WEIGHTED LIMITS FOR POINCARÉ DIFFERENCE EQUATIONS

Rotchana Chieocan

Department of Mathematics, Faculty of Science,
Khon Kaen University,
40002 Khon Kaen, Thailand
rotchana@kku.ac.th

and

Mihály Pituk*

Department of Mathematics,
University of Pannonia,
P.O. Box 158, 8201 Veszprém, Hungary
pitukm@almos.uni-pannon.hu

Abstract. The Poincaré difference equation

$$\mathbf{x}_{n+1} = A_n \mathbf{x}_n, \qquad n \in \mathbb{N},$$

is considered, where A_n, $n \in \mathbb{N}$, are complex square matrices such that the limit $A = \lim_{n\to\infty} A_n$ exists. It is shown that under approriate spectral conditions certain weighted limits of the nonvanishing solutions exist. In the case when the entries of the coefficients A_n, $n \in \mathbb{N}$, and the initial vector $\mathbf{x}_0$ are real our result implies the convergence of the normalized sequence $\frac{\mathbf{x}_n}{\|\mathbf{x}_n\|}$, $n \in \mathbb{N}$, to a normalized eigenvector of the limiting matrix A.

AMS Subject Classification: 39A10; 39A11

Keywords: Poincaré difference equation; growth rate; weighted limit; normalized sequence.

1. Introduction and Main Result

Let $\mathbb{N}$ and $\mathbb{C}$ denote the set of nonnegative integers and the set of complex numbers, respectively. Given a positive integer k, $\mathbb{C}^k$ is the k-dimensional space of complex column vectors with the standard inner product $(\cdot,\cdot)$ and any norm $\|\cdot\|$. The space of $k \times k$ matrices with complex entries is denoted by $\mathbb{C}^{k\times k}$. Let $A \in \mathbb{C}^{k\times k}$. The symbols A^*, $\sigma(A)$ and $\|A\|$ denote the adjoint matrix (the conjugate transpose), the spectrum (the set of eigenvalues) and the operator norm of A, respectively.

Consider the linear difference equation

$$\mathbf{x}_{n+1} = A_n \mathbf{x}_n, \tag{1.1}$$

where the coefficients $A_n \in \mathbb{C}^{k\times k}$, $n \in \mathbb{N}$, are *asymptotically constant*, that is

$$\lim_{n\to\infty} A_n = A \tag{1.2}$$

* Corresponding author. Phone:+36 88 62 4227; Fax:+36 88 62 4521

for some $A \in \mathbb{C}^{k\times k}$. Equations of this type are called *Poincaré difference equations.* Note that Poincaré difference equations have applications in the theory of orthogonal polynomials, continued fractions and numerical mathematics [2], [3].

By a *solution* of (1.1), we mean a sequence $x = (\mathbf{x}_n)_{n\in\mathbb{N}}$ in $\mathbb{C}^k$ such that (1.1) holds for all $n \in \mathbb{N}$. A solution $x = (\mathbf{x}_n)_{n\in\mathbb{N}}$ of (1.1) is said to be *nonvanishing* if $\mathbf{x}_n \neq \mathbf{0}$ for all $n \in \mathbb{N}$. Clearly, if the coefficients $A_n \in \mathbb{C}^{k\times k}$, $n \in \mathbb{N}$, are invertible, then all nontrivial solutions of (1.1) are nonvanishing.

A key result in the study of Eq. (1.1) is the following Perron type theorem (see [7, Theorem 1] or [3, Chap. 8, Theorem 8.46]).

Theorem 1.1. *Suppose (1.2) holds. If $x = (\mathbf{x}_n)_{n\in\mathbb{N}}$ is a nonvanishing solution of (1.1), then the limit*

$$\rho = \rho(x) = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{x}_n\|} \tag{1.3}$$

exists and $\rho = |\lambda|$ for some $\lambda \in \sigma(A)$.

The quantity ρ defined by (1.3) is called the (exponential) *growth rate* of the solution x. Its logarithm is the *Lyapunov exponent.*

For an extension of Theorem 1.1 to a more general class of linear functional difference equations and further related results, see, e.g., [1], [3], [5] and [8].

Let ρ be the growth rate of a nonvanishing solution of Eq. (1.1). By Theorem 1.1, the spectral set

$$\Lambda(\rho) = \{\, \lambda \in \sigma(A) \mid |\lambda| = \rho \,\} \tag{1.4}$$

is nonempty. In this note we will consider the case when the spectral set $\Lambda(\rho)$ is a singleton, $\Lambda(\rho) = \{\lambda\}$, where λ is a simple eigenvalue of A. We will show that in this case an appropriate weighted limit of the solution exists and is equal to the corresponding eigenvector of the limiting matrix A. Our main result is the following theorem.

