LARGE TIME BEHAVIOR OF A LINEAR DIFFERENCE EQUATION WITH RATIONALLY NON-RELATED DELAYS

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Abstract. We consider the linear difference equation

$$x(t) = \sum_{j=1}^{N} a_j x(t - r_j), \qquad t \geq 0,$$

where $a_j > 0$, $1 \leq j \leq N$, and the delays $0 < r_1 < r_2 < \cdots < r_N$ are not rationally related in the sense that r_j/r_k is irrational for some j and k. It is shown that the large time behavior of the continuous solutions can be described in terms of the unique real root of the associated characteristic equation. The proof is based on Newman's Tauberian theorem.

Keywords: difference equation; rationally non-related delays; large time behavior; characteristic equation; dominant root

1. Introduction and the Main Results

In [20] we studied the linear difference equation with distributed delays

$$x(t) = \int_{-r}^{0} x(t+\theta)\, d\eta(\theta), \qquad t \geq 0, \tag{1.1}$$

where $r > 0$ and the kernel $\eta\colon [-r, 0] \to \mathbb{R}$ is a nonconstant nondecreasing function normalized such that $\eta(0) = 0$ and η is continuous from the left at each $\theta \in (-r, 0]$. The integral in (1.1) is a Riemann-Stieltjes integral. It is known [10] that for every continuous function $\phi\colon [-r, 0] \to \mathbb{R}$ such that

$$\phi(0) = \int_{-r}^{0} \phi(\theta)\, d\eta(\theta)$$

Eq. (1.1) has a unique continuous solution $x\colon [-r, \infty) \to \mathbb{R}$ with initial values

$$x(t) = \phi(t), \qquad -r \leq t \leq 0. \tag{1.2}$$

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The main result of [20] says that under the above assumptions the characteristic equation

$$\varDelta(s) = 0, \qquad \varDelta(s) = 1 - \int_{-r}^{0} e^{s\theta}\, d\eta(\theta)$$

has a unique real root λ, and if x is the solution of the initial value problem (1.1) and (1.2), then the function

$$[0,\infty) \ni t \longmapsto x(t)e^{-\lambda t} \in \mathbb{R}$$

is Cesàro-summable to a limit which can be expressed in terms of the initial function ϕ. More precisely, the limit relation

$$\lim_{t\to\infty}\left[\frac{1}{t}\int_0^t x(\tau)e^{-\lambda\tau}\,d\tau\right] = \frac{1}{\varDelta'(\lambda)}\int_{-r}^{0}\left(e^{\lambda\theta}\int_\theta^0 \phi(\tau)e^{-\lambda\tau}\,d\tau\right)d\eta(\theta) \qquad (1.3)$$

holds.

Eq. (1.1) includes as a special case the equation with discrete delays

$$x(t) = \sum_{j=1}^{N} a_j x(t - r_j), \qquad t \geq 0, \qquad (1.4)$$

where

$$a_j > 0, \qquad 1 \leq j \leq N, \qquad (1.5)$$

and

$$0 < r_1 < r_2 < \cdots < r_N. \qquad (1.6)$$

Indeed, if we let $r = r_N$ and

$$\eta(\theta) = \sum_{j=1}^{N} a_j H(\theta + r_j), \qquad -r \leq \theta \leq 0,$$

where $H\colon \mathbb{R} \to \mathbb{R}$ is the step function defined by

$$H(t) = \begin{cases} -1 & \text{for } t \leq 0 \\ 0 & \text{for } t > 0, \end{cases}$$

then Eq. (1.1) reduces to Eq. (1.4). Our aim in this paper is to show that for Eq. (1.4) the asymptotic relation (1.3) can be improved. We will show that if $N \geq 2$, then the large time behavior of the solutions of (1.4) is determined by the fact whether the delays in (1.4) are rationally related or not. Recall that the delays $r_j > 0$, $1 \leq j \leq N$, are *rationally related* if all ratios

$$\frac{r_j}{r_k}, \qquad 1 \leq j \leq N, \quad 1 \leq k \leq N,$$

are rational numbers. It is easily seen that if the delays $r_j > 0$, $1 \leq j \leq N$, are rationally related, then there exists $\rho > 0$ such that all delays are integer multiples of ρ, that is

$$r_j = n_j\rho \qquad \text{for some positive integer } n_j, \quad 1 \leq j \leq N. \qquad (1.7)$$

