arXiv:1412.5778v4 [math.NT] 11 Jul 2016

Links between generalized Montréal-functors

Márton Erdélyi   Gergely Zábrádi *

12th July 2016

Abstract

Let o be the ring of integers in a finite extension $K/\mathbb{Q}_p$ and $G = \mathbf{G}(\mathbb{Q}_p)$ be the $\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-split reductive group $\mathbf{G}$ defined over $\mathbb{Z}_p$ with connected centre and split Borel $\mathbf{B} = \mathbf{TN}$. We show that Breuil's [2] pseudocompact (φ, Γ)-module $D^\vee_\xi(\pi)$ attached to a smooth o-torsion representation π of $B = \mathbf{B}(\mathbb{Q}_p)$ is isomorphic to the pseudocompact completion of the basechange $\mathcal{O}_\mathcal{E} \otimes_{\Lambda(N_0),\ell} \widetilde{D_{SV}}(\pi)$ to Fontaine's ring (via a Whittaker functional $\ell \colon N_0 = \mathbf{N}(\mathbb{Z}_p) \to \mathbb{Z}_p$) of the étale hull $\widetilde{D_{SV}}(\pi)$ of $D_{SV}(\pi)$ defined by Schneider and Vigneras [9]. Moreover, we construct a G-equivariant map from the Pontryagin dual $\pi^\vee$ to the global sections $\mathfrak{Y}(G/B)$ of the G-equivariant sheaf $\mathfrak{Y}$ on G/B attached to a noncommutative multivariable version $D^\vee_{\xi,\ell,\infty}(\pi)$ of Breuil's $D^\vee_\xi(\pi)$ whenever π comes as the restriction to B of a smooth, admissible representation of G of finite length.

Contents

*Both authors wish to thank the Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences for its hospitality where this work was written. The second author was partially supported by a Hungarian OTKA Research grant K-100291 and by the János Bolyai Scholarship of the Hungarian Academy of Sciences.

1 Introduction

1.1 Notations

Let $G = \mathbf{G}(\mathbb{Q}_p)$ be the $\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-split connected reductive group $\mathbf{G}$ defined over $\mathbb{Z}_p$ with connected centre and a fixed split Borel subgroup $\mathbf{B} = \mathbf{TN}$. Put $B := \mathbf{B}(\mathbb{Q}_p)$, $T := \mathbf{T}(\mathbb{Q}_p)$, and $N := \mathbf{N}(\mathbb{Q}_p)$. We denote by Φ_+ the set of roots of T in N, by $\Delta \subset \Phi_+$ the set of simple roots, and by $u_\alpha : \mathbb{G}_a \to N_\alpha$, for $\alpha \in \Phi_+$, a $\mathbb{Q}_p$-homomorphism onto the root subgroup N_α of N such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $x \in \mathbb{Q}_p$ and $t \in T(\mathbb{Q}_p)$, and $N_0 = \prod_{\alpha\in\Phi_+} u_\alpha(\mathbb{Z}_p)$ is a subgroup of $N(\mathbb{Q}_p)$. We put $N_{\alpha,0} := u_\alpha(\mathbb{Z}_p)$ for the image of u_α on $\mathbb{Z}_p$. We denote by T_+ the monoid of dominant elements t in $T(\mathbb{Q}_p)$ such that $\mathrm{val}_p(\alpha(t)) \geq 0$ for all $\alpha \in \Phi_+$, by $T_0 \subset T_+$ the maximal subgroup, by T_{++} the subset of strictly dominant elements, i.e. $\mathrm{val}_p(\alpha(t)) > 0$ for all $\alpha \in \Phi_+$, and we put $B_+ = N_0T_+, B_0 = N_0T_0$. The natural conjugation action of T_+ on N_0 extends to an action on the Iwasawa o-algebra $\Lambda(N_0) = o[[N_0]]$. For $t \in T_+$ we denote this action of t on $\Lambda(N_0)$ by φ_t. The map $\varphi_t\colon \Lambda(N_0) \to \Lambda(N_0)$ is an injective ring homomorphism with a distinguished left inverse $\psi_t\colon \Lambda(N_0) \to \Lambda(N_0)$ satisfying $\psi_t \circ \varphi_t = \mathrm{id}_{\Lambda(N_0)}$ and $\psi_t(u\varphi_t(\lambda)) = \psi_t(\varphi_t(\lambda)u) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$ and $\lambda \in \Lambda(N_0)$.

Each simple root α gives a $\mathbb{Q}_p$-homomorphism $x_\alpha : N \to \mathbb{G}_a$ with section u_α. We denote by $\ell_\alpha : N_0 \to \mathbb{Z}_p$, resp. $\iota_\alpha\colon \mathbb{Z}_p \to N_0$, the restriction of x_α, resp. u_α, to N_0, resp. $\mathbb{Z}_p$.

Since the centre of G is assumed to be connected, there exists a cocharacter $\xi\colon \mathbb{Q}_p^\times \to T$ such that $\alpha \circ \xi$ is the identity on $\mathbb{Q}_p^\times$ for each $\alpha \in \Delta$. We put $\Gamma := \xi(\mathbb{Z}_p^\times) \leq T$ and often denote the action of $s := \xi(p)$ by $\varphi = \varphi_s$.

By a smooth o-torsion representation π of G (resp. of $B = \mathbf{B}(\mathbb{Q}_p)$) we mean a torsion o-module π together with a smooth (ie. stabilizers are open) and linear action of the group G (resp. of B).

For example, $\mathbf{G} = \mathrm{GL}_n$, B is the subgroup of upper triangular matrices, N consists of the strictly upper triangular matrices (1 on the diagonal), T is the diagonal subgroup, $N_0 = \mathbf{N}(\mathbb{Z}_p)$, the simple roots are $\alpha_1, \ldots, \alpha_{n-1}$ where $\alpha_i(\mathrm{diag}(t_1, \ldots, t_n)) = t_it_{i+1}^{-1}$, x_{α_i} sends a matrix to its $(i, i+1)$-coefficient, $u_{\alpha_i}(\cdot)$ is the strictly upper triangular matrix, with $(i, i+1)$-coefficient $\cdot$ and 0 everywhere else.

Let $\ell\colon N_0 \to \mathbb{Z}_p$ (for now) any surjective group homomorphism and denote by $H_0 \lhd N_0$ the kernel of ℓ. The ring $\Lambda_\ell(N_0)$, denoted by $\Lambda_{H_0}(N_0)$ in [9], is a generalisation of the ring $\mathcal{O}_\mathcal{E}$, which corresponds to $\Lambda_{\mathrm{id}}(N_0^{(2)})$ where $N_0^{(2)}$ is the $\mathbb{Z}_p$-points of the unipotent radical of a split Borel subgroup in GL_2. We refer the reader to [9] for the proofs of some of the following claims.

The maximal ideal $\mathcal{M}(H_0)$ of the completed group o-algebra $\Lambda(H_0) = o[[H_0]]$ is generated by ϖ and by the kernel of the augmentation map $o[[H_0]] \to o$.

The ring $\Lambda_\ell(N_0)$ is the $\mathcal{M}(H_0)$-adic completion of the localisation of $\Lambda(N_0)$ with respect to the Ore subset $S_\ell(N_0)$ of elements which are not in the ideal $\mathcal{M}(H_0)\Lambda(N_0)$. The ring $\Lambda(N_0)$ can be viewed as the ring $\Lambda(H_0)[[X]]$ of skew Taylor series over $\Lambda(H_0)$ in the variable $X = [u] - 1$ where $u \in N_0$ and $\ell(u)$ is a topological generator of $\ell(N_0) = \mathbb{Z}_p$. Then $\Lambda_\ell(N_0)$ is viewed as the ring of infinite skew Laurent series $\sum_{n\in\mathbb{Z}} a_nX^n$ over $\Lambda(H_0)$ in the variable X with $\lim_{n\to-\infty} a_n = 0$ for the compact topology of $\Lambda(H_0)$. For a different characterization of this ring in terms of a projective limit $\Lambda_\ell(N_0) \cong \varprojlim_{n,k} \Lambda(N_0/H_k)[1/X]/\varpi^n$ for $H_k \lhd N_0$ normal subgroups contained and open in H_0 satisfying $\bigcap_{k\geq 0} H_k = \{1\}$ see also [13].