Theorem 1.2. *Suppose (1.2) holds. Assume also that the growth rate ρ of a nonvanishing solution $x = (\mathbf{x}_n)_{n\in\mathbb{N}}$ of Eq. (1.1) is the modulus of a simple eigenvalue λ of A and the moduli of all other eigenvalues of A are different from ρ. Let $\mathbf{v}$ be an eigenvector of A corresponding to λ so that the adjoint matrix A^* has a unique eigenvector $\mathbf{w}$ corresponding to $\overline{\lambda}$, the conjugate of λ, such that $(\mathbf{v}, \mathbf{w}) = 1$. Then*

$$\frac{\mathbf{x}_n}{(\mathbf{x}_n, \mathbf{w})} \longrightarrow \mathbf{v}, \qquad n \to \infty, \tag{1.5}$$

and

$$\frac{(\mathbf{x}_{n+1}, \mathbf{w})}{(\mathbf{x}_n, \mathbf{w})} \longrightarrow \lambda, \qquad n \to \infty. \tag{1.6}$$

Remark. As an easy consequence of the limit relations (1.5) and (1.6), we obtain that under the hypotheses of Theorem 1.2, we have

$$\frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|} \longrightarrow \rho, \qquad n \to \infty. \tag{1.7}$$

Note that condition (1.7) is stronger than conclusion (1.3) of Theorem 1.1.

According to Perron's theorem, the spectral condition of Theorem 1.2 is satisfied if ρ is the spectral radius of a positive matrix A (see, e.g., [4, Sec. 1.7]).

Recall that if $\Lambda \subset \sigma(A)$ is a set of eigenvalues of A, then the eigenprojection (spectral projection) of A associated with Λ is defined by

$$P_\Lambda = \sum_{\lambda\in\Lambda} P_\lambda, \tag{1.8}$$

where, for each $\lambda \in \sigma(A)$, the symbol P_λ denotes the eigenprojection of A corresponding to λ. For the definition of P_λ, see, e.g., [4] or [8].

Let $x = (\mathbf{x}_n)_{n\in\mathbb{N}}$ be a nonvanishing solution of (1.1) with growth rate ρ. By Theorem 1.1, the spectral set $\Lambda(\rho)$ defined by (1.4) is nonempty. If we let

$$\Lambda^c(\rho) = \sigma(A) \setminus \Lambda(\rho), \tag{1.9}$$

then $\sigma(A) = \Lambda(\rho) \cup \Lambda^c(\rho)$ and we have the decomposition

$$\mathbf{x}_n = P_{\Lambda(\rho)}\mathbf{x}_n + P_{\Lambda^c(\rho)}\mathbf{x}_n, \qquad n \in \mathbb{N}. \tag{1.10}$$

In the proof of Theorem 1.2, we will need the following result from the proof of Theorem 1.3 in [8].

Lemma 1.3. [8, Lemma 3.2] *Suppose (1.2) holds. If $x = (\mathbf{x}_n)_{n\in\mathbb{N}}$ is a nonvanishing solution of (1.1) with growth rate ρ, then*

$$\lim_{n\to\infty} \frac{\|P_{\Lambda^c(\rho)}\mathbf{x}_n\|}{\|\mathbf{x}_n\|} = 0, \tag{1.11}$$

where $\Lambda^c(\rho)$ is given by (1.9).

Now we can give a proof of Theorem 1.2.

Proof of Theorem 1.2. By the triangle inequality, we have for $n \in \mathbb{N}$,

$$\|\mathbf{x}_n\| - \|P_{\Lambda^c(\rho)}\mathbf{x}_n\| \le \|\mathbf{x}_n - P_{\Lambda^c(\rho)}\mathbf{x}_n\| \le \|\mathbf{x}_n\| + \|P_{\Lambda^c(\rho)}\mathbf{x}_n\|$$

and hence

$$1 - \frac{\|P_{\Lambda^c(\rho)}\mathbf{x}_n\|}{\|\mathbf{x}_n\|} \le \frac{\|\mathbf{x}_n - P_{\Lambda^c(\rho)}\mathbf{x}_n\|}{\|\mathbf{x}_n\|} \le 1 + \frac{\|P_{\Lambda^c(\rho)}\mathbf{x}_n\|}{\|\mathbf{x}_n\|}.$$