According to a result by Medina and the author [14], under condition (1.7) every continuous solution $x : [-r_N, \infty) \to \mathbb{R}$ of (1.4) satisfies the asymptotic relation

$$x(t)e^{-\lambda t} = p(t) + o(1), \qquad t \to \infty, \tag{1.8}$$

where λ is the unique real characteristic root of (1.4) and $p\colon \mathbb{R} \to \mathbb{R}$ is a continuous ρ-periodic function which can be given explicitly in terms of the initial data (see [14, Theorem 3] for details). By a *characteristic root* of (1.4), we mean a root of the *characteristic equation*

$$h(s) = 0, \qquad h(s) = 1 - \sum_{j=1}^{N} a_j e^{-sr_j}. \tag{1.9}$$

As shown in [14], if (1.7) holds, then the transformation $z = e^{s\rho}$ reduces the characteristic equation (1.9) to a polynomial equation and Eq. (1.4) can be regarded as a higher order recurrence equation involving a parameter. If the delays in (1.4) are not rationally related, then no such reduction seems to be possible and the analysis of Eq. (1.4) is more difficult. In this paper, we will consider this interesting case of rationally non-related delays. Our main result is the following improvement of the asymptotic relation (1.3).

Theorem 1. *Suppose that conditions (1.5) and (1.6) hold. Assume also that $N \geq 2$ and the delays r_j, $1 \leq j \leq N$, are not rationally related. Let $x\colon [-r_N, \infty) \to \mathbb{R}$ be a solution of Eq. (1.4) with initial values*

$$x(t) = \phi(t), \qquad -r_N \leq t \leq 0, \tag{1.10}$$

where $\phi : [-r_N, 0] \to \mathbb{R}$ is a continuous function such that

$$\phi(0) = \sum_{j=1}^{N} a_j \phi(-r_j). \tag{1.11}$$

Then

$$\lim_{t\to\infty} \left[\, x(t)e^{-\lambda t} \,\right] = \frac{1}{h'(\lambda)} \sum_{j=1}^{N} a_j \int_{-r_j}^{0} \phi(\theta)e^{-\lambda(\theta + r_j)}\, d\theta, \tag{1.12}$$

where λ is the unique real root of the characteristic function h defined by (1.9).

Remark 1. It is easily verified that if the delays r_j, $\leq j \leq N$, are rationally related and ρ has the meaning from (1.7), then for every continuous ρ-periodic function $p\colon \mathbb{R} \to \mathbb{R}$ the function

$$x(t) = p(t)e^{\lambda t}, \qquad t \geq -r_N,$$

where λ is the unique real characteristic root of (1.4), is a solution of Eq. (1.4). Clearly, if p is nonconstant, then the limit in (1.12) does not exist. Thus, if the delays in (1.4) are rationally related, then the limit in (1.12) in general does not exist.

Remark 2. A close look of Theorem 1 below shows that under the hypotheses of the theorem the unique real characteristic root λ of Eq. (1.4) is *dominant*, that is,

$$\operatorname{Re} s < \lambda \qquad \text{for every characteristic root } s \neq \lambda. \tag{1.13}$$

For a class of neutral functional differential equations Frasson and Verduyn Lunel [7] considered a similar situation and proved an asymptotic result analogous to (1.12). However, it should be noted that instead of (1.13) Frasson and Verduyn Lunel [7] required a stronger assumption. Namely, the existence of a "strictly dominant"real characteristic root λ in the sense that there exists $\epsilon > 0$, the so-called *spectral gap*, such that

$$\operatorname{Re} s < \lambda - \epsilon \qquad \text{for every characteristic root } s \neq \lambda. \tag{1.14}$$

We emphasize that if the delays are rationally independent, then this stronger condition (1.14) does not hold for Eq. (1.4). Indeed, results due to Henry [11] and Avellar and Hale [1] imply that if the delays are rationally independent, then Eq. (1.4) has a sequence of characteristic roots $\{s_n\}_{n=1}^{\infty}$ such that

$$\lambda > \operatorname{Re} s_n \to \lambda \qquad \text{as } n \to \infty.$$

Consequently, although Eq. (1.4) can be regarded as a special case of neutral functional differential equations (see [10]), Theorem 1 cannot be obtained from the results by Frasson and Verduyn Lunel [7]. For similar qualitative results for functional differential equations and finite dimensional difference equations, see [2–6], [9], [16–19] and the references therein.

