Monochromatic bounded degree subgraph partitions

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May 30, 2014

Abstract

Let $\mathcal{F} = \{F_1, F_2, \ldots\}$ be a sequence of graphs such that F_n is a graph on n vertices with maximum degree at most Δ. We show that there exists an absolute constant C such that the vertices of any 2-edge-colored complete graph can be partitioned into at most $2^{C\Delta \log \Delta}$ vertex disjoint monochromatic copies of graphs from $\mathcal{F}$. If each F_n is bipartite, then we can improve this bound to $2^{C\Delta}$; this result is optimal up to the constant C.

1 Introduction

Let K_n be a complete graph on n vertices whose edges are colored with r colors ($r \geq 1$). How many monochromatic cycles (single vertices and edges are considered to be cycles) are needed to partition the vertex set of K_n? This question received much attention in the last few years. Let $p(r)$ denote the minimum number of monochromatic cycles needed to partition the vertex set of any r-colored K_n. It is not obvious that $p(r)$ is a well-defined function. That is, it is not obvious that there always is a

*Research supported in part by National Physical Sciences Consortium Fellowship
†Research supported in part by OTKA Grant No. K104343.

partition whose cardinality is independent of n. However, in [18] Erdős, Gyárfás, and Pyber proved that there exists a constant C such that $p(r) \leq Cr^2 \log r$ (throughout this paper log denotes the natural logarithm). Furthermore, in [18] (see also [26]), the authors conjectured that $p(r) = r$.

The special case $r = 2$ of this conjecture was asked earlier by Lehel and for $n \geq n_0$ was first proved by Łuczak, Rödl, and Szemerédi [41]. Allen improved on the value of n_0 [1] and recently Bessy and Thomassé [4] proved the original conjecture for $r = 2$. For general r the current best bound is due to Gyárfás, Ruszinkó, Sárközy, and Szemerédi [27] who proved that for $n \geq n_0(r)$ we have $p(r) \leq 100r \log r$. For $r = 3$ an approximate version of the conjecture was proved in [28] but, surprisingly, Pokrovskiy [43] found a counterexample to the conjecture. However, in the counterexample, all but one vertex can be covered by r vertex disjoint monochromatic cycles. Thus, a slightly weaker version of the conjecture still can be true, say that, apart from a constant number of vertices, the vertex set can be covered by r vertex disjoint monochromatic cycles.

Let us also note that the above problem was generalized in various directions; for hypergraphs (see [29] and [48]), for complete bipartite graphs (see [18] and [31]), for graphs which are not necessarily complete (see [2] and [47]), and for partitions by monochromatic connected k-regular graphs (see [50] and [51]).

Another area that attracted a lot of interest is the study of Ramsey numbers for bounded degree graphs. For a graph G, the *Ramsey number* $R(G)$ is the smallest positive integer N such that if the edges of a complete graph K_N are partitioned into two color classes then one color class has a subgraph isomorphic to G. The existence of such a positive integer is guaranteed by Ramsey's classical result [45]. Determining $R(G)$ even for very special graphs is notoriously hard (see e.g. [25] or [44]).

In 1975, Burr and Erdős [3] raised the problem that every graph G with n vertices and maximum degree Δ has a linear Ramsey number, so $R(G) \leq C(\Delta)n$, for some constant $C(\Delta)$ depending only on Δ. This was proved by Chvátal, Rödl, Szemerédi and Trotter [9] in one of the earliest applications of Szemerédi's celebrated Regularity Lemma [52]. Because the proof uses the Regularity Lemma, the bound on $C(\Delta)$ is quite weak; it is of tower type in Δ. This was improved by Eaton [17], who proved, using a variant of the Regularity Lemma, that the function $C(\Delta)$ can be taken to be of the form $2^{2^{O(\Delta)}}$.

Soon after, Graham, Rödl, and Ruciński [24] improved this further to $C(\Delta) \leq 2^{O(\Delta \log^2 \Delta)}$ and for bipartite graphs $C(\Delta) \leq 2^{O(\Delta \log \Delta)}$. They also proved that there are bipartite graphs with n vertices and maximum degree Δ for which the Ramsey number is at least $2^{\Omega(\Delta)}n$. Recently, Conlon [10] and, independently, Fox and Sudakov [23] have shown how to remove the $\log \Delta$ factor in the exponent, achieving an essentially best possible bound of $R(G) \leq 2^{O(\Delta)}n$ in the bipartite case. For the non-bipartite graph case, the current best bound is due to Conlon, Fox, and Sudakov [13] $C(\Delta) \le 2^{O(\Delta \log \Delta)}$. Similar results have been proven for hypergraphs: [14, 15, 42] use the Hypergraph Regularity Lemma and [12] improves the bounds by avoiding the Regularity Lemma.