For a finite index subgroup $\mathcal{G}_2$ in a group $\mathcal{G}_1$ we denote by $J(\mathcal{G}_1/\mathcal{G}_2) \subset \mathcal{G}_1$ a (fixed) set of representatives of the left cosets in $\mathcal{G}_1/\mathcal{G}_2$.

1.2 General overview

By now the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is very well understood through the work of Colmez [3], [4] and others (see [1] for an overview). To review Colmez's work let $K/\mathbb{Q}_p$ be a finite extension with ring of integers o, uniformizer ϖ and residue field k. The starting point is Fontaine's [8] theorem that the category of o-torsion Galois representations of $\mathbb{Q}_p$ is equivalent to the category of torsion (φ,Γ)-modules over $\mathcal{O}_{\mathcal{E}} = \varprojlim_h o/\varpi^h((X))$. One of Colmez's breakthroughs was that he managed to relate p-adic (and mod p) representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to (φ,Γ)-modules, too. The so-called "Montréal-functor" associates to a smooth o-torsion representation π of the standard Borel subgroup $B_2(\mathbb{Q}_p)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ a torsion (φ,Γ)-module over $\mathcal{O}_{\mathcal{E}}$. There are two different approaches to generalize this functor to reductive groups G other than $\mathrm{GL}_2(\mathbb{Q}_p)$. We briefly recall these "generalized Montréal functors" here.

The approach by Schneider and Vigneras [9] starts with the set $\mathcal{B}_+(\pi)$ of generating B_+-subrepresentations $W \leq \pi$. The Pontryagin dual $W^\vee = \mathrm{Hom}_o(W, K/o)$ of each W admits a natural action of the inverse monoid B_+^{-1}. Moreover, the action of $N_0 \leq B_+^{-1}$ on $W^\vee$ extends to an action of the Iwasawa algebra $\Lambda(N_0) = o[[N_0]]$. For $W_1, W_2 \in \mathcal{B}_+(\pi)$ we also have $W_1 \cap W_2 \in \mathcal{B}_+(\pi)$ (Lemma 2.2 in [9]) therefore we may take the inductive limit $D_{SV}(\pi) := \varinjlim_{W \in \mathcal{B}_+(\pi)} W^\vee$. In general, $D_{SV}(\pi)$ does not have good properties: for instance it may not admit a canonical right inverse of the T_+-action making $D_{SV}(\pi)$ an étale T_+-module over $\Lambda(N_0)$. However, by taking a resolution of π by compactly induced representations of B, one may consider the derived functors D^i_{SV} of D_{SV} for $i \geq 0$ producing étale T_+-modules $D^i_{SV}(\pi)$ over $\Lambda(N_0)$. Note that the functor D_{SV} is neither left- nor right exact, but exact in the middle. The fundamental open question of [9] whether the topological localizations $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D^i_{SV}(\pi)$ are finitely generated over $\Lambda_\ell(N_0)$ in case when π comes as a restriction of a smooth admissible representation of G of finite length. One can pass to usual 1-variable étale (φ,Γ)-modules—still not necessarily finitely generated—over $\mathcal{O}_{\mathcal{E}}$ via the map $\ell\colon \Lambda_\ell(N_0) \to \mathcal{O}_{\mathcal{E}}$ which step is an equivalence of categories for finitely generated étale (φ,Γ)-modules (Thm. 8.20 in [10]).

More recently, Breuil [2] managed to find a different approach, producing a pseudocompact (ie. projective limit of finitely generated) (φ,Γ)-module $D^\vee_\xi(\pi)$ over $\mathcal{O}_{\mathcal{E}}$ when π is killed by a power ϖ^h of the uniformizer ϖ. In [2] (and also in [9]) ℓ is a *generic* Whittaker functional, namely ℓ is chosen to be the composite map

$$\ell\colon N_0 \to N_0/(N_0 \cap [N,N]) \cong \prod_{\alpha\in\Delta} N_{\alpha,0} \overset{\sum_{\alpha\in\Delta} u_\alpha^{-1}}{\longrightarrow} \mathbb{Z}_p \ .$$

Breuil passes right away to the space of H_0-invariants π^{H_0} of π where H_0 is the kernel of the group homomorphism $\ell\colon N_0 \to \mathbb{Z}_p$. By the assumption that π is smooth, the invariant subspace π^{H_0} has the structure of a module over the Iwasawa algebra $\Lambda(N_0/H_0)/\varpi^h \cong o/\varpi^h[[X]]$. Moreover, it admits a semilinear action of F which is the Hecke action of $s := \xi(p)$: For any $m \in \pi^{H_0}$ we define

$$F(m) := \mathrm{Tr}_{H_0/sH_0s^{-1}}(sm) = \sum_{u \in J(H_0/sH_0s^{-1})} usm \ .$$

So π^{H_0} is a module over the skew polynomial ring $\Lambda(N_0/H_0)/\varpi^h[F]$ (defined by the identity $FX = (sXs^{-1})F = ((X+1)^p-1)F$). We consider those (i) finitely generated $\Lambda(N_0/H_0)/\varpi^h[F]$-submodules $M \subset \pi^{H_0}$ that are (ii) invariant under the action of Γ and are (iii) *admissible* as a $\Lambda(N_0/H_0)/\varpi^h$-module, ie. the Pontryagin dual $M^\vee = \mathrm{Hom}_o(M, o/\varpi^h)$ is finitely generated over $\Lambda(N_0/H_0)/\varpi^h$. Note that this admissibility condition (iii) is equivalent to the usual admissibility condition in smooth representation theory, ie. that for any (or equivalently for a single) open subgroup $N' \leq N_0/H_0$ the fixed points $M^{N'}$ form a finitely generated module over o. We denote by $\mathcal{M}(\pi^{H_0})$ the—via inclusion partially ordered—set of those submodules $M \leq \pi^{H_0}$ satisfying $(i), (ii), (iii)$. Note that whenever M_1, M_2 are in $\mathcal{M}(\pi^{H_0})$ then so is $M_1 + M_2$. It is shown in [4] (see also [5] and Lemma 2.6 in [2]) that for $M \in \mathcal{M}(\pi^{H_0})$ the localized Pontryagin dual $M^\vee[1/X]$ naturally admits a structure of an étale (φ, Γ)-module over $o/\varpi^h((X))$. Therefore Breuil [2] defines

$$D^\vee_\xi(\pi) := \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} M^\vee[1/X] \ .$$

By construction this is a projective limit of usual (φ, Γ)-modules. Moreover, $D^\vee_\xi$ is right exact and compatible with parabolic induction [2]. It can be characterized by the following universal property: For any (finitely generated) étale (φ, Γ)-module over $o/\varpi^h((X)) \cong o/\varpi^h[[\mathbb{Z}_p]][([1]-1)^{-1}]$ (here $[1]$ is the image of the topological generator of $\mathbb{Z}_p$ in the Iwasawa algebra $o/\varpi^h[[\mathbb{Z}_p]]$) we may consider continuous $\Lambda(N_0)$-homomorphisms $\pi^\vee \to D$ via the map $\ell\colon N_0 \to \mathbb{Z}_p$ (in the weak topology of D and the compact topology of $\pi^\vee$). These all factor through $(\pi^\vee)_{H_0} \cong (\pi^{H_0})^\vee$. So we may require these maps be ψ_s- and Γ-equivariant where $\Gamma = \xi(\mathbb{Z}_p \setminus \{0\})$ acts naturally on $(\pi^{H_0})^\vee$ and $\psi_s\colon (\pi^{H_0})^\vee \to (\pi^{H_0})^\vee$ is the dual of the Hecke-action $F\colon \pi^{H_0} \to \pi^{H_0}$ of s on π^{H_0}. Any such continuous ψ_s- and Γ-equivariant map f factors uniquely through $D^\vee_\xi(\pi)$. However, it is not known in general whether $D^\vee_\xi(\pi)$ is nonzero for smooth irreducible representations π of G (restricted to B).