If the sum of the coefficients in Eq. (1.4) is one, then Theorem 1 reduces to the following result on asymptotic constancy.

Theorem 2. *Consider the difference equation*

$$y(t) = \sum_{j=1}^{N} b_j y(t - r_j), \qquad t \geq 0, \tag{1.15}$$

where

$$b_j > 0, \quad 1 \leq j \leq N, \qquad \sum_{j=1}^{N} b_j = 1, \tag{1.16}$$

and the delays r_j, $1 \leq j \leq N$, satisfy condition (1.6). Assume also that $N \geq 2$ and the delays r_j, $1 \leq j \leq N$, are not rationally related. Let $y\colon [-r_N, \infty) \to \mathbb{R}$ be a solution of Eq. (1.15) with initial values

$$y(t) = \psi(t), \qquad -r_N \leq t \leq 0, \tag{1.17}$$

where $\psi : [-r_N, 0] \to \mathbb{R}$ is a continuous function such that

$$\psi(0) = \sum_{j=1}^{N} b_j \psi(-r_j). \tag{1.18}$$

Then

$$\lim_{t\to\infty} y(t) = \left(\sum_{j=1}^{N} b_j \int_{-r_j}^{0} \psi(\theta)\, d\theta \right) \Big/ \left(\sum_{j=1}^{N} b_j r_j \right). \tag{1.19}$$

Remark 3. Suppose that (1.16) holds and the delays r_j, $1 \leq j \leq N$, are rationally related. If ρ has the meaning from (1.7), then every continuous ρ-periodic function is a solution of Eq. (1.15). Consequently, if the delays in (1.15) are rationally related, then the solutions of (1.15) in general are not convergent as $t \to \infty$.

2. Lemmas

As a preparation for the proof of Theorems 1 and 2, we establish some auxiliary results.

Lemma 1. *Suppose that conditions (1.6) and (1.16) hold. Assume also that* $N \geq 2$ *and the delays* r_j*,* $1 \leq j \leq N$*, are not rationally related. Then* $s = 0$ *is a simple root of the characteristic function*

$$g(s) = 1 - \sum_{j=1}^{N} b_j e^{-sr_j}, \qquad s \in \mathbb{C}, \tag{2.1}$$

associated with Eq. (1.15) and all other roots of g *have negative real part.*

Proof. By virtue of (1.16), we have

$$g(0) = 1 - \sum_{j=1}^{N} b_j = 0,$$

and

$$g'(0) = \sum_{j=1}^{N} b_j r_j > 0.$$

Thus, $s = 0$ is a simple root of g. Next we show that all roots of g have nonpositive real part. Suppose by the way of contradiction that $g(s) = 0$ for some $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$. Then

$$1 = \left| \sum_{j=1}^{N} b_j e^{-sr_j} \right| \leq \sum_{j=1}^{N} |b_j e^{-sr_j}| = \sum_{j=1}^{N} b_j e^{-r_j \operatorname{Re} s} < \sum_{j=1}^{N} b_j = 1,$$

a contradiction. Thus, g cannot have a root with positive real part. It remains to show that $s = 0$ is the only root of g on the imaginary axis $\operatorname{Re} s = 0$. Suppose that $g(s) = 0$ for some $s \in \mathbb{C}$ with $\operatorname{Re} s = 0$. Let $\eta = \operatorname{Im} s$ so that $s = i\eta$, where i is the imaginary unit. We need to show that $\eta = 0$. Since $g(i\eta) = 0$, we have

$$1 = \sum_{j=1}^{N} b_j e^{-i\eta r_j} = \sum_{j=1}^{N} b_j \cos(\eta r_j) - i \sum_{j=1}^{N} b_j \sin(\eta r_j).$$

From this, we find that

$$1 = \sum_{j=1}^{N} b_j \cos(\eta r_j) \leq \sum_{j=1}^{N} b_j = 1,$$

and therefore the last inequality must be an equality. Consequently,

$$\cos(\eta r_j) = 1, \qquad 1 \leq j \leq N,$$

and hence

$$\eta r_j = 2 m_j \pi \qquad \text{for some integer } m_j, \qquad 1 \leq j \leq N. \tag{2.2}$$