It is a natural question (initiated by András Gyárfás) to combine the above two problems and ask how many monochromatic members from a bounded degree graph family are needed to partition the vertex set of a 2-edge-colored K_N. In this paper we study this problem. Given a sequence $\mathcal{F} = \{F_1, F_2, \dots\}$ of graphs, we say it is Δ-*bounded* if each F_n is a graph on n vertices with maximum degree at most Δ. In general we say that $\mathcal{F}$ has some graph property if every graph of $\mathcal{F}$ has that property (e.g. $\mathcal{F}$ is bipartite if F_n is bipartite for every n).

We prove the following result on partitions by monochromatic members of $\mathcal{F}$.

Theorem 1. *There exists an absolute constant C such that, for every Δ and every Δ-bounded graph sequence $\mathcal{F}$, every 2-edge-colored complete graph can be partitioned into at most $2^{C\Delta \log \Delta}$ vertex disjoint monochromatic graphs from $\mathcal{F}$.*

Thus, perhaps surprisingly, we have the same phenomenon as for cycles; we can partition into monochromatic graphs from $\mathcal{F}$ such that their average size is roughly the same as the single largest monochromatic graph we can find. In the case of a bipartite $\mathcal{F}$ we can eliminate the $\log \Delta$ factor from the exponent to get the following essentially best possible result.

Theorem 2. *There exists an absolute constant C such that, for every Δ and every bipartite Δ-bounded graph sequence $\mathcal{F}$, every 2-edge-colored complete graph can be partitioned into at most $2^{C\Delta}$ vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$.*

We do not make an effort to optimize the constant C since probably it will be far from optimal anyway. However, in both theorems we must use at least $2^{\Omega(\Delta)}$ parts.

Theorem 3. *There exists an absolute constant c such that, for every Δ, there is a bipartite Δ-bounded graph sequence $\mathcal{F}$ and there is a 2-edge-coloring of K_n so that covering the vertices of K_n using monochromatic copies of graphs from $\mathcal{F}$ requires at least $2^{c\Delta}$ such copies.*

It would be desirable to close the gap between the upper and lower bounds for non-bipartite $\mathcal{F}$ as well, though doing so may require improved bounds for the Ramsey numbers of bounded degree graphs. Furthermore, it would be interesting to extend this problem for more than 2 colors.

Let us also mention one interesting special case of our theorem. The k^{th} *power* of a cycle C is the graph obtained from C by joining every pair of vertices with distance at most k in C. Density questions for powers of cycles have generated a lot of interest; in particular the famous Pósa-Seymour conjecture (see e.g. [7, 19, 20, 21, 22, 34, 37, 38, 40]). Theorem 1 implies the following result on the partition number by monochromatic powers of cycles.

Corollary 1. *There exists an absolute constant C so that for every k every 2-colored complete graph can be partitioned into at most $2^{Ck \log k}$ vertex disjoint monochromatic kth powers of cycles.*

However, we must note that in this case probably the optimal answer is $O(k)$.

2 Notation and tools

For basic graph concepts see the monograph of Bollobás [5].
$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A \cup B$ and $E \subset A \times B$. A proper r-coloring of G is a coloring of its vertices where no two adjacent vertices receive the same color. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G. $N(v)$ is the set of neighbors of $v \in V$. Hence, $|N(v)| = deg(v) = deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum and $\Delta(G)$ for the maximum degree in G. When A, B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. In particular, we write $deg(v, U) = e(\{v\}, U)$ for the number of edges from v to U. For non-empty A and B,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

is the *density* of the graph between A and B.

Definition 1. *The bipartite graph $G = (A, B, E)$ is ε-regular if*

$$X \subset A,\ Y \subset B,\ |X| > \varepsilon|A|,\ |Y| > \varepsilon|B| \quad \textit{imply} \quad |d(X, Y) - d(A, B)| < \varepsilon.$$

We will often say simply that "the pair (A, B) is ε-regular" with the graph G implicit.

Definition 2. *(A, B) is (ε, d, δ)-super-regular if it is ε-regular, satisfies $d(A, B) \geq d$, and*

$$deg(a) \geq \delta|B| \ \forall\, a \in A, \qquad deg(b) \geq \delta|A| \ \forall\, b \in B.$$

An $(\varepsilon, \delta, \delta)$-super-regular pair is simply called (ε, δ)-super-regular (and in this case the density condition is not needed).

