Theorem 7. *For $n \leq 5$ Conjecture 6 holds.*

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Translation invariant Minkowski valuations on lattice polytopes

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(joint work with M. Ludwig)

Two classification theorems were critical in the beginning of the theory of valuations on convex sets: first, the Hadwiger theorem [8] for valuations on convex bodies (that is, compact convex sets) in $\mathbb{R}^n$ and second, the Betke & Kneser theorem [5] for valuations on lattice polytopes (that is, convex polytopes with vertices in $\mathbb{Z}^n$). In recent years, numerous classification results were established for convex-body valued valuations (see, for example, [9, 10, 12, 7, 3, 1, 2, 20, 19, 22, 6, 17, 16, 21, 13]). The aim of this talk is to establish classification results for convex-body valued valuations defined on lattice polytopes. The question leads us to define and classify the discrete Steiner point.

A function z defined on a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ with values in an abelian group (or more generally, an abelian monoid) is a valuation if

$$\mathrm{z}(P) + \mathrm{z}(Q) = \mathrm{z}(P \cup Q) + \mathrm{z}(P \cap Q) \tag{1}$$

whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{F}$ and $\mathrm{z}(\emptyset) = 0$.

An operator $\mathrm{Z} : \mathcal{F} \to \mathcal{K}(\mathbb{R}^n)$ is called a Minkowski valuation if Z satisfies (1) and addition on $\mathcal{K}(\mathbb{R}^n)$ is Minkowski addition; that is,

$$K + L = \{x + y : x \in K,\, y \in L\}.$$

An operator $\mathrm{Z} : \mathcal{F} \to \mathcal{K}(\mathbb{R}^n)$ is called $\mathrm{SL}_n(\mathbb{R})$ equivariant if $\mathrm{Z}(\phi P) = \phi\, \mathrm{Z}\, P$ for $\phi \in \mathrm{SL}_n(\mathbb{R})$ and $P \in \mathcal{F}$. Define $\mathrm{SL}_n(\mathbb{Z})$ equivariance of operators on $\mathcal{P}(\mathbb{Z}^n)$ analogously. For valuations $\mathrm{Z} : \mathcal{P}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ that are $\mathrm{SL}_n(\mathbb{R})$ equivariant and translation invariant, a complete classification has been established. Let $n \geq 2$.

Theorem 1 ([11]). *An operator $\mathrm{Z} : \mathcal{P}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ is an $\mathrm{SL}_n(\mathbb{R})$ equivariant and translation invariant Minkowski valuation if and only if there exists a constant $c \geq 0$ such that for every $P \in \mathcal{P}(\mathbb{R}^n)$, we have*

$$\mathrm{Z}\, P = c(P - P).$$

The aim of this talk is to classify certain types of Minkowski valuations on lattice polytopes. The following result is an analogue of Theorem 1. Let $n \geq 2$.

Theorem 2. *An operator* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathcal{K}(\mathbb{R}^n)$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation invariant Minkowski valuation if and only if there exist constants* a, $b \geq 0$ *such that for every* $P \in \mathcal{P}(\mathbb{Z}^n)$*, we have*

$$\mathrm{Z}\, P = a(P - \ell_1(P)) + b(-P + \ell_1(P)).$$

Here for a lattice polytope P, the point $\ell_1(P)$ is its discrete Steiner point that is a new notion. It is defined as the one-homogeneous part of the Ehrhart expansion of the discrete moment vector $\ell(P) = \sum_{x \in P \cap \mathbb{Z}^n} x$; namely,

$$\ell(\lambda P) = \sum_{i=0}^{n+1} l_i(P) \lambda^i \quad \text{for } \lambda \in \mathbb{N}.$$

That such an expansion exists follows from results by McMullen [14]. The discrete Steiner point is characterized in the following result.

Theorem 3. *A function* $\mathrm{z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *and translation equivariant valuation if and only if* $\mathrm{z} = \ell_1$.

Theorem 3 corresponds to the well-known characterization of the classical Steiner point due to Schneider [18].

An operator $\mathrm{Z} : \mathcal{F} \to \mathcal{K}(\mathbb{R}^n)$ is called $\mathrm{SL}_n(\mathbb{R})$ contravariant if $\mathrm{Z}(\phi P) = \phi^{-t}\, \mathrm{Z}\, P$ for $\phi \in \mathrm{SL}_n(\mathbb{R})$ and $P \in \mathcal{F}$, where ϕ^{-t} is the inverse of the transpose of ϕ. Define $\mathrm{SL}_n(\mathbb{Z})$ contravariance of operators on $\mathcal{P}(\mathbb{Z}^n)$ analogously. For $\mathrm{SL}_n(\mathbb{R})$ contravariant Minkowski valuations on $\mathcal{P}(\mathbb{R}^n)$, a complete classification has been established. Let $n \geq 2$.

Theorem 4 ([11])**.** *An operator* $\mathrm{Z} : \mathcal{P}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ *is an* $\mathrm{SL}_n(\mathbb{R})$ *contravariant and translation invariant Minkowski valuation if and only if there exists a constant* $c \geq 0$ *such that for every* $P \in \mathcal{P}(\mathbb{R}^n)$*, we have*

$$\mathrm{Z}\, P = c\, \Pi\, P.$$

Here $\Pi\, P$ is the so-called projection body of P. For operators on lattice polytopes, we obtain the following result (here we do not quote the slightly more complicated case $n = 2$).

Theorem 5. *For* $n \geq 3$*, an operator* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathcal{K}(\mathbb{R}^n)$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *contravariant and translation invariant Minkowski valuation if and only if then there exists a constant* $c \geq 0$ *such that for every* $P \in \mathcal{P}(\mathbb{Z}^n)$*, we have*

$$\mathrm{Z}\, P = c\, \Pi\, P.$$

Open problems

(1) Characterize all $\mathrm{SL}_n(\mathbb{Z})$ equivariant Minkowski valuations on at most n-dimensional lattice polytopes.
(2) Characterize all $\mathrm{SL}_n(\mathbb{Z}+i\mathbb{Z})$ equivariant and translation invariant Minkowski valuations on at most $2n$-dimensional lattice polytopes where $\mathbb{Z} + i\mathbb{Z}$ stands for the Gauß integers.

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Random polytopes: scaling limits and variance asymptotics

Pierre Calka
(joint work with T. Schreiber and J. Yukich)

This talk is based on a joint work with Tomasz Schreiber and Joe Yukich and on several joint works with Joe Yukich, including a work in progress.

The study of so-called *random polytopes* defined as convex hulls of independent and identically distributed random points in $\mathbb{R}^d$, $d \geq 2$, has started more than 50 years ago. Focus has quickly turned to the description of their asymptotic behavior when the size of the input goes to infinity. In two seminal works published in 1963 and 1964 [9, 10], A. Rényi and R. Sulanke obtained explicit formulae for the