This implies that $\eta = 0$. Indeed, if η was different from zero, then (2.2) would imply that

$$\frac{r_j}{r_k} = \frac{m_j}{m_k}$$

is rational for all j, $k \in \{1, \dots, N\}$ contradicting the assumption that the delays r_j, $1 \leq j \leq N$, are not rationally related. Thus, $\eta = 0$. $\square$

Lemma 2. *Suppose that conditions (1.6) and (1.16) hold. Let* $y\colon [-r_N,\infty)\to\mathbb{R}$ *be the solution of the initial value problem (1.15) and (1.17), where* $\psi : [-r_N,0]\to\mathbb{R}$ *is a continuous function satisfying (1.18). Then* y *is uniformly continuous on* $[0,\infty)$ *and*

$$\sup_{t\ge 0}|y(t)|\le \max_{-r_N\le t\le 0}|\psi(t)|. \tag{2.3}$$

Proof. For every $\delta>0$ and for every positive integer n, define

$$\omega_\delta(n)=\sup\{\,|y(t_2)-y(t_1)|\mid t_1,\,t_2\in[-r_N,nr_1],\ 0<t_2-t_1<\delta\,\}.$$

From the triangle inequality

$$|y(t_2)-y(t_1)|\le|y(t_2)|+|y(t_1)|\qquad\text{whenever } t_1,\,t_2\in[-r_N,nr_1],$$

it follows that

$$\omega_\delta(n)\le 2\max_{-r_N\le t\le nr_1}|y(t)|<\infty.$$

We will show that if $\delta<r_1$, then the sequence $\{\omega_\delta(n)\}_{n=1}^\infty$ is nonincreasing. Suppose that $\delta\in(0,r_1)$ and n is a fixed positive integer. Choose $t_1,\,t_2\in[-r_N,(n{+}1)r_1]$ such that $0<t_2-t_1<\delta$. Two cases may occur depending on whether $t_2\le nr_1$ or $nr_1<t_2\le(n+1)r_1$.

Case 1. Suppose that $t_2\le nr_1$. Since $t_2-t_1>0$, we have that $t_1<t_2\le nr_1$. From the definition of $\omega_\delta(n)$, we obtain

$$|y(t_2)-y(t_1)|\le\omega_\delta(n). \tag{2.4}$$

Case 2. Now suppose that $nr_1<t_2\le(n+1)r_1$. Taking into account that $0<t_2-t_1<\delta$, we have

$$t_2>t_1>t_2-\delta>nr_1-\delta\ge r_1-\delta>0.$$

Consequently, we can use Eq. (1.15) to obtain

$$y(t_2)-y(t_1)=\sum_{j=1}^N b_jy(t_2-r_j)-\sum_{j=1}^N b_jy(t_1-r_j)=\sum_{j=1}^N b_j\big(\,y(t_2-r_j)-y(t_1-r_j)\,\big).$$

Hence

$$|y(t_2)-y(t_1)|\le\sum_{j=1}^N b_j|y(t_2-r_j)-y(t_1-r_j)|. \tag{2.5}$$

For each $j\in\{1,\dots,N\}$, we have

$$t_1-r_j<t_2-r_j\le t_2-r_1\le(n+1)r_1-r_1=nr_1$$

and

$$(t_2-r_j)-(t_1-r_j)=t_2-t_1\in(0,\delta).$$

This, together with the definition of $\omega_\delta(n)$, implies

$$|y(t_2-r_j)-y(t_1-r_j)|\le\omega_\delta(n),\qquad 1\le j\le N.$$

Consequently, the right-hand side of (2.5) is not greater than

$$\sum_{j=1}^{N} b_j\, \omega_\delta(n) = \omega_\delta(n).$$

Thus, in both Cases 1 and 2, Inequality (2.4) holds. Since $t_1,\, t_2 \in [-r_N, (n+1)r_1]$, $0 < t_2 - t_1 < \delta$, were arbitrary, this implies

$$\omega_\delta(n+1) \le \omega_\delta(n).$$

In particular,

$$\omega_\delta(n) \le \omega_\delta(1) \qquad \text{for every positive integer } n. \tag{2.6}$$

Using (2.6), we can easily show that y is uniformly continuous on $[0,\infty)$. Let $\epsilon > 0$ be given. Since y is continuous on the compact interval $[-r_N, r_1]$, it is uniformly continuous there. Consequently, there exists $\delta > 0$ such that

$$|y(t_2) - y(t_1)| < \frac{\epsilon}{2} \qquad \text{whenever } t_1,\, t_2 \in [-r_N, r_1] \text{ and } 0 < t_2 - t_1 < \delta. \tag{2.7}$$