The way Colmez goes back to representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ requires the following construction. From any (φ, Γ)-module over $\mathcal{E} = \mathcal{O}_\mathcal{E}[1/p]$ and character $\delta\colon \mathbb{Q}_p^\times \to o^\times$ Colmez constructs a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant sheaf $\mathfrak{Y}\colon U \mapsto D \boxtimes_\delta U$ ($U \subseteq \mathbb{P}^1$ open) of K-vectorspaces on the projective space $\mathbb{P}^1(\mathbb{Q}_p) \cong \mathrm{GL}_2(\mathbb{Q}_p)/B_2(\mathbb{Q}_p)$. This sheaf has the following properties: (i) the centre of $\mathrm{GL}_2(\mathbb{Q}_p)$ acts via δ on $D \boxtimes_\delta \mathbb{P}^1$; (ii) we have $D \boxtimes_\delta \mathbb{Z}_p \cong D$ as a module over the monoid $\begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ (where we regard $\mathbb{Z}_p$ as an open subspace in $\mathbb{P}^1 = \mathbb{Q}_p \cup \{\infty\}$). Moreover, whenever D is 2-dimensional and δ is the character corresponding to the Galois representation of $\bigwedge^2 D$ via local class field theory then the G-representation of global sections $D \boxtimes_\delta \mathbb{P}^1$ admits a short exact sequence

$$0 \to \Pi(\check{D})^\vee \to D \boxtimes \mathbb{P}^1 \to \Pi(D) \to 0$$

where $\Pi(\cdot)$ denotes the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\check{D} = \mathrm{Hom}(D, \mathcal{E})$ is the dual (φ, Γ)-module.

In [10] the functor $D \mapsto \mathfrak{Y}$ is generalized to arbitrary $\mathbb{Q}_p$-split reductive groups G with connected centre. Assume that $\ell = \ell_\alpha : N_0 \to N_{\alpha,0} \cong \mathbb{Z}_p$ is the projection onto the root subgroup corresponding to a fixed simple root $\alpha \in \Delta$. Then we have an action of the monoid T_+ on the ring $\Lambda_\ell(N_0)$ as we have $tH_0t^{-1} \leq H_0$ for any $t \in T_+$. Let D be an étale (φ, Γ)-module finitely generated over $\mathcal{O}_\mathcal{E}$ and choose a character $\delta\colon \mathrm{Ker}(\alpha) \to o^\times$. Then we may let the monoid $\xi(\mathbb{Z}_p \setminus \{0\})\,\mathrm{Ker}(\alpha) \leq T$ (containing T_+) act on D via the character δ of $\mathrm{Ker}(\alpha)$

and via the natural action of $\mathbb{Z}_p \setminus \{0\} \cong \varphi^{\mathbb{N}_0} \times \Gamma$ on D. This way we also obtain a T_+-action on $\Lambda_\ell(N_0) \otimes_{u_\alpha} D$ making $\Lambda_\ell(N_0) \otimes_{u_\alpha} D$ an étale T_+-module over $\Lambda_\ell(N_0)$. In [10] a G-equivariant sheaf $\mathfrak{Y}$ on G/B is attached to D such that its sections on $\mathcal{C}_0 := N_0 w_0 B/B \subset G/B$ is B_+-equivariantly isomorphic to the étale T_+-module $(\Lambda_\ell(N_0) \otimes_{u_\alpha} D)^{bd}$ over $\Lambda(N_0)$ consisting of bounded elements in $\Lambda_\ell(N_0) \otimes_{u_\alpha} D$ (for a more detailed overview see section 4.1).

1.3 Summary of our results

Our first result is the construction of a noncommutative multivariable version of $D^\vee_\xi(\pi)$. Let π be a smooth o-torsion representation of B such that $\varpi^h \pi = 0$. The idea here is to take the invariants π^{H_k} for a family of open normal subgroups $H_k \leq H_0$ with $\bigcap_{k\geq 0} H_k = \{1\}$. Now Γ and the quotient group N_0/H_k act on π^{H_k} (we choose H_k so that it is normalized by both Γ and N_0). Further, we have a Hecke-action of s given by $F_k := \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot)$. As in [2] we consider the set $\mathcal{M}_k(\pi^{H_k})$ of finitely generated $\Lambda(N_0/H_k)[F_k]$-submodules of π^{H_k} that are stable under the action of Γ and admissible as a representation of N_0/H_k. In section 2.1 we show that for any $M_k \in \mathcal{M}_k(\pi^{H_k})$ there is an étale (φ,Γ)-module structure on $M_k^\vee[1/X]$ over the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$. So the projective limit

$$D^\vee_{\xi,\ell,\infty}(\pi) := \varprojlim_{k\geq 0} \varprojlim_{M_k \in \mathcal{M}_k(\pi^{H_k})} M_k^\vee[1/X]$$

is an étale (φ,Γ)-module over $\Lambda_\ell(N_0)/\varpi^h = \varprojlim_k \Lambda(N_0/H_k)/\varpi^h[1/X]$. More-over, we also give a natural isomorphism $D^\vee_{\xi,\ell,\infty}(\pi)_{H_0} \cong D^\vee_\xi(\pi)$ showing that $D^\vee_{\xi,\ell,\infty}(\pi)$ corresponds to $D^\vee_\xi(\pi)$ via (the projective limit of) the equivalence of categories in Thm. 8.20 in [10]. Moreover, the natural map $\pi^\vee \to D^\vee_{\xi,\ell}(\pi)$ factors through the projection map $D^\vee_{\xi,\ell,\infty}(\pi) \twoheadrightarrow D^\vee_{\xi,\ell}(\pi) = D^\vee_{\xi,\ell,\infty}(\pi)_{H_0}$. Note that this shows that $D^\vee_{\xi,\ell,\infty}(\pi)$ is naturally attached to π—not just simply via the equivalence of categories (loc. cit.)—in the sense that any ψ- and Γ-equivariant map from $\pi^\vee$ to an étale (φ,Γ)-module over $o/\varpi^h(\!(X)\!)$ factors uniquely through the corresponding multivariable (φ,Γ)-module. This fact is used crucially in the subsequent sections of this paper.

In section 2.2 we develop these ideas further and show that the natural map $\pi^\vee \to D^\vee_{\xi,\ell,\infty}(\pi)$ factors through the map $\pi^\vee \to D_{SV}(\pi)$. In fact, we show (Prop. 2.14) that $D^\vee_{\xi,\ell,\infty}(\pi)$ has the following universal property: Any continuous ψ_s- and Γ-equivariant map $f\colon D_{SV}(\pi) \to D$ into a finitely generated étale (φ,Γ)-module D over $\Lambda_\ell(N_0)$ factors uniquely through $\mathrm{pr} = \mathrm{pr}_\pi\colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$. The association $\pi \mapsto \mathrm{pr}_\pi$ is a natural transformation between the functors D_{SV} and $D^\vee_{\xi,\ell,\infty}$. One application is that Breuil's functor $D^\vee_\xi$ vanishes on compactly induced representations of B (see Corollary 2.13).

In order to be able to compute $D^\vee_{\xi,\ell,\infty}(\pi)$ (hence also $D^\vee_\xi(\pi)$) from $D_{SV}(\pi)$ we introduce the notion of the *étale hull* of a $\Lambda(N_0)$-module with a ψ-action of T_+ (or of a submonoid $T_* \leq T_+$). Here a $\Lambda(N_0)$-module D with a ψ-action of T_+ is the analogue of a (ψ,Γ)-module over $o[\![X]\!]$ in this multivariable noncommutative setting. The étale hull $\widetilde{D}$ of D (together with a canonical map $\iota\colon D \to \widetilde{D}$) is characterized by the universal property that any ψ-equivariant map $f\colon D \to D'$ into an étale T_+-module D' over $\Lambda(N_0)$ factors uniquely through ι. It can be constructed as a direct limit $\varinjlim_{t\in T_+} \varphi_t^* D$ where $\varphi_t^* D = \Lambda(N_0) \otimes_{\varphi_t,\Lambda(N_0)} D$ (Prop. 2.21). We show (Thm. 2.28 and the remark thereafter) that the pseudocompact completion of

$\Lambda_\ell(N_0)\otimes_{\Lambda(N_0)}\widetilde{D_{SV}}(\pi)$ is canonically isomorphic to $D^\vee_{\xi,\ell,\infty}(\pi)$ as they have the same universal property.