Letting $n \to \infty$ in the last system of inequalities and using (1.10) and (1.11), we obtain

$$\frac{\|P_{\Lambda(\rho)}\mathbf{x}_n\|}{\|\mathbf{x}_n\|} = \frac{\|\mathbf{x}_n - P_{\Lambda^c(\rho)}\mathbf{x}_n\|}{\|\mathbf{x}_n\|} \longrightarrow 1, \qquad n \to \infty. \tag{1.12}$$

By assumption, $\Lambda(\rho) = \{\lambda\}$ and hence

$$P_{\Lambda(\rho)}\mathbf{x} = P_\lambda \mathbf{x}, \qquad \mathbf{x} \in \mathbb{C}^k, \ \ n \in \mathbb{N}.$$

Since λ is a simple eigenvalue of A, according to [4, Problem 3.19], the eigenprojection P_λ is given explicitly by

$$P_\lambda \mathbf{x} = (\mathbf{x}, \mathbf{w})\, \mathbf{v}, \qquad \mathbf{x} \in \mathbb{C}^k,$$

where $\mathbf{v}$ is an eigenvector of A corresponding to λ and $\mathbf{w}$ is an eigenvector of the adjoint matrix A^* corresponding to its eigenvalue $\overline{\lambda}$ such that $(\mathbf{v}, \mathbf{w}) = 1$. Hence

$$P_{\Lambda(\rho)}\mathbf{x}_n = (\mathbf{x}_n, \mathbf{w})\, \mathbf{v}, \qquad n \in \mathbb{N}. \tag{1.13}$$

This, together with (1.12), implies

$$\frac{|(\mathbf{x}_n, \mathbf{w})|}{\|\mathbf{x}_n\|} \longrightarrow \frac{1}{\|\mathbf{v}\|}, \qquad n \to \infty. \tag{1.14}$$

In particular, $(\mathbf{x}_n, \mathbf{w}) \neq 0$ for all large n. Using (1.13) in (1.10), we obtain

$$\mathbf{x}_n = (\mathbf{x}_n, \mathbf{w})\, \mathbf{v} + P_{\Lambda^c(\rho)}\mathbf{x}_n, \qquad n \in \mathbb{N}.$$

From this, we find that

$$\frac{\mathbf{x}_n}{(\mathbf{x}_n, \mathbf{w})} = \mathbf{v} + \frac{P_{\Lambda^c(\rho)}\mathbf{x}_n}{(\mathbf{x}_n, \mathbf{w})} \tag{1.15}$$

for all large n. From (1.11) and (1.14), we obtain

$$\frac{\|P_{\Lambda^c(\rho)}\mathbf{x}_n\|}{|(\mathbf{x}_n,\mathbf{w})|}=\frac{\|P_{\Lambda^c(\rho)}\mathbf{x}_n\|}{\|\mathbf{x}_n\|}\frac{\|\mathbf{x}_n\|}{|(\mathbf{x}_n,\mathbf{w})|}\longrightarrow 0,\qquad n\to\infty.$$

This, together with (1.15), implies (1.5). It remains to show (1.6). Rewrite Eq. (1.1) as

$$\mathbf{x}_{n+1}=A\mathbf{x}_n+B_n\mathbf{x}_n,\qquad n\in\mathbb{N},\tag{1.16}$$

where $B_n=A_n-A$ for $n\in\mathbb{N}$. By virtue of (1.2),

$$\|B_n\|\to 0,\qquad n\to\infty.\tag{1.17}$$

From (1.16), we find that

$$(\mathbf{x}_{n+1},\mathbf{w})=(A\mathbf{x}_n,\mathbf{w})+(B_n\mathbf{x}_n,\mathbf{w}),\qquad n\in\mathbb{N}.\tag{1.18}$$

Taking into account that for $n\in\mathbb{N}$,

$$(A\mathbf{x}_n,\mathbf{w})=(\mathbf{x}_n,A^*\mathbf{w})=(\mathbf{x}_n,\overline{\lambda}\mathbf{w})=\lambda\,(\mathbf{x}_n,\mathbf{w}),$$

Eq.(1.18) can be written in the form

$$(\mathbf{x}_{n+1},\mathbf{w})=\lambda(\mathbf{x}_n,\mathbf{w})+(B_n\mathbf{x}_n,\mathbf{w}),\qquad n\in\mathbb{N}.$$