Without loss of generality, we may (and do) assume that $\delta < r_1$. From (2.7), we obtain

$$\omega_\delta(1) \le \frac{\epsilon}{2}. \tag{2.8}$$

Suppose that $t_1,\, t_2 \in [0,\infty)$ and $0 < t_2 - t_1 < \delta$. Choose a positive integer n such that $n \ge t_2/r_1$ so that $t_1 < t_2 \le nr_1$. From the definition of $\omega_\delta(n)$ and Inequalities (2.6) and (2.8), we find that

$$|y(t_2) - y(t_1)| \le \omega_\delta(n) \le \omega_\delta(1) \le \frac{\epsilon}{2} < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves that y is uniformly continuous on $[0,\infty)$.

It remains to show (2.3). Define

$$M = \max_{-r_N \le t \le 0} |\psi(t)|.$$

Let $\epsilon > 0$. We claim that

$$|y(t)| < M + \epsilon \qquad \text{for all } t \ge 0. \tag{2.9}$$

Suppose by the way of contradiction that (2.9) does not hold. Since $|y(0)| = |\psi(0)| \le M < M + \epsilon$, there exists $t_1 > 0$ such that

$$|y(t)| < M + \epsilon \quad \text{for all } t \in [0, t_1) \qquad \text{and} \qquad |y(t_1)| = M + \epsilon.$$

From this and Eq. (1.15), we obtain

$$M + \epsilon = |y(t_1)| = \left|\sum_{j=1}^{N} b_j y(t_1 - r_j)\right| \le \sum_{j=1}^{N} b_j |y(t_1 - r_j)| < \sum_{j=1}^{N} b_j (M+\epsilon) = M + \epsilon,$$

a contradiction. Consequently, (2.9) holds and hence

$$\sup_{t\ge 0} |y(t)| \le M + \epsilon.$$

Conclusion (2.3) follows by letting $\epsilon \to 0$ in the last inequality. $\square$

The proof of our main results will be based on Newman's Tauberian theorem for the Laplace transform (see [13, Chap. XVI, Lemma 2.2] or [12, Sec. 1.2]). For the readers' convenience, we state it in Lemma 3 below. It is noteworthy that Newman's beautiful result can be used to give a simple short proof of the famous Prime Number Theorem (see [12], [13, Chap. XVI] or [15] for details).

Lemma 3. *Let $z\colon [0,\infty) \to \mathbb{R}$ be a continuous and bounded function so that the Laplace transform*

$$\tilde{z}(s) = \int_0^\infty e^{-st} z(t)\, dt$$

is well-defined and holomorphic in the open half-plane $\operatorname{Re} s > 0$. Suppose that $\tilde{z}$ can be extended as a holomorphic function to a neighborhood of every point on the imaginary axis. Then the improper Riemann integral

$$\int_0^\infty z(t)\, dt \tag{2.10}$$

converges and is equal to $\tilde{z}(0)$.

We will also need the following variant of a result due to Barbălat.

Lemma 4. *Let $z\colon [0,\infty) \to \mathbb{R}$ be a continuous function such that the improper Riemann integral (2.10) converges. If z is uniformly continuous on $[0,\infty)$, then*

$$\lim_{t\to\infty} z(t) = 0.$$

Lemma 4 follows from [8, Lemma 1.2.3] applied to the function $g\colon [0,\infty) \to \mathbb{R}$ defined by $g(t) = \int_0^t z(\tau)\, d\tau$ for $t \geq 0$.

3. Proofs of the Main Results

First we give a proof of Theorem 2.

Proof of Theorem 2. By Lemma 2, the solution y of the initial value problem (1.15) and (1.17) is bounded on $[0,\infty)$. Consequently, the Laplace transform

$$\tilde{y}(s) = \int_0^\infty e^{-st} y(t)\, dt$$

is well-defined and holomorphic in the open half-plane $\operatorname{Re} s > 0$. Taking the Laplace transform of both sides of Eq. (1.15) and using the shifting property, we find that

$$\tilde{y}(s) = \sum_{j=1}^N b_j \left(e^{-sr_j} \tilde{y}(s) + \int_{-r_j}^0 \psi(\theta) e^{-s(\theta + r_j)}\, d\theta \right)$$

whenever $\operatorname{Re} s > 0$. Hence

$$\tilde{y}(s) g(s) = f(s) \qquad \text{whenever } \operatorname{Re} s > 0, \tag{3.1}$$

where g is the characteristic function given by (2.1) and

$$f(s) = \sum_{j=1}^N b_j \int_{-r_j}^0 \psi(\theta) e^{-s(\theta + r_j)}\, d\theta.$$