In order to go back to representations of G we need an étale action of T_+ on $D^\vee_{\xi,\ell,\infty}(\pi)$, not just of $\xi(\mathbb{Z}_p\setminus\{0\})$. This is only possible if $tH_0t^{-1}\leq H_0$ for all $t\in T_+$ which is not the case for generic ℓ. So in section 3 we equip $D^\vee_{\xi,\ell,\infty}(\pi)$ with an étale action of T_+ (extending that of $\xi(\mathbb{Z}_p\setminus\{0\})\leq T_+$) in case $\ell=\ell_\alpha$ is the projection of N_0 onto a root subgroup $N_{\alpha,0}\cong\mathbb{Z}_p$ for some simple root α in Δ. Moreover, we show (Prop. 3.8) that the map $\mathrm{pr}\colon D_{SV}(\pi)\to D^\vee_{\xi,\ell,\infty}(\pi)$ is ψ-equivariant for this extended action, too. Note that $D^\vee_{\xi,\ell,\infty}(\pi)$ may not be the projective limit of finitely generated étale T_+-modules over $\Lambda_\ell(N_0)$ as we do not necessarily have an action of T_+ on $M^\vee_\infty[1/X]$ for $M\in\mathcal{M}(\pi^{H_0})$, only on the projective limit. So the construction of a G-equivariant sheaf on G/B with sections on $\mathcal{C}_0=N_0w_0B/B\subset G/B$ isomorphic to a dense B_+-stable $\Lambda(N_0)$-submodule $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ of $D^\vee_{\xi,\ell,\infty}(\pi)$ is not immediate from the work [10] as only the case of finitely generated modules over $\Lambda_\ell(N_0)$ is treated in there. However, as we point out in section 4.1 the most natural definition of bounded elements in $D^\vee_{\xi,\ell,\infty}(\pi)$ works: The $\Lambda(N_0)$-submodule $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ is defined as the union of ψ-invariant compact $\Lambda(N_0)$-submodules of $D^\vee_{\xi,\ell,\infty}(\pi)$. This section is devoted to showing that the image of $\widetilde{\mathrm{pr}}\colon\widetilde{D_{SV}}(\pi)\to D^\vee_{\xi,\ell,\infty}(\pi)$ is contained in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ (Cor. 4.4) and that the constructions of [10] can be carried over to this situation (Prop. 4.7). We denote the resulting G-equivariant sheaf on G/B by $\mathfrak{Y}=\mathfrak{Y}_{\alpha,\pi}$.

Now consider the functors $(\cdot)^\vee\colon\pi\mapsto\pi^\vee$ and the composite

$$\mathfrak{Y}_{\alpha,\cdot}(G/B)\colon\pi\mapsto D^\vee_{\xi,\ell,\infty}(\pi)\mapsto\mathfrak{Y}_{\alpha,\pi}(G/B)$$

both sending smooth, admissible o/ϖ^h-representations of G of finite length to topological representations of G over o/ϖ^h. The main result of our paper (Thm. 4.17) is a natural transformation $\beta_{G/B}$ from $(\cdot)^\vee$ to $\mathfrak{Y}_{\alpha,\cdot}$. This generalizes Thm. IV.4.7 in [4]. The proof of this relies on the observation that the maps $\mathcal{H}_g\colon D^\vee_{\xi,\ell,\infty}(\pi)^{bd}\to D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ in fact come from the G-action on $\pi^\vee$. More precisely, for any $g\in G$ and $W\in\mathcal{B}_+(\pi)$ we have maps

$$(g\cdot)\colon(g^{-1}W\cap W)^\vee\to(W\cap gW)^\vee$$

where both $(g^{-1}W\cap W)^\vee$ and $(W\cap gW)^\vee$ are naturally quotients of $W^\vee$. We show in (the proof of) Prop. 4.16 that these maps fit into a commutative diagram

$$\begin{tikzcd}
W^\vee \arrow[r] \arrow[rr, bend left] \arrow[d, "\mathrm{pr}_W"] & (g^{-1}W\cap W)^\vee \arrow[r, "g\cdot"'] \arrow[d] & (W\cap gW)^\vee \arrow[d] \\
D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \arrow[r] \arrow[rr, bend right] & \mathrm{res}^{\mathcal{C}_0}_{g^{-1}\mathcal{C}_0\cap\mathcal{C}_0}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}) \arrow[r, "g\cdot"] & \mathrm{res}^{\mathcal{C}_0}_{\mathcal{C}_0\cap g\mathcal{C}_0}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd})
\end{tikzcd}$$

allowing us to construct the map $\beta_{G/B}$. The proof of Thm. 4.17 is similar to that of Thm. IV.4.7 in [4]. However, unlike that proof we do not need the full machinery of "standard presentations" in Ch. III.1 of [4] which is not available at the moment for groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$.

2 Comparison of Breuil's functor with that of Schneider and Vigneras

2.1 A $\Lambda_\ell(N_0)$-variant of Breuil's functor

Our first goal is to associate a (φ,Γ)-module over $\Lambda_\ell(N_0)$ (not just over $\mathcal{O}_\mathcal{E}$) to a smooth o-torsion representation π of G in the spirit of [2] that corresponds to $D^\vee_\xi(\pi)$ via the equivalence of categories of [10] between (φ,Γ)-modules over $\mathcal{O}_\mathcal{E}$ and over $\Lambda_\ell(N_0)$.

Let H_k be the normal subgroup of N_0 generated by $s^kH_0s^{-k}$, ie. we put

$$H_k = \langle n_0 s^k H_0 s^{-k} n_0^{-1} \mid n_0 \in N_0\rangle \ .$$

H_k is an open subgroup of H_0 normal in N_0 and we have $\bigcap_{k\geq 0} H_k = \{1\}$. Denote by F_k the operator $\mathrm{Tr}_{H_k/sH_ks^{-1}}\circ(s\cdot)$ on π and consider the skew polynomial ring $\Lambda(N_0/H_k)/\varpi^h[F_k]$ where $F_k\lambda = (s\lambda s^{-1})F_k$ for any $\lambda\in\Lambda(N_0/H_k)/\varpi^h$. The set of finitely generated $\Lambda(N_0/H_k)[F_k]$-submodules of π^{H_k} that are stable under the action of Γ and admissible as a representation of N_0/H_k is denoted by $\mathcal{M}_k(\pi^{H_k})$.

Lemma 2.1. *We have $F=F_0$ and $F_k\circ\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot) = \mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot)\circ F_0$ as maps on π^{H_0}.*

Proof. We compute

$$\begin{aligned} F_k\circ\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot) = \mathrm{Tr}_{H_k/sH_ks^{-1}}\circ(s\cdot)\circ\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot) &= \\ \mathrm{Tr}_{H_k/sH_ks^{-1}}\circ\mathrm{Tr}_{sH_ks^{-1}/s^{k+1}H_0s^{-k-1}}\circ(s^{k+1}\cdot) &= \\ \mathrm{Tr}_{H_k/s^{k+1}H_0s^{-k-1}}\circ(s^{k+1}\cdot) &= \\ \mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ\mathrm{Tr}_{s^kH_0s^{-k}/s^{k+1}H_0s^{-k-1}}\circ(s^{k+1}\cdot) &= \\ \mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot)\circ\mathrm{Tr}_{H_0/sH_0s^{-1}}\circ(s\cdot) &= \\ \mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\cdot)\circ F_0 \ . \quad \square \end{aligned}$$

□

Note that if $M\in\mathcal{M}(\pi^{H_0})$ then $\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^kM)$ is a $s^kN_0s^{-k}H_k$-subrepresentation of π^{H_k}. So in view of the above Lemma we define M_k to be the N_0-subrepresentation of π^{H_k} generated by $\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^kM)$, ie. $M_k := N_0\,\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^kM)$. By Lemma 2.1 M_k is a $\Lambda(N_0/H_k)/\varpi^h[F_k]$-submodule of π^{H_k}.