We will use frequently the following well-known property of regular pairs claiming that subsets of a regular pair also form a regular pair with somewhat weaker parameters.

Lemma 1 (Slicing Lemma, Fact 1.5 in [39]). *Let (A, B) be an $(\varepsilon, d, 0)$-super-regular pair (i.e. one with no minimum degree constraint), and, for some $\beta > \varepsilon$, let $A' \subset A$, $|A'| \geq \beta|A|$, $B' \subset B$, $|B'| \geq \beta|B|$. Then (A', B') is an $(\varepsilon', d', 0)$-super-regular pair with $\varepsilon' = \max\{\varepsilon/\beta, 2\varepsilon\}$ and $|d' - d| < \varepsilon$.*

Definition 3. *Given a k-partite graph $G = (V, E)$ with k-partition $V = V_1 \cup \ldots \cup V_k$, the k-cylinder $V_1 \times \ldots \times V_k$ is ε-regular ((ε, d, δ)-super-regular or (ε, δ)-super-regular) if all the $\binom{k}{2}$ pairs of subsets (V_i, V_j), $1 \leq i < j \leq k$, are ε-regular ((ε, d, δ)-super-regular or (ε, δ)-super-regular). Given $0 < \alpha < 1$, the k-cylinder $V_1 \times \ldots \times V_k$ is α-balanced if, for every $i < j$, $||V_i| - |V_j|| \leq \alpha min(|V_i|, |V_j|)$.*

Instead of the Regularity Lemma of Szemerédi [52] we will use the following lemma which Conlon and Fox [11] argued as a consequence of the Duke, Lefmann, and Rödl weak Regularity Lemma [16].

Lemma 2 ([16] and Lemma 5.3 in [11]). *For each $0 < \varepsilon < 1/2$, any graph $G = (V, E)$ on at least k vertices has an ε-regular k-cylinder with parts of equal size (i.e. 0-balanced); the size of each part is at least $\frac{1}{2k}\varepsilon^{k^2\varepsilon^{-5}}|V|$.*

We will use the following corollary of this lemma.

Lemma 3 (Lemma 5.4 in [11]). *For each $0 < \varepsilon < 1/2$, any 2-colored complete graph on at least 2^{2k} vertices has, in one of the colors (say in red), an $(\varepsilon, 1/2, 0)$-super-regular 0-balanced k-cylinder (i.e. one with no minimum degree constraint and parts of equal size), where the size of each part is at least $\frac{1}{2(2^{2k})}\varepsilon^{2^{4k}\varepsilon^{-5}}n$.*

Indeed, to get this one applies Lemma 2 for the red subgraph with 2^{2k} in place of k to get an ε-regular 2^{2k}-cylinder. Then we may consider the complete graph whose vertices i correspond to the parts of the cylinder V_i and we color the edge (i, j) by the majority color in the pair (V_i, V_j). We then apply $R(K_k) \leq 2^{2k}$ and use the fact that, if (V_i, V_j) is regular in one color, then it is also regular in the other color.

We will also use the Hajnal-Szemerédi Theorem on equitable proper colorings. A proper coloring is *equitable* if the numbers of vertices in any two color classes differ by at most one.

Lemma 4 (Hajnal-Szemerédi Theorem [30]). *Any graph with maximum degree at most Δ has an equitable proper coloring with $\Delta + 1$ colors.*

For a simpler proof see also [33].

Our other main tool is a quantitative version of the Blow-up Lemma (see [35, 36]).

Lemma 5 (Quantitative Blow-up Lemma). *There exists an absolute constant C_{BL} such that, given a graph R of order $r \geq 2$ and positive parameters δ and Δ, for any $0 < \varepsilon < (\frac{\delta d^{\Delta}}{r\Delta})^{C_{BL}}$ the following holds. Let N be an arbitrary positive integer, and let us replace the vertices of R with pairwise disjoint N-sets $V_1, V_2, \ldots, V_r$ (blowing up). We construct two graphs on the same vertex-set $V = \bigcup V_i$. The graph $R(N)$ is obtained by replacing all edges of R with copies of the complete bipartite graph $K_{N,N}$, and a sparser graph G is constructed by replacing the edges of R with some (ε, d, δ)-super-regular pairs. If a graph H with $\Delta(H) \leq \Delta$ is embeddable into $R(N)$, then it is embeddable into G.*

Thus, roughly speaking, regular cylinders behave as complete partite graphs from the viewpoint of embedding bounded degree subgraphs. Note that, in either of the proofs [35, 36], the dependence of ε on the other parameters was not computed explicitly. To prove Lemma 5 one has to go through the proof, say from [36], and make all the dependencies of the parameters explicit. All the details are presented in [49]. Note that for the proof of Theorem 1 we could use $d = \delta$, we need the stronger version for the proof of Theorem 2, as δ may be much smaller than d.

