# A LIMIT BOUNDARY VALUE PROBLEM FOR A NONLINEAR DIFFERENCE EQUATION

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Abstract. We give an asymptotic description of the monotone increasing solutions of a limit boundary value problem for a class of nonlinear difference equations with continuous arguments and rationally non-related shifts.

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## 1. Introduction

In [5] Mallet-Paret studied the existence and some properties of the monotone increasing solutions $x : \mathbb{R} \to \mathbb{R}$ of the limit boundary value problem (LBVP for short)

$$-c\,x'(\xi) = F(x(\xi + r_1), x(\xi + r_2), \dots, x(\xi + r_N)), \tag{1.1}$$

$$\lim_{\xi\to-\infty} x(\xi) = -1, \qquad \lim_{\xi\to\infty} x(\xi) = 1, \tag{1.2}$$

where $c \in \mathbb{R}$ and the quantities $r_j \in \mathbb{R}$, $1 \le j \le N$, the so-called *shifts,* and the nonlinearity $F\colon \mathbb{R}^N \to \mathbb{R}$ satisfy the following standing assumptions:

(i) $N \ge 2$, $r_1 = 0$ and $r_j \ne r_k$ whenever $1 \le j < k \le N$.

(ii) $F : \mathbb{R}^N \to \mathbb{R}$ is a continuously differentiable function such that the partial derivatives $D_jF$, $1 \le j \le N$, are locally Lipschitz continuous.

(iii) $D_jF(u) > 0$ whenever $u \in \mathbb{R}^N$ and $2 \le j \le N$.

(iv) There exists $q \in (-1, 1)$ such that the function $\Phi\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(x) = F(x, x, \dots, x), \qquad x \in \mathbb{R},$$

satisfies the following conditions:

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$$
\begin{aligned}
&\Phi(x) > 0 \qquad \text{for } x \in (-\infty, -1) \cup (q, 1),\\
&\Phi(x) < 0 \qquad \text{for } x \in (-1, q) \cup (1, \infty),\\
&\Phi(-1) = \Phi(q) = \Phi(1) = 0.
\end{aligned}
$$

(v) We have that

$$\Phi'(-1) < 0, \qquad \Phi'(q) > 0, \qquad \text{and} \qquad \Phi'(1) < 0.$$

Throughout the paper, the terms *monotone increasing* and *monotone decreasing* are used as synonyms for nondecreasing and nonincreasing, respectively.

Note that assumption (iv) implies that the Eq. (1.1) has exactly three equilibria, $x = -1$, $x = q$ and $x = 1$. If $c \neq 0$ then Eq (1.1) is a functional differential equation of mixed type (including both delayed and advanced arguments), while in the case when $c = 0$ Eq. (1.1) reduces to a difference equation.

Under the above hypotheses, Mallet-Paret [5] gave the following asymptotic description of the monotone increasing solutions of LBVP (1.1)–(1.2).

**Theorem 1.1.** [5, Theorem 2.2] *If $c \neq 0$ and $x\colon \mathbb{R} \to \mathbb{R}$ is a monotone increasing solution of LBVP (1.1)–(1.2), then there exist $C_\pm > 0$ and $\epsilon > 0$ such that*

$$x(\xi) = \begin{cases} -1 + C_- e^{\lambda_-^u \xi} + O(e^{(\lambda_-^u + \epsilon)\xi}), & \xi \to -\infty, \\ 1 - C_+ e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s - \epsilon)\xi}), & \xi \to \infty, \end{cases} \tag{1.3}$$

*where $\lambda_-^u \in (0, \infty)$ is the unique positive eigenvalue of the linearization of Eq. (1.1) about the equilibrium $x = -1$,*

$$-cx'(\xi) = \sum_{j=1}^{N} D_j F(\kappa(-1))\, x(\xi + r_j), \tag{1.4}$$

*and $\lambda_+^s \in (-\infty, 0)$ is the unique negative eigenvalue of the linearization of Eq. (1.1) about the equilibrium $x = 1$,*

$$-cx'(\xi) = \sum_{j=1}^{N} D_j F(\kappa(1))\, x(\xi + r_j), \tag{1.5}$$

*where*

$$\kappa(x) = (x, x, \ldots, x) \in \mathbb{R}^N, \qquad x \in \mathbb{R}. \tag{1.6}$$