Lemma 2.2. *For any $M\in\mathcal{M}(\pi^{H_0})$ the N_0-subrepresentation M_k lies in $\mathcal{M}_k(\pi^{H_k})$.*

Proof. Let $\{m_1,\dots,m_r\}$ be a set of generators of M as a $\Lambda(N_0/H_0)/\varpi^h[F]$-module. We claim that the elements $\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^km_i)$ $(i=1,\dots,r)$ generate M_k as a module over $\Lambda(N_0/H_k)/\varpi^h[F_k]$. Since both H_k and $s^kH_0s^{-k}$ are normalized by $s^kN_0s^{-k}$, for any $u\in N_0$ we have

$$\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^kus^{-k}\cdot) = (s^kus^{-k}\cdot)\circ\mathrm{Tr}_{H_k/s^kH_0s^{-k}} \ . \tag{1}$$

Therefore by continuity we also have

$$\mathrm{Tr}_{H_k/s^kH_0s^{-k}}\circ(s^k\lambda s^{-k}\cdot) = (s^k\lambda s^{-k}\cdot)\circ\mathrm{Tr}_{H_k/s^kH_0s^{-k}}$$

for any $\lambda \in \Lambda(N_0/H_0)/\varpi^h$. Now writing any $m \in M$ as $m = \sum_{j=1}^r \lambda_j F^{i_j} m_j$ we compute

$$\mathrm{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k \sum_{j=1}^r \lambda_j F^{i_j} m_j) = \sum_{j=1}^r (s^k \lambda s^{-k}) F_k^{i_j} \, \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^k m_j) \in$$

$$\in \sum_{j=1}^r \Lambda(N_0/H_k)/\varpi^h[F_k] \, \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^k m_j) \ .$$

For the stability under the action of Γ note that Γ normalizes both H_k and $s^kH_0s^{-k}$ and the elements in Γ commute with s.

Since M is admissible as an N_0-representation, s^kM is admissible as a representation of $s^kN_0s^{-k}$. Further by (1) the map $\mathrm{Tr}_{H_k/s^kH_0s^{-k}}$ is $s^kN_0s^{-k}$-equivariant therefore its image is also admissible. Finally, M_k can be written as a finite sum

$$\sum_{u \in J(N_0/s^kN_0s^{-k}H_k)} u \, \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)$$

of admissible representations of $s^kN_0s^{-k}$ therefore the statement. □ □

Lemma 2.3. *Fix a simple root $\alpha \in \Delta$ such that $\ell(N_{\alpha,0}) = \mathbb{Z}_p$. Then for any $M \in \mathcal{M}(\pi^{H_0})$ the kernel of the trace map*

$$\mathrm{Tr}_{H_0/H_k} \colon Y_k := \sum_{u \in J(N_{\alpha,0}/s^kN_{\alpha,0}s^{-k})} u \, \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM) \to N_0F^k(M) \qquad (2)$$

is finitely generated over o. In particular, the length of $Y_k^\vee[1/X]$ as a module over $o/\varpi^h((X))$ equals the length of $M^\vee[1/X]$.

Proof. Since any $u \in N_{\alpha,0} \leq N_0$ normalizes both H_0 and H_k and we have $N_{\alpha,0}H_0 = N_0$ by the assumption that $\ell(N_{\alpha,0}) = \mathbb{Z}_p$, the image of the map (2) is indeed $N_0F^k(M)$. Moreover, by the proof of Lemma 2.6 in [2] the quotient $M/N_0F^k(M)$ is finitely generated over o. Therefore we have $M^\vee[1/X] \cong (N_0F^k(M))^\vee[1/X]$ as a module over $o/\varpi^h((X))$. In particular, their length are equal:

$$l := \mathrm{length}_{o/\varpi^h((X))} M^\vee[1/X] = \mathrm{length}_{o/\varpi^h((X))} (N_0F^k(M))^\vee[1/X] \ .$$

We compute

$$\begin{aligned} l = \mathrm{length}_{o/\varpi^h((X))} M^\vee[1/X] &= \mathrm{length}_{o/\varpi^h((\varphi^k(X)))} (s^kM)^\vee[1/X] \geq \\ &\geq \mathrm{length}_{o/\varpi^h((\varphi^k(X)))} (\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM))^\vee[1/X] = \\ = \mathrm{length}_{o/\varpi^h((X))} (o/\varpi^h[[X]] &\otimes_{o/\varpi^h[[\varphi^k(X)]]} \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM))^\vee[1/X] \geq \\ &\geq \mathrm{length}_{o/\varpi^h((X))} Y_k^\vee[1/X] \ . \end{aligned}$$

By the existence of a surjective map (2) we must have equality in the above inequality everywhere. Therefore we have $\mathrm{Ker}(\mathrm{Tr}_{H_0/H_k})^\vee[1/X] = 0$, which shows that $\mathrm{Ker}(\mathrm{Tr}_{H_0/H_k})$ is finitely generated over o, because M is admissible, and so is $\mathrm{Ker}(\mathrm{Tr}_{H_0/H_k}) \leq M$. □ □

The kernel of the natural homomorphism

$$\Lambda(N_0/H_k)/\varpi^h \to \Lambda(N_0/H_0)/\varpi \cong k[[X]]$$

is a nilpotent prime ideal in the ring $\Lambda(N_0/H_k)/\varpi^h$. We denote the localization at this ideal by $\Lambda(N_0/H_k)/\varpi^h[1/X]$. For the justification of this notation note that any element in $\Lambda(N_0/H_k)/\varpi^h[1/X]$ can uniquely be written as a formal Laurent-series $\sum_{n\gg-\infty} a_n X^n$ with coefficients a_n in the finite group ring $o/\varpi^h[H_0/H_k]$. Here X—by an abuse of notation—denotes the element $[u_0]-1$ for an element $u_0 \in N_{\alpha,0} \leq N_0$ with $\ell(u_0) = 1 \in \mathbb{Z}_p$. The ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ admits a conjugation action of the group Γ that commutes with the operator φ defined by $\varphi(\lambda) := s\lambda s^{-1}$ (for $\lambda \in \Lambda(N_0/H_k)/\varpi^h[1/X]$). A (φ,Γ)-module over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ is a finitely generated module over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ together with a semilinear commuting action of φ and Γ. Note that φ is no longer injective on the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ for $k \geq 1$, in particular it is not flat either. However, we still call a (φ,Γ)-module D_k over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ étale if the natural map

$$1\otimes\varphi\colon \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} D_k \to D_k$$

is an isomorphism of $\Lambda(N_0/H_k)/\varpi^h[1/X]$-modules. For any $M \in \mathcal{M}(\pi^{H_0})$ we put

$$M_k^\vee[1/X] := \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\Lambda(N_0/H_k)/\varpi^h} M_k^\vee$$

where $(\cdot)^\vee$ denotes the Pontryagin dual $\mathrm{Hom}_o(\cdot, K/o)$.

The group N_0/H_k acts by conjugation on the finite $H_0/H_k \lhd N_0/H_k$. Therefore the kernel of this action has finite index. In particular, there exists a positive integer r such that $s^r N_{\alpha,0} s^{-r} \leq N_0/H_k$ commutes with H_0/H_k. Therefore the group ring $o/\varpi^h((\varphi^r(X)))[H_0/H_k]$ is contained as a subring in $\Lambda(N_0/H_k)/\varpi^h[1/X]$.