Hence

$$\frac{(\mathbf{x}_{n+1},\mathbf{w})}{(\mathbf{x}_n,\mathbf{w})}=\lambda+\frac{(B_n\mathbf{x}_n,\mathbf{w})}{(\mathbf{x}_n,\mathbf{w})},\qquad n\in\mathbb{N}.\tag{1.19}$$

Further, using the l_2-norm on $\mathbb{C}^k$, the Schwarz inequality implies for $n\in\mathbb{N}$,

$$|(B_n\mathbf{x}_n,\mathbf{w})|\le\|B_n\mathbf{x}_n\|\|\mathbf{w}\|\le\|B_n\|\|\mathbf{x}_n\|\|\mathbf{w}\|.$$

This, together with (1.14) and (1.17), yields

$$\frac{|(B_n\mathbf{x}_n,\mathbf{w})|}{|(\mathbf{x}_n,\mathbf{w})|}\le\|\mathbf{w}\|\|B_n\|\frac{\|\mathbf{x}_n\|}{|(\mathbf{x}_n,\mathbf{w})|}\longrightarrow 0,\qquad n\to\infty.$$

Letting $n\to\infty$ in (1.19) and using the last limit relation, we conclude that (1.6) holds. □

As a consequence of Theorems 1.1 and 1.2, we have the following result.

Theorem 1.4. *Suppose (1.2) holds. Assume also that the eigenvalues $\lambda_1,\lambda_2\dots,\lambda_k$ of the limiting matrix A have distinct moduli and let $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_k$ be the corresponding eigenvectors of A. Denote by $\mathbf{w}_1,\mathbf{w}_2,\dots,\mathbf{w}_k$ the dual basis of $\mathbb{C}^k$ consisting of eigenvectors of A^* corresponding to the eigenvalues $\overline{\lambda_1},\overline{\lambda_2},\dots,\overline{\lambda_k}$ so that*

$$(\mathbf{v}_i,\mathbf{w}_j)=\delta_{ij}=\begin{cases}0 & \text{if } i\ne j\\ 1 & \text{if } i=j.\end{cases}$$

Then for every nonvanishing solution $x=(\mathbf{x}_n)_{n\in\mathbb{N}}$ of Eq. (1.1) there exists $j\in\{1,2,\dots,k\}$ such that

$$\frac{\mathbf{x}_n}{(\mathbf{x}_n,\mathbf{w}_j)}\longrightarrow\mathbf{v}_j,\qquad n\to\infty,\tag{1.20}$$

and

$$\frac{(\mathbf{x}_{n+1},\mathbf{w}_j)}{(\mathbf{x}_n,\mathbf{w}_j)}\longrightarrow\lambda_j,\qquad n\to\infty.\tag{1.21}$$

Theorem 1.4 is an improvement of a result by Máté and Nevai (see [6, Theorem 2]).

In the special case when the limiting matrix A is diagonal,

$$A=\operatorname{diag}\{\lambda_1,\lambda_2,\dots,\lambda_k\},\tag{1.22}$$

where the diagonal elements $\lambda_1,\lambda_2,\dots,\lambda_k$ have distinct moduli, i.e.

$$|\lambda_i|\ne|\lambda_j|\qquad\text{whenever } i\ne j,\tag{1.23}$$

the eigenvectors of A and the associated eigenvectors of A^* from Theorem 1.4 can be chosen as $\mathbf{v}_j=\mathbf{w}_j=\mathbf{e}_j$, where $\mathbf{e}_j$ is the jth canonical basis vector of $\mathbb{C}^k$. In this case Theorem 1.4 yields the following corollary.

Corollary 1.5. *Suppose (1.2), (1.22) and (1.23) hold. Then for every nonvanishing solution* $x = (\mathbf{x}_n)_{n\in\mathbb{N}}$ *of Eq. (1.1) there exists* $j \in \{1, 2, \ldots, k\}$ *such that as* $n \to \infty$

$$\frac{\mathbf{x}_{n,i}}{\mathbf{x}_{n,j}} \longrightarrow 0 \qquad \text{whenever } i \neq j,\ i \in \{1, 2, \ldots, k\}, \tag{1.24}$$

and

$$\frac{\mathbf{x}_{n+1,j}}{\mathbf{x}_{n,j}} \longrightarrow \lambda_j, \tag{1.25}$$

where $\mathbf{x}_{n,j}$ *denotes the* j*th coordinate of* $\mathbf{x}_n$*.*

2. Convergence of the Normalized Solution

If the coefficients in Eq. (1.1) are constant matrices, $A_n = A$ for all $n \in \mathbb{N}$, then Eq. (1.1) reduces to the constant coefficient equation