*If $c = 0$ and $x\colon \mathbb{R} \to \mathbb{R}$ is a monotone increasing solution of LBVP (1.1)–(1.2), then*

$$
\begin{aligned}
\lim_{\xi \to -\infty} \frac{1}{\xi} \log(1 + x(\xi)) &= \lambda_-^u, \qquad \text{if } r_{\min} > 0, \\
\lim_{\xi \to \infty} \frac{1}{\xi} \log(1 - x(\xi)) &= \lambda_+^s, \qquad \text{if } r_{\min} < 0,
\end{aligned} \tag{1.7}
$$

*where*

$$r_{\min} = \min_{1 \le j \le N} r_j, \qquad r_{\max} = \max_{1 \le j \le N} r_j,$$

*and $\lambda_-^u$, $\lambda_+^s$ have the same meaning as before.*

Note that the existence and uniqueness of the eigenvalues $\lambda_-^u$ and $\lambda_+^s$ is part of the conclusion of the theorem.

Clearly, if $c = 0$, in the case of difference equations, the asymptotic formulas for $x(\xi)$ are not as sharp as in the the case when $c \neq 0$. Our aim in this paper is to show that in the case $c = 0$ the limit relations (1.7) for the monotone increasing solutions of LBVP (1.1)–(1.2) can be improved.

## 2. Main Result

If $c = 0$ then there is an important difference between the cases of rationally related shifts and rationally non-related shifts. Recall that the shifts $r_j$, $1 \le j \le N$, are *rationally related* if all ratios

$$\frac{r_j}{r_k}, \qquad 1 \le j < k \le N,$$

are rational. In this case there exists $\nu > 0$ such that all shifts $r_j$, $1 \le j \le N$, are integer multiples of $\nu$ and Eq. (1.1) can be reduced to a higher order recurrence equation. If the shifts in (1.1) are not rationally related, then no such reduction seems to be possible and the problem becomes more difficult and interesting. In this paper, we will restrict ourselves to the case $c = 0$ and rationally non-related shifts. We will prove the following improvement of the limit relations (1.7) of Theorem 1.1.

**Theorem 2.1.** *Suppose that $c = 0$ and the shifts $r_j$, $1 \le j \le N$, are not rationally related. If $x\colon \mathbb{R} \to \mathbb{R}$ is a monotone increasing solution of LBVP (1.1)–(1.2) then there exist constants $C_\pm > 0$ such that*

$$x(\xi) = \begin{cases} -1 + C_- e^{\lambda_-^u \xi} + o(e^{\lambda_-^u \xi}), & \xi \to -\infty, \quad \text{if } r_{\max} > 0, \\ 1 - C_+ e^{\lambda_+^s \xi} + o(e^{\lambda_+^s \xi}), & \xi \to \infty, \quad \text{if } r_{\min} < 0, \end{cases} \tag{2.1}$$

*where $\lambda_-^u$ and $\lambda_+^s$ have the same meaning as in Theorem 1.1.*

Before we give a proof of Theorem 2.1, we establish an auxiliary result for the linear difference equation

$$\sum_{j=1}^{N} A_j(\xi) y(\xi + r_j) = 0, \tag{2.2}$$

where the shifts $r_j$, $1 \le j \le N$, satisfy condition (i) of Section 1 and the coefficients $A_j\colon \mathbb{R} \to \mathbb{R}$, $1 \le j \le N$, are locally integrable functions with the following properties:

(a) There exist constants

$$\begin{aligned} \alpha_j,\, \beta_j \in \mathbb{R}, &\qquad 1 \le j \le N, \\ \alpha_j > 0, &\qquad 2 \le j \le N, \end{aligned} \tag{2.3}$$

such that

$$\alpha_j \le A_j(\xi) \le \beta_j, \qquad \xi \in \mathbb{R}, \quad 1 \le j \le N. \tag{2.4}$$

(b) The limits

$$A_{j\pm} = \lim_{\xi \to \pm\infty} A_j(\xi), \qquad 1 \le j \le N, \tag{2.5}$$

exist (in $\mathbb{R}$), and the convergence is exponentially fast, that is, for some $k > 0$, we have

$$A_j(\xi) = A_{j\pm} + O(e^{-k|\xi|}), \qquad \xi \to \pm\infty, \quad 1 \le j \le N. \tag{2.6}$$

(c) The sum of the limits in (2.5) is negative, that is,

$$A_{\Sigma\pm} = \sum_{j=1}^{N} A_{j\pm} < 0. \tag{2.7}$$

Recall that under condition (2.5) Eq. (2.2) is said to be *asymptotically autonomous as* $\xi \to \pm\infty$ and the constant coefficient equation

$$\sum_{j=1}^{N} A_{j\pm}\, y(\xi + r_j) = 0 \tag{2.8}$$

is called the *limiting equation* of (2.2) as $\xi \to \pm\infty$. The *eigenvalues* of (2.8) are the roots of the *characteristic equation*

$$\Delta_{\pm}(s) = 0, \qquad \text{where } \Delta_{\pm}(s) = \sum_{j=1}^{N} A_{j\pm}\, e^{s r_j}. \tag{2.9}$$

The proof of Theorems 2.1 will be based on the following proposition.