Lemma 2.4. *As modules over the group ring $o/\varpi^h((\varphi^r(X)))[H_0/H_k]$ we have an isomorphism*

$$M_k^\vee[1/X] \to o/\varpi^h((\varphi^r(X)))[H_0/H_k] \otimes_{o/\varpi^h((\varphi^r(X)))} Y_k^\vee[1/X] \ .$$

In particular, $M_k^\vee[1/X]$ is induced as a representation of the finite group H_0/H_k, so the reduced (Tate-) cohomology groups $\tilde{H}^i(H', M_k^\vee[1/X])$ vanish for all subgroups $H' \leq H_0/H_k$ and $i \in \mathbb{Z}$.

Proof. By the definition of M_k we have a surjective $o/\varpi^h[[\varphi^r(X)]][H_0/H_k]$-linear map

$$f\colon o/\varpi^h[[\varphi^r(X)]][H_0/H_k] \otimes_{o/\varpi^h[[\varphi^r(X)]]} Y_k \to M_k$$

sending $\lambda\otimes y$ to λy for $\lambda \in o/\varpi^h[[\varphi^r(X)]][H_0/H_k]$ and $y \in Y_k$. By taking the Pontryagin dual of f and inverting X we obtain an injective $o/\varpi^h((\varphi^r(X)))[H_0/H_k]$-homomorphism

$$\begin{aligned} f^\vee[1/X]\colon M_k^\vee[1/X] &\to (o/\varpi^h[[\varphi^r(X)]][H_0/H_k] \otimes_{o/\varpi^h[[\varphi^r(X)]]} Y_k)^\vee[1/X] \cong \\ &\cong o/\varpi^h((\varphi^r(X)))[H_0/H_k] \otimes_{o/\varpi^h((\varphi^r(X)))} (Y_k^\vee[1/X]) \ . \end{aligned}$$

On the other hand, by construction the action of the group H_0/H_k on the domain of f is via the action on the first term which is a regular left-translation action. Therefore the H_0/H_k-invariants can be computed as the image of the trace map:

$$(o/\varpi^h[[\varphi^r(X)]][H_0/H_k] \otimes_{o/\varpi^h[[\varphi^r(X)]]} Y_k)^{H_0/H_k} = (\sum_{h\in H_0/H_k} h) \otimes Y_k \ .$$

The composite of f with the bijection

$$(\sum_{h\in H_0/H_k} h)\otimes \mathrm{id}_{Y_k}\colon Y_k \xrightarrow{\sim} (\sum_{h\in H_0/H_k} h)\otimes Y_k$$

is the trace map on Y_k whose kernel is finitely generated over o by Lemma 2.3. In particular, the kernel of the restriction of f to the H_0/H_k-invariants is finitely generated over o. Dually, we find that $f^\vee[1/X]$ becomes surjective after taking H_0/H_k-coinvariants. Since $M_k^\vee[1/X]$ is a finite dimensional representation of the finite p-group H_0/H_k over the local artinian ring $o/\varpi^h((X))$ with residual characteristic p, the map $f^\vee[1/X]$ is in fact an isomorphism as its cokernel has trivial H_0/H_k-coinvariants. □ □

Denote by $H_{k,-}/H_k$ the kernel of the group homomorphism

$$s(\cdot)s^{-1}\colon N_0/H_k \to N_0/H_k\ .$$

It is a normal subgroup contained in the finite subgroup $H_0/H_k \leq N_0/H_k$ since $s(\cdot)s^{-1}$ is the multiplication by p map on $N_0/H_0 \cong \mathbb{Z}_p$ which is injective. If k is big enough so that H_k is contained in sH_0s^{-1} then we have $H_{k,-} = s^{-1}H_ks$, otherwise we always have $H_{k,-} = H_0 \cap s^{-1}H_ks$. The ring homomorphism

$$\varphi\colon \Lambda(N_0/H_k)/\varpi^h \to \Lambda(N_0/H_k)/\varpi^h$$

factors through the quotient map $\Lambda(N_0/H_k)/\varpi^h \twoheadrightarrow \Lambda(N_0/H_{k,-})/\varpi^h$. We denote by $\tilde\varphi$ the induced ring homomorphism

$$\tilde\varphi\colon \Lambda(N_0/H_{k,-})/\varpi^h \to \Lambda(N_0/H_k)/\varpi^h\ .$$

Note that $\tilde\varphi$ is injective and makes $\Lambda(N_0/H_k)/\varpi^h$ a free module of rank

$$\begin{aligned}\nu := |\operatorname{Coker}(s(\cdot)s^{-1}\colon N_0/H_k \to N_0/H_k)| &= \\ = p|\operatorname{Coker}(s(\cdot)s^{-1}\colon H_0/H_k \to H_0/H_k)| &= \\ = p|\operatorname{Ker}(s(\cdot)s^{-1}\colon H_0/H_k \to H_0/H_k)| = p|H_{k,-}/H_k|&\end{aligned}$$

over $\Lambda(N_0/H_{k,-})/\varpi^h$ since the kernel and cokernel of an endomorphism of a finite group have the same cardinality.

Lemma 2.5. *We have a series of isomorphisms of $\Lambda(N_0/H_k)/\varpi^h[1/X]$-mod-ules*

$$\begin{aligned}\mathrm{Tr}^{-1} = \mathrm{Tr}^{-1}_{H_{k,-}/H_k}\colon (\Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h} M_k)^\vee[1/X] &\overset{(1)}{\to} \\ \overset{(1)}{\to} \mathrm{Hom}_{\Lambda(N_0/H_k),\varphi}(\Lambda(N_0/H_k), M_k^\vee[1/X]) &\overset{(2)}{\to} \\ \overset{(2)}{\to} \mathrm{Hom}_{\Lambda(N_0/H_{k,-}),\tilde\varphi}(\Lambda(N_0/H_k), (M_k^\vee[1/X])^{H_{k,-}}) &\overset{(3)}{\to} \\ \overset{(3)}{\to} \Lambda(N_0/H_k)\otimes_{\Lambda(N_0/H_{k,-}),\tilde\varphi} M_k^\vee[1/X]^{H_{k,-}} &\overset{(4)}{\to} \\ \overset{(4)}{\to} \Lambda(N_0/H_k)\otimes_{\Lambda(N_0/H_{k,-}),\tilde\varphi} (M_k^\vee[1/X])_{H_{k,-}} &\overset{(5)}{\to} \\ \overset{(5)}{\to} \Lambda(N_0/H_k)/\varpi^h\otimes_{\Lambda(N_0/H_k)/\varpi^h,\varphi} M_k^\vee[1/X]&\ .\end{aligned}$$

Proof. (1) follows from the adjoint property of $\otimes$ and Hom. The second isomorphism follows from noting that the action of the ring $\Lambda(N_0/H_k)$ over itself via φ factors through the quotient $\Lambda(N_0/H_{k,-})$ therefore $H_{k,-}$ acts trivially on $\Lambda(N_0/H_k)$ via this map. So any module-homomorphism $\Lambda(N_0/H_k) \to M_k^\vee[1/X]$ lands in the $H_{k,-}$-invariant part $M_k^\vee[1/X]^{H_{k,-}}$ of $M_k^\vee[1/X]$. The third isomorphism follows from the fact that $\Lambda(N_0/H_k)$ is a free module over $\Lambda(N_0/H_{k,-})$ via $\tilde{\varphi}$. The fourth isomorphism is given by (the inverse of) the trace map $\mathrm{Tr}_{H_{k,-}/H_k}\colon (M_k^\vee[1/X])_{H_{k,-}} \to M_k^\vee[1/X]^{H_{k,-}}$ which is an isomorphism by Lemma 2.4. The last isomorphism follows from the isomorphism $(M_k^\vee[1/X])_{H_{k,-}} \cong \Lambda(N_0/H_{k,-}) \otimes_{\Lambda(N_0/H_k)} M_k^\vee[1/X]$. □ □

Remark. Here φ always acted only on the ring $\Lambda(N_0/H_k)$, hence denoting φ_t the action $n \mapsto tnt^{-1}$ for a fixed $t \in T_+$ and choosing k large enough such that $tH_0t^{-1} \geq H_k$ we get analogously an isomorphism

$$\mathrm{Tr}^{-1}_{t^{-1}H_kt/H_k}\colon (\Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi_t,\Lambda(N_0/H_k)/\varpi^h} M_k)^\vee[1/X] \to \\ \to \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h,\varphi_t} M_k^\vee[1/X] \ .$$

One of the key points of Lemma 2.4 is that the trace map on $M_k^\vee[1/X]$ induces a bijection between $M_k^\vee[1/X]_{H_{k,-}}$ and $M_k^\vee[1/X]^{H_{k,-}}$ as noted in the isomorphism (4) above. We shall use this fact later on.