In particular we will need the following consequence of the Blow-up Lemma.

Lemma 6. *There exists an absolute constant C_{BL} such that, given positive parameters δ, d, and Δ, and given a Δ-bounded graph sequence $\mathcal{F}$, for any $0 < \varepsilon < \frac{1}{2}\left(\frac{(\frac{\delta}{2})(\frac{d}{2})^{\Delta}}{\Delta(\Delta+2)}\right)^{C_{BL}}$ the following holds. Let $G = (V, E)$ be a $(\Delta+2)$-partite graph with $(\Delta+2)$-partition $V = V_1 \cup \ldots \cup V_{\Delta+2}$, where the cylinder $V_1 \times \ldots \times V_{\Delta+2}$ is (ε, d, δ)-super-regular and ε-balanced. Then we can partition the vertex set into at most $(\Delta+3)$ vertex disjoint copies of graphs from $\mathcal{F}$.*

Indeed, note first that by the Blow-up Lemma (Lemma 5 with $r = \Delta + 2$), if the cylinder is 0-balanced, then it is enough to check the statement for the complete $(\Delta+2)$-partite graph with the same partite sets. But then the Hajnal-Szemerédi Theorem (Lemma 4) implies that we may cover the cylinder with a single graph from $\mathcal{F}$ (even if we had only $\Delta + 1$ partite sets instead of $\Delta + 2$).

If the cylinder is not 0-balanced (but ε-balanced), then first we eliminate the small discrepancies among the sizes. For each index $i \in [\Delta+2]$ define $v_i = |V_i|$ and take $v = \max_i(v_i)$. Then, for each set $S \subseteq [\Delta+2]$ of size $\Delta+1$, define $w_S = v - v_i$ where i is the unique index not contained in S. For each such set S, by the Blow-up Lemma, we may find a copy of a graph from $\mathcal{F}$ that uses w_S vertices from each V_i with $i \in S$. Thus, we use $(\Delta+2)$ such graphs, one for each S.

After this procedure, the number of vertices remaining in V_i is

$$v_i - \sum_{S: i \in S} w_S = v_i - \sum_S w_S + w_{[\Delta+2]\setminus\{i\}} = v - \sum_S w_S \quad (\geq (1-(\Delta+2)\varepsilon)v)\,. \tag{1}$$

That is, after this procedure, the cylinder is 0-balanced, and we may cover the remaining vertices with a single graph from $\mathcal{F}$ by the Blow-up Lemma. Indeed, the Slicing Lemma (Lemma 1) and (1) imply that the remaining cylinder is $(2\varepsilon, d/2, \delta/2)$-super-regular and 0-balanced so we may indeed apply the Blow-up Lemma (Lemma 5). We use a total of at most $(\Delta + 3)$ graphs.

3 Proof of Theorem 1

For technical reasons, it will be convenient to prove a stronger version of Theorem 1.

Theorem 4. *There exists an absolute constant C so that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are Δ-bounded graph sequences and every graph in $\mathcal{F}_1, \mathcal{F}_2$ has an equitable proper coloring with χ_1, χ_2 colors, respectively, then every 2-edge-colored complete graph can be partitioned into at most $2^{C(\chi_1+\chi_2+\Delta)\log\Delta}$ vertex disjoint red copies of graphs from $\mathcal{F}_1$ and monochromatic blue copies of graphs from $\mathcal{F}_2$.*

To see that this implies Theorem 1, note again that by the Hajnal-Szemerédi Theorem (Lemma 4) any graph of maximum degree at most Δ has an equitable proper coloring with $\Delta + 1$ colors.

To avoid redundancy, for the rest of this section $\mathcal{F}_1$ and $\mathcal{F}_2$ will be Δ-bounded sequences.