**Proposition 2.2.** *Suppose that the shifts $r_j$, $1 \le j \le N$, are not rationally related and $r_{\min} < 0$. Let $y\colon [r_{\min}, \infty) \to (0,\infty)$ be a positive, monotone decreasing function satisfying Eq. (2.2) for $\xi \ge 0$. Assume that conditions (2.3) and (2.4) hold but only for $\xi \ge 0$. Assume also that Eq. (2.2) is asymptotically autonomous as $\xi \to \infty$ and the convergence is exponentially fast in the sense that conditions (2.5) and (2.6) hold but only for $\xi \to \infty$. Finally, assume that the sum $A_{\Sigma+}$ in (2.7) is negative. Then there exists a constant $C_+ > 0$ such that*

$$y(\xi) = C_+ e^{\lambda_+^s \xi} + o(e^{\lambda_+^s \xi}), \qquad \xi \to \infty, \tag{2.10}$$

*where $\lambda_+^s$ is the unique negative eigenvalue of the limiting equation of (2.2) as $\xi \to \infty$.*

*The analogous result for the positive, monotone increasing solutions of Eq. (2.2) on $(-\infty, 0]$ as $\xi \to -\infty$, namely*

$$y(\xi) = C_- e^{\lambda_-^u \xi} + o(e^{\lambda_-^u \xi}), \qquad \xi \to -\infty, \tag{2.11}$$

*where $C_- > 0$ and $\lambda_-^u$ is the unique positive eigenvalue of the limiting equation of (2.2) as $\xi \to -\infty$, also holds when $r_{\max} > 0$, (2.3) and (2.4) are assumed for $\xi \le 0$, (2.5) and (2.6) hold but only for $\xi \to -\infty$ and $A_{\Sigma+} < 0$ is replaced with $A_{\Sigma-} < 0$.*

In the following lemmas we summarize some known results which will be used in the proof of Proposition 2.2. The first result is a consequence of [5, Lemma 4.2] and [5, Proposition 4.3].

**Lemma 2.3.** *Suppose that the shifts $r_j$, $\le j \le N$, are not rationally related and $r_{\min} < 0$. Assume that $A_{j+} > 0$, $2 \le j \le N$, and the sum $A_{\Sigma+}$ in (2.7) is negative. Then the characteristic function $\Delta_+$ defined by (2.9) has a unique negative root denoted by $\lambda_+^s$. Moreover, this root is simple and all other roots of $\Delta_+$ have real parts different from $\lambda_+^s$.*

*The analogous result for $\Delta_-$ also holds when $r_{\max} > 0$, $A_{j+} > 0$, $2 \le j \le N$, is replaced with $A_{j-} > 0$, $2 \le j \le N$, $A_{\Sigma+} < 0$ is replaced with $A_{\Sigma-} < 0$ and $\lambda_+^s$ is replaced with $\lambda_-^u$, the unique positive root of $\Delta_-$.*

The next result was proved in [5, Proposition 5.4].

**Lemma 2.4.** *Suppose that $r_{\min} < 0$ and let $y\colon [r_{\min}, \infty) \to (0, \infty)$ be a positive, monotone decreasing function satisfying Eq. (2.2) for $\xi \geq 0$. Assume that conditions (2.3) and (2.4) hold but only for $\xi \geq 0$. Assume also that Eq. (2.2) is asymptotically autonomous as $\xi \to \infty$ in the sense that the limits in (2.5) exist but only for $\xi \to \infty$. Finally, assume the sum $A_{\Sigma+}$ in (2.7) is negative. Then*

$$\lim_{\xi \to \infty} \frac{1}{\xi} \log y(\xi) = \lambda_+^s, \tag{2.12}$$

*where $\lambda_+^s$ is the unique negative eigenvalue of the limiting equation of (2.2) as $\xi \to \infty$.*

*The analogous result as $\xi \to -\infty$ also holds when $r_{\max} > 0$ and $\lambda_+^s$ is replaced with $\lambda_-^u$, the unique positive eigenvalue of the limiting equation of (2.2) as $\xi \to -\infty$.*

For some related results on asymptotically autonomous equations, see, e.g., [2], [4], [6], [7], [8], [9].