$$\mathbf{x}_{n+1} = A\mathbf{x}_n, \qquad n \in \mathbb{N}. \tag{2.1}$$

It is easily verified that if A has k different positive eigenvalues, then for every nontrivial solution $x = (\mathbf{x}_n)_{n\in\mathbb{N}}$ of Eq. (2.1) the normalized sequence $\frac{\mathbf{x}_n}{\|\mathbf{x}_n\|}$ converges to a normalized eigenvector of A as $n \to \infty$. This convergence is sometimes called as the *ergodic Poincaré property* of the solution [3]. In this section, we will extend the last property of the constant coefficient equation (2.1) to the perturbed equation (1.1) under the additional assumption that the entries of the coefficient matrices A_n, $n \in \mathbb{N}$, are real. The importance of the additional assumption will be illustrated by an example.

In the sequel $\mathbb{R}^k$ denotes the k-dimensional space of real column vectors with any norm $\|\cdot\|$. As a consequence of Theorem 1.2, we establish the following result.

Theorem 2.1. *Suppose (1.2) holds, where matrices* A_n, $n \in \mathbb{N}$, *are real. Assume also that the initial vector* $\mathbf{x}_0$ *and hence* $\mathbf{x}_n$, $n \in \mathbb{N}$, *is also real. Finally, assume that the growth rate* ρ *of the solution* $x = (\mathbf{x}_n)_{n\in\mathbb{N}}$ *of Eq. (1.1) given by (1.3) is a simple positive eigenvalue of* A *and the moduli of all other eigenvalues of* A *are different from* ρ*. Let* $\mathbf{v} \in \mathbb{R}^k$ *be a normalized eigenvector of* A *corresponding to* ρ, $A\mathbf{v} = \rho\mathbf{v}$, $\|\mathbf{v}\| = 1$*. Then*

$$\text{either} \quad \frac{\mathbf{x}_n}{\|\mathbf{x}_n\|} \longrightarrow \mathbf{v}, \qquad \text{or} \quad \frac{\mathbf{x}_n}{\|\mathbf{x}_n\|} \longrightarrow -\mathbf{v} \tag{2.2}$$

as $n \to \infty$*.*

Proof. Since A_n, $n \in \mathbb{N}$, are real matrices, so are the limiting matrix A and its adjoint A^*. By assumptions, ρ is a positive eigenvalue of A, therefore the normalized eigenvector $\mathbf{v}$ of A corresponding to ρ and the eigenvector $\mathbf{w}$ of A^* corresponding ρ can be chosen real and such that $(\mathbf{v}, \mathbf{w}) = 1$. In particular, the inner products $(\mathbf{x}_n, \mathbf{w})$, $n \in \mathbb{N}$, are also real. By Theorem 1.2, we have

$$\frac{(\mathbf{x}_{n+1}, \mathbf{w})}{(\mathbf{x}_n, \mathbf{w})} \longrightarrow \rho > 0, \qquad n \to \infty. \tag{2.3}$$

Therefore, either

$$(\mathbf{x}_n, \mathbf{w}) > 0 \qquad \text{for all large } n, \tag{2.4}$$

or

$$(\mathbf{x}_n, \mathbf{w}) < 0 \qquad \text{for all large } n. \tag{2.5}$$

Using Theorem 1.2 again, we conclude that

$$\frac{\mathbf{x}_n}{(\mathbf{x}_n, \mathbf{w})} \longrightarrow \mathbf{v}, \qquad n \to \infty. \tag{2.6}$$

Hence

$$\frac{\|\mathbf{x}_n\|}{|(\mathbf{x}_n, \mathbf{w})|} \longrightarrow \|\mathbf{v}\| = 1, \qquad n \to \infty. \tag{2.7}$$

From

$$\frac{\mathbf{x}_n}{\|\mathbf{x}_n\|} = \frac{\mathbf{x}_n}{(\mathbf{x}_n, \mathbf{w})} \frac{(\mathbf{x}_n, \mathbf{w})}{\|\mathbf{x}_n\|}, \qquad n \in \mathbb{N},$$

and the limit relations (2.6) and (2.7) we find that in case (2.4)

$$\frac{\mathbf{x}_n}{\|\mathbf{x}_n\|} \longrightarrow \mathbf{v}, \qquad n \to \infty,$$

while in case (2.5)

$$\frac{\mathbf{x}_n}{\|\mathbf{x}_n\|} \longrightarrow -\mathbf{v}, \qquad n \to \infty.$$

□

Theorems 1.1 and 2.1 yield the following corollary.