We have three main tools for finding monochromatic copies of graphs. One of these is the Blow-up Lemma (Lemma 6). Another is applying bounds on Ramsey numbers:

Lemma 7. *There exists an absolute constant C_1 such that given a 2-edge-colored K_n and an $\varepsilon > 0$, we may cover all but an εn fraction of the vertices using vertex disjoint monochromatic red copies of graphs from $\mathcal{F}_1$ and blue copies of graphs from $\mathcal{F}_2$ while using at most $2^{C_1\Delta\log\Delta}\log(1/\varepsilon)$ such copies.*

Proof. By the bound in [13] (see Theorem 2.6), any 2-edge-colored K_n contains either a red copy of F_1 or a blue copy of F_2 for any two graphs F_1, F_2 with maximum degree at most Δ that are on $n' = 2^{-C_1\Delta\log\Delta}n$ (assume for simplicity that this is an integer) vertices. Pick any graph in $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ with n' vertices, find a copy in an appropriate color of one of these in our K_n, remove the vertices of this copy from the graph, and recurse on the remaining $n - n'$ vertices. Note the number of remaining vertices is $(1-2^{-C_1\Delta\log\Delta})n \leq e^{-2^{-C_1\Delta\log\Delta}}n$, so after repeating this process $2^{C_1\Delta\log\Delta}\log(1/\varepsilon)$ times, we are left with a graph on at most $e^{-\log(1/\varepsilon)}n = \varepsilon n$ vertices, as desired. $\square$

The final tool we have is a simple greedy embedding of bounded degree bipartite graphs into very dense bipartite graphs.

Lemma 8. *Given a bipartite graph $H = (A, B, E)$ where every vertex of B has degree at most Δ, a bipartite graph $G = (A', B', E')$ where every vertex of A' has degree at least $(1 - 1/(2\Delta))|B'|$ and where $|B'| \geq 2|B|$, and an injection $\phi : A \to A'$, the function ϕ extends to an injective homomorphism from H to G.*

Proof. We will embed the vertices of $B = \{b_1, b_2, \dots, b_k\}$ one at a time (in this order). Once we have embedded $b_1, \dots, b_i$, we show how to embed $b = b_{i+1}$. Note that b must be embedded in a way consistent with its neighbors; i.e., b must be contained in the common neighborhood (in G) of $\phi(N_H(b))$. Since b has degree at most Δ and all of the vertices in A' have degree at least $(1-\frac{1}{2\Delta})|B'|$, by a union bound the number of vertices in B' consistent with the neighbors of b is at least

$$|B'| - \Delta\frac{1}{2\Delta}|B'| = \frac{|B'|}{2} \geq |B|.$$

Since we have embedded only $i < |B|$ vertices so far, at least one of the above $|B|$ vertices has not yet had any vertex embedded to it; to this vertex we embed b. When this procedure embeds b_k, the embedding is complete, and is an injective homomorphism that extends ϕ by construction. $\square$

We will combine these three tools to prove Theorem 4. We basically follow the *greedy-absorbing* proof technique that originated in [18] and is used in many papers in this area (e.g. [27], [31], [51]). We establish the desired bound in the following steps.

- Step 1: First we find a special object, a regular cylinder, that is dense in a color (say red) as given by Lemma 3.
- Step 2: Then we remove the cylinder and by iterating Ramsey's Theorem as it was done in Lemma 7 we cover most of the remaining vertices with monochromatic copies of graphs from $\mathcal{F}_1, \mathcal{F}_2$.
- Step 3: Finally we add the few vertices that are neither in the cylinder nor covered by monochromatic copies to the cylinder; since there will be few vertices added, this will not affect the regularity of the cylinder and the cylinder will absorb these vertices. Indeed, if it were the case that all of the vertices in the cylinder had large enough degrees in red to all the other partite sets, then the Blow-up Lemma (Lemma 6) would allow us to cover all of them with a few

red subgraphs from $\mathcal{F}_1$. By regularity, there may be few vertices that fail to meet this minimum degree condition, and they must have large degree in blue to one of the sets in the cylinder. Inductively, we will partition these remaining vertices into either red copies of graphs in $\mathcal{F}_1$ or blue copies of graphs obtainable by taking a graph F_2 from $\mathcal{F}_2$ and removing an equitable color class. Then we will use the fact that these vertices have large degree in blue and Lemma 8 to "glue in" the missing parts of the F_2 graphs by using some vertices from the cylinder.

We now proceed with proving Theorem 4. The proof is by induction on $\chi_1 + \chi_2$. If either χ_1 or χ_2 is 1, then the result is trivial, as every graph in the corresponding collection is an independent set. Otherwise, assume both χ_1 and χ_2 are at least 2. Let any 2-edge-colored K_n be given. We may assume that n is at least $2^{2(\Delta+2)}$ since otherwise we can cover it by isolated vertices. We may also assume $\Delta \geq 2$. Let

$$\varepsilon = 2^{-C_2\Delta}, k = \Delta + 2, \text{ and } \eta = \frac{1}{2(2^{2k})}\left(\frac{\varepsilon}{2}\right)^{2^{4k}(\frac{\varepsilon}{2})^{-5}}, \tag{2}$$

with some sufficiently large absolute constant C_2 (independent of Δ). Apply Lemma 3 with parameter $\varepsilon/2$ to the 2-coloring to get a 0-balanced k-cylinder $V = V_1 \cup \cdots \cup V_k$, where, for each $i, 1 \leq i \leq k$, we have $|V_i| \geq \eta|V|$, and there is a color, say red, so that the cylinder is $(\varepsilon/2, 1/2, 0)$-super-regular in the red subgraph.