By Lemma 1, all roots of g have nonpositive real part. Consequently, (3.1) can be written in the form

$$\tilde{y}(s) = \frac{f(s)}{g(s)} \qquad \text{whenever } \operatorname{Re} s > 0. \tag{3.2}$$

Clearly, both f and g are entire functions. Further, according to Lemma 1, $s = 0$ is the only root of g on the imaginary axis and $g'(0) \neq 0$. This, together with (3.2), implies that $\tilde{y}$ can be extended as a holomorphic function to a neighborhood of every point on the imaginary axis with the possible exception of $s = 0$ at which $\tilde{y} = f/g$ has at most simple pole. This means that the Laurent series of $\tilde{y} = f/g$ at $s = 0$ has the form

$$\tilde{y}(s) = \frac{f(s)}{g(s)} = \sum_{j=-1}^{\infty} C_j s^j \qquad \text{whenever } 0 < |s| < \epsilon, \tag{3.3}$$

where $\epsilon > 0$ is sufficiently small and

$$C_{-1} = \operatorname*{Res}_{s=0} \frac{f(s)}{g(s)} = \frac{f(0)}{g'(0)} = \left(\sum_{j=1}^{N} b_j \int_{-r_j}^{0} \psi(\theta)\, d\theta \right) \Big/ \left(\sum_{j=1}^{N} b_j r_j \right). \tag{3.4}$$

Define

$$z(t) = y(t) - C_{-1}, \qquad t \geq 0. \tag{3.5}$$

By Lemma 2, y and hence z is bounded and uniformly continuous on $[0, \infty)$. Further, taking the Laplace transform of z, we find that

$$\tilde{z}(s) = \tilde{y}(s) - \frac{C_{-1}}{s} \qquad \text{whenever } \operatorname{Re} s > 0. \tag{3.6}$$

As noted before, $\tilde{y}$ and hence $\tilde{z}$ can be extended as a holomorphic function to a neighborhood of every nonzero point on the imaginary axis. Moreover, from (3.3) and (3.6), we see that $\tilde{z}$ can be extended as a holomorphic function also to the ϵ-neighborhood of $s = 0$ by

$$\tilde{z}(s) = \sum_{j=0}^{\infty} C_j s^j \qquad \text{whenever } |s| < \epsilon.$$

By the applications of Newman's theorem (see Lemma 3), we conclude that the improper integral (2.10) converges. As noted before, z is uniformly continuous on $[0, \infty)$ and therefore Barbălat's lemma (see Lemma 4) implies that

$$\lim_{t \to \infty} z(t) = 0.$$

Hence

$$\lim_{t \to \infty} y(t) = C_{-1},$$

which, in view of (3.4), is equivalent to (1.19). $\square$

Using Theorem 2, we can give a short proof of Theorem 1.

Proof of Theorem 1. Let h be the characteristic function defined by (1.9). By virtue of (1.5), we have

$$h'(\tau) = \sum_{j=1}^{N} a_j r_j e^{-\tau r_j} > 0 \qquad \text{for all } \tau \in (-\infty, \infty).$$

Consequently, h is strictly increasing on $(-\infty, \infty)$. Since

$$\lim_{\tau \to -\infty} h(\tau) = -\infty \qquad \text{and} \qquad \lim_{\tau \to \infty} h(\tau) = 1,$$

h has a unique real root λ. Let x be the solution of (1.4) with initial values (1.10). Define

$$y(t) = x(t) e^{-\lambda t}, \qquad t \geq -r_N.$$

Then y is a solution of the initial value problem (1.15) and (1.17) with

$$b_j = a_j e^{-\lambda r_j}, \qquad 1 \leq j \leq N,$$

and

$$\psi(t) = \phi(t) e^{-\lambda t} \qquad - r_N \leq t \leq 0.$$

It is easily verified that the hypotheses of Theorem 2 are satisfied. By the application of Theorem 2, we conclude that (1.19) holds which is only a reformulation of conclusion (1.12) of Theorem 1. □

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