We will also need two basic results from the theory of Laplace transform. The first result, rediscovered in [5, Lemma 3.5] (see also [1]), is sometimes called as the Pringsheim-Landau theorem (see, e.g., [10]).

**Lemma 2.5.** *Suppose that $y\colon [0, \infty) \to [0, \infty)$ is a nonnegative measurable function such that the abscissa of convergence $\sigma_c$ of the Laplace transform*

$$\tilde{y}(s) = \int_0^\infty y(\xi) e^{-s\xi}\, d\xi \tag{2.13}$$

*is finite. Then $\tilde{y}(s)$ cannot be extended as a holomorphic function to any neighborhood of $s = \sigma_c$.*

The basic tool in the proof of Proposition 2.2 will be the following variant of Ikehara's Tauberian theorem [10] (see [3, Proposition 2.3]).

**Lemma 2.6.** *Let $y\colon [0, \infty) \to (0, \infty)$ be a positive, monotone decreasing function such that its Laplace transform $\tilde{y}(s)$ converges in the half-plane $\operatorname{Re} s > \sigma$ for some $\sigma \in (-\infty, 0)$. Assume that for some constant $C$ the function*

$$\tilde{y}(s) - \frac{C}{s - \sigma} \tag{2.14}$$

*can be extended as a holomorphic function to a neighborhood of every point of the vertical line $\operatorname{Re} s = \sigma$. Then*

$$\lim_{\xi \to \infty} y(\xi) e^{-\sigma\xi} = C. \tag{2.15}$$

Now we are in a position to give a proof of Proposition 2.2.

*Proof of Proposition 2.2.* Consider the case when $\xi \to \infty$. Rewrite Eq. (2.2) as

$$\sum_{j=1}^{N} A_{j+} y(\xi + r_j) = h(\xi), \tag{2.16}$$

where

$$h(\xi) = \sum_{j=1}^{N} (A_{j+} - A_j(\xi)) y(\xi + r_j).$$

Let $\sigma_c$ be the abscissa of convergence of the Laplace transform $\tilde{y}(s)$ of the positive, monotone decreasing solution $y$ of (2.2). Conclusion (2.12) of Lemma 2.4 implies that the Laplace transform $\tilde{y}(s)$ converges for $s \in (\lambda_+^s, \infty)$ and diverges for $s \in (-\infty, \lambda_+^s)$. Hence $\sigma_c = \lambda_+^s$. The asymptotic relations (2.6) and (2.12) imply that the Laplace transform $\tilde{h}(s)$ of $h$ converges for $\operatorname{Re} s > \lambda_+^s - k$. Taking the Laplace transform of Eq. (2.16), we obtain for $\operatorname{Re} s > \lambda_+^s$,

$$\varDelta_+(s)\tilde{y}(s) = \psi(s) + \tilde{h}(s),$$

with $\varDelta_+$ as in (2.9) and

$$\psi(s) = -\sum_{j=1}^{N} A_{j+} \int_{-r_j}^{0} y(\xi + r_j) e^{-s\xi}\, d\xi.$$

Taking into account that $\psi$ is an entire function, we obtain

$$\varDelta_+(s)\tilde{y}(s) = f(s) \qquad \text{for } \operatorname{Re} s > \lambda_+^s, \tag{2.17}$$

where

$$f(s) = \psi(s) + \tilde{h}(s) \qquad \text{is holomorphic for } \operatorname{Re} s > \lambda_+^s - k. \tag{2.18}$$