By Lemma 7, using at most $2^{C_1\Delta\log\Delta}\log(2/(\varepsilon^2\eta))$ copies, we may cover all but $\varepsilon^2\eta n/2$ vertices of $K_n \setminus V$ with monochromatic graphs in the appropriate color from $\mathcal{F}_1, \mathcal{F}_2$. Take the remaining $\varepsilon^2\eta n/2$ vertices and add them to the cylinder in such a way that the cylinder remains as balanced as possible. Note that since we added at most $\varepsilon^2\eta n/2$ vertices in this way, the resulting cylinder $V' = V_1' \cup \cdots \cup V_k'$ is $(\varepsilon, 1/3, 0)$-super-regular in red.

Take $\delta = \frac{1}{2\Delta}$. We classify the vertices of V' based on their red degrees. We say that a vertex v is *good for i* if either $v \in V_i'$ or the red degree of v to V_i' is at least $\delta|V_i'|/2$, and we say that it is *good* if it is good for every V_i'. By regularity, at most an ε fraction of the vertices of V_j' fail to be good for i, and so at most an ε fraction of the vertices of V' fail to be good for V_i'. Define, for each i, B_i to be the set of vertices that are not good for i but are good for every j that is smaller than i. By construction, the B_i's partition the vertices of V' that are not good, and every vertex in B_i has red degree to V_i' at most $\delta|V_i'|/2$. Remove the vertices in $\cup_{i=1}^k B_i$ from the cylinder. Denote the resulting partite sets by $V_i'', 1 \leq i \leq k$. Since

$$|V_i''| \geq (1 - \epsilon k)|V_i'| \geq (1 - \delta/2)|V_i'|,$$

we have that every vertex in B_i has red degree at most $\delta|V_i''|$ to V_i''. Therefore, it has blue degree at least $(1-\delta)|V_i''|$ to V_i''. Furthermore, since we removed at most $k\varepsilon|V_i'|$

($\ll \delta|V_i'|/4$ if C_2 is large enough) vertices from each V_i' and the remaining vertices were all good, the Slicing Lemma (Lemma 1) implies that the resulting cylinder $V'' = V_1'' \cup \cdots \cup V_k''$ is $(2\varepsilon, 1/4, \delta/4)$-super-regular in red.

Define a Δ-bounded sequence $\mathcal{F}_2'$ of graphs by taking, for each m, a graph from $\mathcal{F}_2$ on $\left\lceil \frac{\chi_2}{\chi_2-1} m \right\rceil$ vertices, taking an equitable proper coloring of this graph into χ_2 parts, and removing a part of size $\lceil m/(\chi_2 - 1) \rceil$. The resulting graph has m vertices and an equitable proper coloring into $(\chi_2 - 1)$ parts. Therefore, by induction, we may partition each B_i into at most $2^{C(\chi_1+\chi_2+\Delta-1)\log\Delta}$ red copies of graphs from $\mathcal{F}_1$ and blue copies of graphs from $\mathcal{F}_2'$. Denote by $F_1', F_2', \ldots, F_\ell'$ the blue copies of graphs from $\mathcal{F}_2'$ in such a partition. Each F_i' may be obtained from some F_i in $\mathcal{F}_2$ by removing an equitable color class S_i from F_i. Define a bipartite graph H whose vertex sets are $A = V(F_1') \cup V(F_2') \cup \cdots \cup V(F_\ell')$ and $B = S_1 \cup S_2 \cup \cdots \cup S_\ell$ so that the edges leaving $V(F_i')$ are to S_i and, along with the edges of the monochromatic F_i', form a copy of F_i. Note that $|V(F_i)| \leq 2|V(F_i')| + 1 \leq 3|V(F_i')|$, and so $|B| \leq 3|B_i| \leq 3k\varepsilon|V_i'|$. By Lemma 8, there is an embedding of B into some $V_i''' \subseteq V_i''$ so that, along with the identity embedding on A, it forms a homomorphism from H into the blue edges of G. This embedding extends every monochromatic copy of a graph in $\mathcal{F}_2'$ to a monochromatic copy of a graph in $\mathcal{F}_2$.

