ON ENDOMORPHISMS OF THE EINSTEIN GYROGROUP IN ARBITRARY DIMENSION

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Abstract. We determine the automorphisms and the continuous endomorphisms of the Einstein gyrogroup in arbitrary dimension. This generalizes a recent result of L. Molnár and D. Virosztek, who have determined the continuous endomorphisms in the three-dimensional case.

1. The Einstein gyrogroup

The n-dimensional Einstein gyrogroup is the open unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$, endowed with the binary operation of velocity addition from the special theory of relativity (with the speed of light taken to be 1):

$$u \oplus v = \frac{1}{1+(u,v)}\left(u + \sqrt{1-|u|^2}\cdot v + \frac{(u,v)}{1+\sqrt{1-|u|^2}}\cdot u\right).$$

Here (u,v) is the inner product of u and v, and $|u| = \sqrt{(u,u)}$ is the usual Euclidean norm.

Note that $|u \oplus v| < 1$ if $|u| < 1$ and $|v| < 1$, so $(\mathbb{B}^n, \oplus)$ is an algebraic structure. It satisfies certain axioms that make it a gyrogroup [7]. We shall not need all of the axioms, but let us observe that $u \oplus 0 = 0 \oplus u = u$ and $u \oplus (-u) = 0$ for all $u \in \mathbb{B}^n$. The operation $\oplus$ is not associative, but $(-u) \oplus (u \oplus v) = v$ holds for all u and v in $\mathbb{B}^n$.

The Einstein gyrogroup is closely related to hyperbolic geometry. If we think of $\mathbb{B}^n$ as the Cayley–Klein–Beltrami model of hyperbolic n-space, then the map $v \mapsto u \oplus v$ is an isometry of hyperbolic n-space for any fixed u. When $u \neq 0$, this isometry maps the halfline starting at 0 and passing through u (henceforth referred to as halfline $0u$) onto its sub-halfline starting at u.

This implies a well-known fact about commutativity in the Einstein gyrogroup:

Proposition 1. *Let $u, v \in \mathbb{B}^n$. Then the equality $u \oplus v = v \oplus u$ holds if and only if u and v are linearly dependent (in the usual sense of vector algebra in $\mathbb{R}^n$).*

Proof. If u and v are linearly dependent, then they belong to a diameter of the ball $\mathbb{B}^n$. This diameter represents a line L in hyperbolic space. The $L \to L$ maps $w \mapsto u \oplus w$ and $w \mapsto v \oplus w$ are translations of L, so they commute. Hence, $u \oplus v = u \oplus (v \oplus 0) = v \oplus (u \oplus 0) = v \oplus u$ as claimed.

If u and v are linearly independent, then they span a two-dimensional plane, which intersects $\mathbb{B}^n$ in a disc. This disc represents a hyperbolic plane.

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Hyperbolic isometries preserve angles. Thus, the halfline $u(u \oplus v)$ forms the same angle with the halfline $0u$ as $0v$ does. Hence,

$$\angle 0u(u \oplus v) = \pi - \angle u0v.$$

Similarly,

$$\angle 0v(v \oplus u) = \pi - \angle u0v.$$

If, by way of contradiction, we have $u \oplus v = v \oplus u = w$, then corresponding sides of the triangles $u0v$ and vwu have equal length, making the two triangles congruent and implying

$$\angle u0v = \angle vwu.$$

But then the four angles of the quadrilateral $u0vw$ sum to 2π, which is impossible in the hyperbolic plane. □

Corollary 2. *The points $x, y, z \in \mathbb{B}^n$ are collinear if and only if*

$$((-x) \oplus y) \oplus ((-x) \oplus z) = ((-x) \oplus z) \oplus ((-x) \oplus y). \tag{1}$$

Proof. In the Cayley–Klein–Beltrami model, lines of hyperbolic space are represented by chords of the ball $\mathbb{B}^n$. Thus, x, y, z are collinear in the ordinary sense of Euclidean geometry if and only if they are collinear as points of hyperbolic space.

The map $w \mapsto (-x) \oplus w$ is an isometry of hyperbolic space, so it preserves collinearity. Thus x, y and z are collinear if and only if 0, $(-x) \oplus y$ and $(-x) \oplus z$ are. The claim now follows from Proposition 1. □

2. Endomorphisms and automorphisms

An endomorphism of the n-dimensional Einstein gyrogroup is a map $f : \mathbb{B}^n \to \mathbb{B}^n$ such that

$$f(u \oplus v) = f(u) \oplus f(v) \tag{2}$$

for all $u, v \in \mathbb{B}^n$. An automorphism is a bijective endomorphism.

Note that any endomorphism f satisfies $f(0) = 0$. Indeed, $0 \oplus 0 = 0$, whence $f(0) \oplus f(0) = f(0) = f(0) \oplus 0$. But $v \mapsto f(0) \oplus v$ is bijective, so $f(0) = 0$.

When $n = 1$, the Einstein gyrogroup is a group. It is isomorphic to the additive group $(\mathbb{R}, +)$ of real numbers. Endomorphisms of this group have been extensively studied, they go under the name of additive functions. Most of them are non-continuous. Moreover, most of the automorphisms of $(\mathbb{R}, +)$ are also non-continuous. In fact, the continuous endomorphisms are precisely the linear functions $x \mapsto ax : \mathbb{R} \to \mathbb{R}$ with fixed $a \in \mathbb{R}$, and there are many further automorphisms, let alone endomorphisms.