We denote the composite of the five isomorphisms in Lemma 2.5 by Tr^{-1} emphasising that all but (4) are tautologies. Our main result in this section is the following generalization of Lemma 2.6 in [2].

Proposition 2.6. *The map*

$$\mathrm{Tr}^{-1} \circ (1 \otimes F_k)^\vee[1/X]\colon \\ M_k^\vee[1/X] \to \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X] \tag{3}$$

is an isomorphism of $\Lambda(N_0/H_k)/\varpi^h[1/X]$-modules. Therefore the natural action of Γ and the operator

$$\begin{aligned} \varphi\colon M_k^\vee[1/X] &\to M_k^\vee[1/X] \\ f &\mapsto (\mathrm{Tr}^{-1} \circ (1 \otimes F_k)^\vee[1/X])^{-1}(1 \otimes f) \end{aligned}$$

make $M_k^\vee[1/X]$ into an étale (φ,Γ)-module over the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$.

Proof. Since M_k is finitely generated over $\Lambda(N_0/H_k)/\varpi^h[F_k]$ by Lemma 2.2, the cokernel C of the map

$$1 \otimes F_k\colon \Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h} M_k \to M_k \tag{4}$$

is finitely generated as a module over $\Lambda(N_0/H_k)/\varpi^h$. Further, it is admissible as a representation of N_0 (again by Lemma 2.2), therefore C is finitely generated over o. In particular, we have $C^\vee[1/X] = 0$ showing that (3) is injective.

For the surjectivity put $Y_k := \sum_{u \in J(N_{\alpha,0}/s^kN_{\alpha,0}s^{-k})} u\,\mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)$. This is an $o/\varpi^h[[X]]$-submodule of M_k. By Lemma 2.3 we have

$$\mathrm{length}_{o/\varpi^h(\!(\varphi^r(X))\!)}(Y_k^\vee[1/X]) = \\ = |N_{\alpha,0} : s^rN_{\alpha,0}s^{-r}|\mathrm{length}_{o/\varpi^h(\!(X)\!)}(Y_k^\vee[1/X]) = p^rl \ .$$

By Lemma 2.4 we obtain

$$\mathrm{length}_{o/\varpi^h((\varphi^r(X)))} M_k^\vee[1/X] = \\ = |H_0 : H_k| \cdot \mathrm{length}_{o/\varpi^h((\varphi^r(X)))} Y_k^\vee[1/X] = |H_0 : H_k| p^r l \ .$$

Consider the ring homomorphism

$$\varphi\colon \Lambda(N_0/H_k)/\varpi^h[1/X] \to \Lambda(N_0/H_k)/\varpi^h[1/X] \ . \tag{5}$$

Its image is the subring $\Lambda(sN_0s^{-1}H_k/H_k)/\varpi^h[1/\varphi(X)]$ over which the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ is a free module of rank $\nu = |N_0 : sN_0s^{-1}H_k| = p|H_{k,-} : H_k|$. So we obtain

$$\begin{aligned} p\mathrm{length}_{o((\varphi^r(X)))}\Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X] = \\ = \mathrm{length}_{o((\varphi^{r+1}(X)))}\Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X] = \\ = \nu\mathrm{length}_{o((\varphi^{r+1}(X)))}\Lambda(sN_0s^{-1}H_k/H_k)/\varpi^h[1/\varphi(X)] \\ \otimes_{\varphi,\Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X] \overset{(*)}{=} \\ = \nu\mathrm{length}_{o((\varphi^r(X)))} M_k^\vee[1/X]_{H_{k,-}} = \\ = \nu\mathrm{length}_{o((\varphi^r(X)))}(o/\varpi^h[H_0/H_{k,-}] \otimes_{o/\varpi^h} Y_k^\vee[1/X]) = \\ = \nu|H_0 : H_{k,-}|p^r l = p|H_0 : H_k|p^r l = p\mathrm{length}_{o/\varpi^h((\varphi^r(X)))} M_k^\vee[1/X] \ . \end{aligned}$$

Here the equality $(*)$ follows from the fact that the map φ induces an isomorphism between $\Lambda(N_0/H_{k,-})/\varpi^h[1/X]$ and $\Lambda(sN_0s^{-1}H_k/H_k)/\varpi^h[1/\varphi(X)]$ sending the subring $o((\varphi^r(X)))$ isomorphically onto $o((\varphi^{r+1}(X)))$.

This shows that (3) is an isomorphism as it is injective and the two sides have equal length as modules over the artinian ring $o/\varpi^h((X))$. $\square$ $\square$

Remark. We also obtain in particular that the map (4) has finite kernel and cokernel. Hence there exists a finite $\Lambda(N_0/H_k)/\varpi^h$-submodule $M_{k,*}$ of M_k such that the kernel of $1 \otimes F_k$ is contained in the image of $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_{k,*}$ in $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_k$. We denote by M_k^* the image of $1 \otimes F_k$.

Note that for $k = 0$ we have $M_0 = M$. Let now $0 \leq j \leq k$ be two integers. By Lemma 2.4 the space of H_j-invariants of M_k is equal to $\mathrm{Tr}_{H_j/H_k}(M_k)$ upto finitely generated modules over o. On the other hand, we compute

$$\begin{aligned} N_0F_j^{k-j}(M_j) = N_0 \mathrm{Tr}_{H_j/s^{k-j}H_js^{j-k}} \circ (s^{k-j}\cdot) \circ \mathrm{Tr}_{H_j/s^jH_0s^{-j}}(s^jM) = \\ = N_0 \mathrm{Tr}_{H_j/s^kH_0s^{-k}}(s^kM) = N_0 \mathrm{Tr}_{H_j/H_k} \circ \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM) = \\ = \mathrm{Tr}_{H_j/H_k}(N_0 \mathrm{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)) = \mathrm{Tr}_{H_j/H_k}(M_k) \end{aligned}$$

since both H_k and H_j are normal in N_0 whence we have $(u\cdot) \circ \mathrm{Tr}_{H_j/H_k} = \mathrm{Tr}_{H_j/H_k} \circ (u\cdot)$ for all $u \in N_0$. So taking H_j/H_k-coinvariants of $M_k^\vee[1/X]$, we have a natural identification

$$M_k^\vee[1/X]_{H_j/H_k} \cong (M_k^{H_j/H_k})^\vee[1/X] \cong \\ \cong (\mathrm{Tr}_{H_j/H_k}(M_k))^\vee[1/X] = (N_0F_j^{k-j}(M_j))^\vee[1/X] \cong M_j^\vee[1/X] \tag{6}$$

induced by the inclusion $N_0F_j^{k-j}(M_j) \subseteq M_k^{H_j} \subseteq M_k$. The last identification follows from the fact that $M_j/N_0F_j^{k-j}(M_j)$ is finitely generated over o as noted in the beginning of the proof of Proposition 2.6 applied to j instead of k.

Lemma 2.7. *We have* $\mathrm{Tr}_{H_j/H_k}\circ F_k = F_j\circ \mathrm{Tr}_{H_j/H_k}$.