Corollary 2.2. *Suppose (1.2) holds, where matrices A_n, $n \in \mathbb{N}$, are real. Assume also that the eigenvalues $\rho_1, \rho_2, \ldots, \rho_k$ of the limiting matrix A are positive and mutually different and let $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_k$ be the corresponding normalized real eigenvectors of A. Then for every nonvanishing solution $x = (\mathbf{x}_n)_{n\in\mathbb{N}}$ of (1.1) with real initial vector $\mathbf{x}_0$ there exists $j \in \{1, 2, \ldots, k\}$ such that*

$$\text{either} \quad \frac{\mathbf{x}_n}{\|\mathbf{x}_n\|} \longrightarrow \mathbf{v}_j, \qquad \text{or} \quad \frac{\mathbf{x}_n}{\|\mathbf{x}_n\|} \longrightarrow -\mathbf{v}_j \tag{2.8}$$

as $n \to \infty$.

The following example shows that in Theorem 2.1 and Corollary 2.2 it is important that the coefficients A_n, $n \in \mathbb{N}$, are real matrices.

Example. Consider the scalar equation

$$x_{n+1} = e^{i\varphi_n} x_n, \qquad n \in \mathbb{N}, \tag{2.9}$$

where i is the imaginary unit, $\varphi_0 = 4\pi$, and

$$\varphi_k = \frac{\pi}{2^{n-1}} \qquad \text{whenever } 2^n \le k < 2^{n+1}, \quad n = 0, 1, 2, \ldots, \tag{2.10}$$

so that $\varphi_1 = 2\pi$, $\varphi_2 = \varphi_3 = \pi$, $\varphi_4 = \varphi_5 = \varphi_6 = \varphi_7 = \frac{\pi}{2}$, $\varphi_8 = \varphi_9 = \cdots = \varphi_{15} = \frac{\pi}{4}$, etc. Eq. (2.9) is a special case of Eq. (1.1) when $k = 1$ and $A_n = e^{i\varphi_n}$, $n \in \mathbb{N}$. Clearly, $\varphi_n \to 0$ as $n \to \infty$ and therefore (1.2) holds with $A = 1$. The only (positive) eigenvalue of $A = 1$ is $\rho = 1$. According to Eq. (2.9), x_{n+1} can be interpreted as the point in the complex plane obtained from point x_n by a couterclockwise rotation with angle φ_n. It is easily verified that if $x_0 = 1$, then $|x_n| = 1$ for all $n \in \mathbb{N}$ and the set of accumulation points of the solution $(x_n)_{n\in\mathbb{N}}$ is the whole unit circle $|z| = 1$. Therefore the normalized sequence $\frac{x_n}{|x_n|} = x_n$ has no limit as $n \to \infty$.

Acknowledgments

This work was done while Rotchana Chieochan was visiting the Department of Mathematics, University of Pannonia, Hungary, supported by the Hungarian Scholarship Board Balassi Institue.

M. Pituk was supported in part by the Hungarian National Foundation for Scientific Research (OTKA) Grant no. K 73274.

References

1. S. Bodine and D. A. Lutz, *Exponentially asymptotically constant systems of difference equations with an application to hyperbolic equilibria*, Journal of Difference Equations and Applications **15** Issue 8 and 9 (2009), 821–832.
2. J. Čermák, *The stability and asymptotic properties of the θ-methods for the pantograph equation*, IMA Journal of Numerical Analysis, **31** (4) (2011), 1533–1551.

3. S. Elaydi, *An Introduction to Difference Equations,* Undergraduate Texts in Mathematics, Springer, New York, 3rd edition, 2005.
4. T. Kato, *A Short Introduction to Perturbation Theory for Linear Operators*, Springer, New York, 1982.
5. H. Matsunaga and S. Murakami, *Asymptotic behavior of solutions of functional difference equations*, Journal of Mathematical Analysis and Applications **305** (2005), 391–410.
6. A. Máté and P. Nevai, *A generalization of Poincaré's theorem for recurrence equations*, Journal of Approximation Theory **63** (1990), 92–97.
7. M. Pituk, *More on Poincaré's and Perron's theorems for difference equations*, Journal of Difference Equations and Applications **8** no. 3 (2002), 201–216.
8. M. Pituk, *A link between the Perron-Frobenius theorem and Perron's theorem for difference equations*, Linear Algebra and its Applications **434** (2011), 490–500.