The only vertices we have not covered with monochromatic copies of graphs from $\mathcal{F}_1$ or $\mathcal{F}_2$ are the vertices in each set of the form $V_i'' \setminus V_i'''$. Since again we removed an additional at most $k\varepsilon|V_i'|$ ($\ll \delta|V_i''|/8$ if C_2 is large enough) vertices from each V_i'', the Slicing Lemma (Lemma 1) again implies that the remaining cylinder is $(2\varepsilon, 1/4, \delta/8)$-super-regular in red. Then this remaining cylinder may be covered by at most $k+1 = \Delta + 3$ red subgraphs from $\mathcal{F}_1$ by Lemma 6. Note that the conditions of the lemma are satisfied if C_2 is a sufficiently large absolute constant. Indeed, we have to check the following inequality

$$\varepsilon = \frac{1}{2^{C_2\Delta}} < \frac{1}{4}\left(\frac{(\frac{1}{32\Delta})(\frac{1}{8})^{\Delta}}{\Delta(\Delta+2)}\right)^{C_{BL}} = \frac{1}{4(32\Delta^2(\Delta+2)8^{\Delta})^{C_{BL}}},$$

which is true if C_2 is sufficiently large compared to C_{BL}.

The total number of monochromatic subgraphs used in the partition is at most

$$2^{C_1\Delta\log\Delta}\log(2/(\varepsilon^2\eta)) + (\Delta+2)2^{C(\chi_1+\chi_2+\Delta-1)\log\Delta} + (\Delta+3) =$$

$$= 2^{C_1\Delta\log\Delta}\log(2/(\varepsilon^2\eta)) + \frac{(\Delta+2)}{2^{C\log\Delta}}2^{C(\chi_1+\chi_2+\Delta)\log\Delta} + (\Delta+3) \leq 2^{C(\chi_1+\chi_2+\Delta)\log\Delta},$$

as desired, if C is sufficiently large compared to C_1 and C_2. Indeed, here by using (2) we have

$$\log(2/(\varepsilon^2\eta)) = \log(2/(\varepsilon^2)) + \log(1/\eta) =$$

$= (2C_2\Delta + 1)\log 2 + \log 2 + 2(\Delta + 2)\log 2 + 2^{4(\Delta+2)+5(C_2\Delta+1)}(C_2\Delta + 1)\log 2 \ll 2^{C\Delta}.$

□

4 Proof of Theorem 2

The proof is almost identical to the proof of Theorem 1 above. First, a major difference is that in Lemma 7 we may use the better Ramsey bound for bipartite graphs, thus giving us the improved bound $2^{C_1\Delta}\log(1/\varepsilon)$. Second, here the induction has only one step, after one step the chromatic number goes down to 1 and we may cover each B_i by one graph. This gives us the bound

$$2^{C_1\Delta}\log(2/(\varepsilon^2\eta)) + (\Delta + 2) + (\Delta + 3) \leq 2^{C\Delta},$$

as desired. Note that, rather than using the Hajnal-Szemerédi theorem, given any Δ-bounded bipartite graph sequence $\mathcal{F}$, we may create a new sequence G_n of Δ-bounded graphs that have proper equitable 2-colorings so that G_n is a union of at most 3 graphs from $\mathcal{F}$; we do this by taking G_n to be two copies of $F_{n/2}$ if n is even and two copies of $F_{(n-1)/2}$ and a copy of F_1 if n is odd. This would allow us to work with $k = 3$ (instead of $\Delta + 2$), so we may find a regular 3-cylinder in Step 1. □

We should also note the above argument is not particular to bipartite graphs; it works for χ-partite graphs if χ is a constant (e.g. tripartite graphs); for any constant χ we get a constant $C(\chi)$ and the bound above becomes $2^{C(\chi)\Delta}$.

5 Proof of Theorem 3

We wish to show that there exists a Δ-bounded bipartite sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ and, for n sufficiently large, a two-edge-coloring of K_n that cannot be partitioned into fewer than $2^{\Omega(\Delta)}$ monochromatic copies of graphs from $\mathcal{F}$. To see this, for every n take G_n to be a bipartite graph on n vertices of degree at most Δ and, for n sufficiently large, with Ramsey number at least $2^{\Omega(\Delta)}n$, as given by the result of Graham, Rödl and Ruciński [24]. We define F_{2^i} recursively; take $F_{2^0} = G_1$. Then define F_{2^i} to be the disjoint union of $F_{2^{i-1}}$ with $G_{2^{i-1}}$. For integers of the form $2^i + j$ with $j < 2^i$, define F_{2^i+j} to be the disjoint union of F_{2^i} with an independent set on j vertices. Under this definition, each F_n is a bipartite graph on n vertices with maximum degree at most Δ. Furthermore, for $n_0 < n_1$, F_{n_0} is a subgraph of F_{n_1}. Finally, taking i to be the largest integer with $2^i \leq n$, F_n contains a copy of $G_{2^{i-1}}$ and so has Ramsey number at least $2^{\Omega(\Delta)}2^{i-1} = 2^{\Omega(\Delta)}n$ (for n sufficiently large). Take $\mathcal{F} = \{F_1, F_2, \ldots\}$. Now, for N sufficiently large, take a 2-edge-coloring of a complete graph on $2^{\Omega(\Delta)}N$ vertices without a monochromatic copy of F_N (this is possible by the condition on the