By Lemma 2.3, $\lambda_+^s$ is the only root of $\varDelta_+$ on the vertical line $\operatorname{Re} s = \lambda_+^s$. Therefore (2.17) and (2.18) imply that $\tilde{y}(s)$ can be extended as a holomorphic function to a neighborhood of every point of the vertical line $\operatorname{Re} s = \lambda_+^s$ with the exception of $s = \lambda_+^s$ by

$$\tilde{y}(s) = \frac{f(s)}{\varDelta_+(s)}.$$

Since $s = \lambda_+^s$ is a simple root of $\varDelta_+$ (see Lemma 2.3), the last identity shows that $\tilde{y}(s)$ has at most simple pole at $s = \lambda_+^s$ so that the Laurent series of $\tilde{y}(s)$ at $s = \lambda_+^s$ has the form

$$\tilde{y}(s) = \sum_{j=-1}^{\infty} C_j (s - \lambda_+^s)^j \qquad \text{whenever } 0 < |s - \lambda_+^s| < \epsilon, \tag{2.19}$$

where $\epsilon > 0$ is sufficiently small and

$$C_{-1} = \operatorname*{Res}_{s=\lambda_+^s} \frac{f(s)}{\varDelta_+(s)} = \frac{f(\lambda_+^s)}{\varDelta_+'(\lambda_+^s)}. \tag{2.20}$$

Therefore the function

$$\tilde{y}(s) - \frac{C_{-1}}{s - \lambda_+^s}$$

has a removable singularity at $s = \lambda_+^s$ and the assumptions of Lemma 2.6 hold with $\sigma = \lambda_+^s$ and $C = C_{-1}$. By the application of Lemma 2.6 we conclude that

$$\lim_{\xi \to \infty} y(\xi) e^{-\lambda_+^s \xi} = C_{-1}.$$

Thus, (2.10) holds with $C_+ = C_{-1}$. It remains to show that $C_{-1} > 0$. As a limit of a positive function, $C_{-1} \geq 0$. Suppose by the way of contradiction that $C_{-1} = 0$. Then (2.19) implies that $\tilde{y}(s)$ can be extended as a holomorphic function to the $\epsilon$-neighborhood of $s = \lambda_+^s = \sigma_c$ contradicting Lemma 2.5. Thus, $C_{-1} > 0$.

The analogous result for solutions on $(-\infty, 0]$ can be obtained after a change of variable $\xi \to -\xi$. □

Based on Proposition 2.2, we can give a simple short proof of Theorem 2.1.

*Proof of Theorem 2.1.* We will consider only the case $\xi \to \infty$, as the proof for $\xi \to -\infty$ is similar. Let $x\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing solution of LBVP (1.1)–(1.2). Then

$$y(\xi) = 1 - x(\xi), \qquad \xi \in \mathbb{R},$$

is a nonnegative, monotone decreasing function. From [5, Lemma 3.3], it follows that $y$ is positive. As shown in [5], $y$ is a solution of the linear Eq. (2.2) with

$$A_j(\xi) = \int_0^1 D_j F(t\pi(x,\xi) + (1-t)\kappa(1))\, dt,$$

where

$$\pi(x,\xi) = (x(\xi + r_1), x(\xi + r_2), \ldots, x(\xi + r_N))$$

and $\kappa$ is defined by (1.6). This follows from the formula

$$F(v) - F(w) = \int_0^1 \frac{dF(tv + (1-t)w)}{dt}\, dt = \sum_{j=1}^N \left( \int_0^1 D_j F(tv + (1-t)w)\, dt \right) (v_j - w_j)$$

for any $v$, $w \in \mathbb{R}^N$ and from the fact that $F(\kappa(1)) = \Phi(1) = 0$ (see condition (iv) in Section 1). Assumption (iii) of Section 1 implies that conditions (2.3) and (2.4) hold. From (1.2) and the continuity of the partial derivatives of $F$, it follows that limits in (2.5) exist for $\xi \to \infty$ and

$$A_{j+} = D_j F(\kappa(1)), \qquad 1 \leq j \leq N.$$

Thus, the limiting equation of (2.2) as $\xi \to \infty$ coincides with the linearized equation (1.5) with $c = 0$. Choose $\delta > 0$ such that $\lambda_+^s + \delta < 0$. Then the second limit relation in (1.7) implies the asymptotic estimate

$$1 - x(\xi) = O(e^{(\lambda_+^s + \delta)\xi}), \qquad \xi \to \infty.$$

This, together with the local Lipschitz continuity of the partial derivatives $D_j F$, $1 \leq j \leq N$, implies that the convergence in (2.5) is exponentially fast, namely, condition (2.6) holds with $k = -(\lambda_+^s + \delta)$ for $\xi \to \infty$. Finally, the last inequality in condition (v) of Section 1 implies that the sum $A_{\Sigma+}$ in (2.7) is negative. Thus, we have verified all hypotheses of Proposition 2.2. Therefore Proposition 2.2 applies and its conclusion (2.10) is only a reformulation the limit relation (2.1) for $\xi \to \infty$. □

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