Henceforth, we assume $n \geq 2$.

Theorem 3. *For $n \geq 2$, automorphisms of the Einstein gyrogroup $(\mathbb{B}^n, \oplus)$ are precisely the restrictions to $\mathbb{B}^n$ of the orthogonal transformations of $\mathbb{R}^n$.*

Proof. Orthogonal transformations of $\mathbb{R}^n$ preserve the inner product and therefore the Euclidean norm, so they map $\mathbb{B}^n$ bijectively onto itself and satisfy (2) for all u and v.

Conversely, if f is an automorphism, then so is its inverse f^{-1}. By Corollary 2, both f and f^{-1} map collinear points to collinear points. I.e., f — as a self-map of hyperbolic space — maps any line onto a line. In other words, f is a collineation of hyperbolic space. By a well-known result sometimes referred to as the fundamental theorem of hyperbolic geometry [1, 3, 6, 8], any collineation is an isometry for $n \geq 2$. So f is an isometry. It is well known that in the Cayley–Klein–Beltrami model, any isometry of hyperbolic n-space fixing 0 is represented by the restriction of an orthogonal transformation. □

We now turn to endomorphisms. We urge the reader to solve

Problem 4. For $n \geq 2$, is every endomorphism of the Einstein gyrogroup continuous?

Meanwhile, we wish to classify continuous endomorphisms. For $n = 3$, which is the most relevant to physics, this was done by L. Molnár and D. Virosztek [5], while the general case was posed by them as an open problem. Their result relies on a chain of reinterpretations of $(\mathbb{B}^3, \oplus)$. The first step in the chain is an observation of S. Kim [2]: $(\mathbb{B}^3, \oplus)$ is bicontinuously isomorphic to $(\mathbb{D}, \odot)$, where $\mathbb{D}$ is the set of 2-square regular density matrices and $A \odot B$ is $\sqrt{A}B\sqrt{A}$ divided by its trace. Molnár and Virosztek show that this in turn is bicontinuously isomorphic to $(\mathbb{P}_2^1, \boxdot)$, where $\mathbb{P}_2^1$ is the set of 2-square positive definite matrices with determinant 1, and $A \boxdot B = \sqrt{A}B\sqrt{A}$. Then they invoke a result from their previous paper [4, Theorem 1] and deduce from it the classification of the continuous endomorphisms of $(\mathbb{P}_2^1, \boxdot)$.

From Theorem 3 of the present paper, using the bicontinuous isomorphisms mentioned above (but in the opposite direction), we infer

Corollary 5. *Every automorphism of* $(\mathbb{D}, \odot)$ *or* $(\mathbb{P}_2^1, \boxdot)$ *is continuous.*

Turning to arbitrary dimension, we have

Theorem 6. *For* $n \geq 2$, *continuous endomorphisms of the Einstein gyrogroup* $(\mathbb{B}^n, \oplus)$ *are precisely the restrictions to* $\mathbb{B}^n$ *of orthogonal transformations of* $\mathbb{R}^n$ *and the map that sends everything to 0.*

For $n = 3$, this recovers the classifications of continuous endomorphisms of $(\mathbb{B}^3, \oplus)$, $(\mathbb{D}, \odot)$ and $(\mathbb{P}_2^1, \boxdot)$ given by Molnár and Virosztek in [5].

Proof. It is clear that orthogonal transformations and the identically zero map are continuous endomorphisms.

Conversely, let f be an arbitrary continuous endomorphism.

If f is injective, then it is an open map, so its image contains a neighbourhood of 0. But this neighbourhood generates $\mathbb{B}^n$ under $\oplus$, and the image of f is closed under $\oplus$, so f must be surjective, i.e., f is an automorphism. The claim now follows from Theorem 3.

If f is not injective, then we have a pair $u \neq v$ with $f(u) = f(v) = f(u \oplus ((-u) \oplus v)) = f(u) \oplus f((-u) \oplus v)$. Let $x = (-u) \oplus v$, then $f(x) = 0$ but $x \neq 0$. The diameter L passing through x is a subgroup isomorphic to $(\mathbb{R}, +)$. We may choose an isomorphism such that x corresponds to 1. It is easy to see that $f(y) = 0$ for every point of the diameter L that corresponds to a rational number. But then, by continuity, $f(y) = 0$ for all y on the diameter L. Thus, f is constant on sets of the form $a \oplus L$ and $L \oplus b$. The former sets are lines in hyperbolic n-space, i.e., chords of the ball $\mathbb{B}^n$.

The chord $a \oplus L$ passes through a and is parallel to L if a is orthogonal to L. The latter sets, when $b \notin L$, are hypercycles in hyperbolic n-space, or half-ellipses in $\mathbb{B}^n$. The half-ellipse $L \oplus b$ connects the two ends of its major axis L and passes through b. It follows that f is constant on any two-dimensional open half-disk whose boundary diameter is L. By continuity, $f = 0$ everywhere. $\square$

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