Ramsey number). Since the sequence of graphs is increasing, this coloring also does not contain a monochromatic copy of any F_n for $n > N$. Therefore, any partition of the vertex set into monochromatic copies of graphs from $\mathcal{F}$ must use at least $2^{\Omega(\Delta)}$ such copies. $\square$

6 Concluding Remarks

There are various interesting potential generalizations of Theorem 1. One may ask if the theorem holds for r colors for any positive integer r.

Conjecture 1. *For every positive integer r there exists a constant C_r (depending on r) such that, for every Δ-bounded sequence $\mathcal{F}$, every r-edge-colored complete graph can be partitioned into at most $2^{\Delta^{C_r}}$ vertex disjoint monochromatic graphs from $\mathcal{F}$.*

Since bounds on Ramsey numbers were key in proving the theorem for $r = 2$, it is worth noting that Conlon, Fox, and Sudakov [13] proved that, for any fixed number of colors r, for any graph G on n vertices of maximum degree Δ the Ramsey number on r colors $R_r(G)$ is at most $2^{C_r\Delta^2}n$. The primary difficulty is replacing the step that uses Lemma 8 to account for the larger number of colors.

Recently Böttcher, Kohayakawa, Taraz, and Würfl [6] proved a generalization of the Blow-up Lemma for graphs of bounded arrangeability without vertices of large degree. An a-arrangeable graph is one in which the vertices may be ordered such that the neighbors to the right of any vertex v have at most a neighbors to the left of v in total. They generalize the Blow-up Lemma from graphs of bounded degree to n-vertex graphs of bounded arrangeability with maximum degree at most $\frac{\sqrt{n}}{\log n}$. Furthermore, Chen and Schelp [8] proved that for every a there is some constant $C(a)$ so that the Ramsey number of any a-arrangeable graph on n vertices is at most $C(a)n$. The best bound that is known for $C(a)$, again due to Graham, Rödl and Ruciński [24], is $C(a) \leq 2^{Ca(\log a)^2}$. One may hope to combine these two results to get another possible generalization of Theorem 1.

We say a sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ is a-nicely-arrangeable if each F_n is a graph on n vertices that is a-arrangeable with maximum degree at most $\frac{\sqrt{n}}{\log(n)}$. Using techniques similar to those found in this paper, one can prove:

Theorem 5. *There exists an absolute constant C so that, for every positive integer a and every a-nicely-arrangeable sequence $\mathcal{F}$, every 2-edge-colored complete graph can be partitioned into at most 2^{Ca^6} vertex disjoint monochromatic graphs from $\mathcal{F}$.*

The primary change from the techniques in this paper necessary to prove the above theorem is to adapt the use of Lemma 8. Currently, we use it to take a pair of graph sequences $\mathcal{F}_1$ and $\mathcal{F}_2$ along with a nearly-complete bipartite graph, recurse on one of the parts of the bipartite graph to find copies of graphs either from $\mathcal{F}_1$ or from graphs obtained from $\mathcal{F}_2$ by removing a color class, and extend the smaller graphs using the nearly-complete bipartite graph. Instead, in the proof for arrangeable graphs, we recurse on one of the parts of the nearly-complete bipartite graph to find copies of graphs from $\mathcal{F}_1$ and $\mathcal{F}_2$ (without any color class removed), along with another nearly-complete bipartite graph. This gives a nearly-complete tripartite graph, and we may continue to recurse until we have a nearly-complete multipartite graph into which we may embed our a-nicely-arrangeable graphs.

Finally, let us mention that since by now both the Regularity Lemma and the Blow-up Lemma has been generalized to hypergraphs (see [46] and [32], respectively), perhaps we can generalize our result to hypergraphs as well.

Acknowledgements. The first author is indebted to Jacob Fox and the second author to András Gyárfás and Endre Szemerédi for helpful conversations on the topic.

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