2) = V$, there is a partition of V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$ such that

- *$m_0 \leq \ell \leq M_0$, $|V_1| = |V_2| = \cdots = |V_\ell| = L$, $|V_0| < \varepsilon n$,*
- *apart from at most $\varepsilon\binom{\ell}{2}$ exceptional pairs, all pairs $G_s|_{V_i \times V_j}$ are ε-regular, where $1 \leq i < j \leq \ell$ and $1 \leq s \leq 2$.*

Proof: Let $\varepsilon \ll \rho \ll \eta \ll 1$, m_0 sufficiently large compared to $1/\varepsilon$ and M_0 obtained from Lemma 1. Let G be a graph on $n > M_0$ vertices such that for any two non-adjacent vertices x and y of G, we have $deg(x) + deg(y) \geq (\frac{3}{2} + \eta)n$. Consider a 2-edge-coloring of G, that is $G = G_1 \cup G_2$. We apply Lemma 1 to G. We obtain a partition of V, that is $V = \cup_{0 \leq i \leq \ell} V_i$.

Define the following *reduced graph* G^R: The vertices $p_1, \ldots, p_\ell$ of G^R correspond to the clusters, and there is an edge between vertices p_i and p_j if the pair (V_i, V_j) is either exceptional[4], or if it is ε-regular in both G_1 and G_2 with density in G exceeding ρ. Thus note that G^R is an ε-perturbed graph where a non-edge is a regular pair where the density is at most ρ. The edge p_ip_j is colored by the color, which is used on most edges from $G[V_i, V_j]$ (the bipartite subgraph of G with edges between V_i and V_j). If the pair is non-exceptional, then the density of this majority color is still at least $\rho/2$ in $G[V_i, V_j]$. This defines a 2-edge-coloring $G^R = G_1^R \cup G_2^R$.

We claim that G^R satisfies a similar Ore-type condition: for any two non-adjacent vertices p_i and p_j of G^R, we have $deg_{G^R}(p_i) + deg_{G^R}(p_j) \geq (\frac{3}{2} + \frac{\eta}{2})\ell$. Indeed, let p_i and p_j be non-adjacent in G^R and consider the corresponding clusters V_i and V_j. By definition the number of non-edges in $G[V_i, V_j]$ is at least $(1-\rho)|V_i||V_j| = (1-\rho)L^2$. For each of these non-edges we can use the Ore-condition in G so we get the following estimate

$$\sum_{u \in V_i} \sum_{v \in V_j} (deg_G(u) + deg_G(v)) \geq (1-\rho)L^2(\frac{3}{2} + \eta)n.$$

On the other hand we can get the following upper bound for this quantity

$$\sum_{u \in V_i} \sum_{v \in V_j} (deg_G(u) + deg_G(v)) \leq L^3(deg_{G^R}(p_i) + deg_{G^R}(p_j)) + 2\varepsilon n L^2 + 2\rho n L^2,$$

where the last 2 error terms come from the edges to V_0, and from the regular pairs with density at most ρ. However, from this we get

$$deg_{G^R}(p_i) + deg_{G^R}(p_j) > (\frac{3}{2} + \frac{\eta}{2})\frac{n}{L} \geq (\frac{3}{2} + \frac{\eta}{2})\ell,$$

as desired.

[4] That is, ε-irregular in G_1 or in G_2. Also, these edges are marked exceptional in G^R.

Applying Theorem 4 to the 2-colored and ε-perturbed G^R, we get a connected matching in $(G_1^R)^-$ and a vertex-disjoint connected matching in $(G_2^R)^-$, which together cover most of G^R. Finally, we lift the connected matchings back to cycles in the original graph using the following[5] lemma in our context.

Lemma 2. *Assume that there is a monochromatic connected matching M (say in $(G_1^R)^-$) saturating at least $c|V(G^R)|$ vertices of G^R, for some positive constant c. Then in the original G there is a monochromatic cycle in G_1 covering at least $c(1-3\varepsilon)n$ vertices.*

This completes the proof. Indeed, the number of vertices left uncovered in G is at most $f(\varepsilon)n \leq \eta n$, using our choice of ε. Here the uncovered parts come from Theorem 4, Lemma 2 and V_0. $\square$

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