Proof. We compute

$$\mathrm{Tr}_{H_j/H_k}\circ F_k = \mathrm{Tr}_{H_j/H_k}\circ \mathrm{Tr}_{H_k/sH_ks^{-1}}\circ(s\cdot) =$$
$$\mathrm{Tr}_{H_j/sH_ks^{-1}}\circ(s\cdot) = \mathrm{Tr}_{H_j/sH_js^{-1}}\circ \mathrm{Tr}_{sH_js^{-1}/sH_ks^{-1}}(s\cdot) =$$
$$\mathrm{Tr}_{H_j/sH_js^{-1}}\circ(s\cdot)\,\mathrm{Tr}_{H_j/H_k} = F_j\circ \mathrm{Tr}_{H_j/H_k}\ . \qquad \square$$

$\square$

Proposition 2.8. *The identification* (6) *is* φ *and* Γ*-equivariant.*

Proof. For fixed j it suffices to treat the case when k is large enough so that we have $H_{k,-} = s^{-1}H_ks$. Indeed, for fixed j and k we may choose a larger integer $k' > k$ with $H_{k',-} = s^{-1}H_{k'}s$ and the φ- and Γ equivariance of the identifications $M_k^\vee[1/X]\cong M_{k'}^\vee[1/X]_{H_{k'}/H_k}$ and $M_j^\vee[1/X]\cong M_{k'}^\vee[1/X]_{H_{k'}/H_j}$ will imply that of

$$M_j^\vee[1/X]\cong M_{k'}^\vee[1/X]_{H_{k'}/H_j} = (M_{k'}^\vee[1/X]_{H_{k'}/H_j})_{H_k/H_j}\cong M_k^\vee[1/X]_{H_k/H_j}\ .$$

So from now on we assume $H_k\leq sH_0s^{-1}\leq sN_0s^{-1}$. As Γ acts both on M_k and M_j by multiplication coming from the action of Γ on π, the map (6) is clearly Γ-equivariant. In order to avoid confusion we are going to denote the map φ on $M_k^\vee[1/X]$ (resp. on $M_j^\vee[1/X]$) temporarily by φ_k (resp. by φ_j). Let f be in $M_k^\vee$ such that its restriction to $M_{k,*}$ is zero (see the Remark after Prop. 2.6). We regard f as an element in $(M_k^*/M_{k,*})^\vee\leq (M_k^*)^\vee$. We are going to compute $\varphi_k(f)$ and $\varphi_j(f_{|\mathrm{Tr}_{H_j/H_k}(M_k^*)})$ explicitly and find that the restriction of $\varphi_k(f)$ to $\mathrm{Tr}_{H_j/H_k}(M_k^*)$ is equal to $\varphi_j(f_{|\mathrm{Tr}_{H_j/H_k}(M_k^*)})$. Note that we have an isomorphism $M_k^\vee[1/X]\cong M_k^{*\vee}[1/X]\cong (M_k^*/M_{k,*})^\vee[1/X]$ (resp. $M_j^\vee[1/X]\cong \mathrm{Tr}_{H_j/H_k}(M_k^*)^\vee[1/X]$) obtained from the Remark after Prop. 2.6.

Let $m\in M_k^*\leq M_k$ be in the form

$$m = \sum_{u\in J((N_0/H_k)/s(N_0/H_k)s^{-1})} uF_k(m_u)$$

with elements $m_u\in M_k$ for $u\in J((N_0/H_k)/s(N_0/H_k)s^{-1})$. By the remark after Proposition 2.6 M_k^* is a finite index submodule of M_k. Note that the elements m_u are unique upto $M_{k,*}+\mathrm{Ker}(F_k)$. Therefore $\varphi_k(f)\in (M_k^*)^\vee$ is well-defined by our assumption that $f_{|M_{k,*}}=0$ noting that the kernel of F_k equals the kernel of $\mathrm{Tr}_{H_{k,-}/H_k}$ since the multiplication by s is injective and we have $F_k = s\circ \mathrm{Tr}_{H_{k,-}/H_k}$. So we compute

$$\varphi_k(f)(m) = ((1\otimes F_k)^\vee)^{-1}(\mathrm{Tr}_{H_{k,-}/H_k}(1\otimes f))(m) =$$
$$= ((1\otimes F_k)^\vee)^{-1}(1\otimes \mathrm{Tr}_{H_{k,-}/H_k}(f))(\sum_{u\in J((N_0/H_k)/s(N_0/H_k)s^{-1})} uF_k(m_u)) =$$
$$= ((1\otimes F_k)^\vee)^{-1}(1\otimes \mathrm{Tr}_{H_{k,-}/H_k}(f))(\sum_u 1\otimes F_k(u\otimes m_u)) =$$
$$= (1\otimes \mathrm{Tr}_{H_{k,-}/H_k}(f))(\sum_{u\in J((N_0/H_k)/s(N_0/H_k)s^{-1})}(u\otimes m_u)) =$$
$$= \mathrm{Tr}_{H_{k,-}/H_k}(f)(F_k^{-1}(u_0F_k(m_{u_0}))) = f(\mathrm{Tr}_{H_{k,-}/H_k}((s^{-1}u_0s)m_{u_0})) \qquad (7)$$

where u_0 is the single element in $J(N_0/sN_0s^{-1})$ corresponding to the coset of 1. The other terms in the above sum vanish as $1 \otimes \mathrm{Tr}_{H_{k,-}/H_k}(f)$ is supported on $1 \otimes M_k$ by definition. In order to simplify notation put f_* for the restriction of f to $\mathrm{Tr}_{H_j/H_k}(M_k)$ and

$$U := J(N_0/sN_0s^{-1}) \cap H_j sN_0s^{-1} \ .$$

Note that we have $0 = \varphi_j(f_*)(uF_j(m'))$ for all $m' \in M_j$ and

$$u \in J(N_0/sN_0s^{-1}) \setminus U \ .$$

Therefore using Lemma 2.7 we obtain

$$\begin{aligned}
\varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k} m) = \varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k} \sum_{u \in J(N_0/sN_0s^{-1})} uF_k(m_u)) = \\
= \varphi_j(f_*)(\sum_{u \in J(N_0/sN_0s^{-1})} uF_j \circ \mathrm{Tr}_{H_j/H_k}(m_u)) = \\
= \sum_{u \in U} f(\mathrm{Tr}_{H_{j,-}/H_j}(s^{-1}\overline{u}s \, \mathrm{Tr}_{H_j/H_k}(m_u))) = \\
= \sum_{u \in U} f(s^{-1}\overline{u}s \, \mathrm{Tr}_{H_{j,-}/H_k}(m_u))
\end{aligned} \tag{8}$$

where for each $u \in U$ we choose a fixed $\overline{u}$ in $sN_0s^{-1} \cap H_j u$. Note that $f(s^{-1}\overline{u}s \, \mathrm{Tr}_{H_{j,-}/H_k}(m_u))$ does not depend on this choice: If $\overline{u_1} \in sN_0s^{-1} \cap H_j u$ is another choice then we have $(\overline{u_1})^{-1}\overline{u} \in sN_0s^{-1} \cap H_j$ whence $s^{-1}(\overline{u_1})^{-1}\overline{u}s$ lies in $H_{j,-} = N_0 \cap s^{-1}H_j s$ so we have

$$\begin{aligned}
s^{-1}\overline{u}s \, \mathrm{Tr}_{H_{j,-}/H_k}(m_u) = s^{-1}\overline{u_1}ss^{-1}(\overline{u_1})^{-1}\overline{u}s \, \mathrm{Tr}_{H_{j,-}/H_k}(m_u) = \\
= s^{-1}\overline{u_1}s \, \mathrm{Tr}_{H_{j,-}/H_k}(m_u) \ .
\end{aligned}$$

Moreover, the equation (8) also shows that $\varphi_j(f_*)$ is a well-defined element in $(\mathrm{Tr}_{H_j/H_k}(M_k^*))^\vee$. On the other hand, for the restriction of $\varphi_k(f)$ to $\mathrm{Tr}_{H_j/H_k}(M_k)$ we compute

$$\begin{aligned}
\varphi_k(f)(\mathrm{Tr}_{H_j/H_k} m) = \varphi_k(f)(\sum_{w \in J(H_j/H_k)} w \sum_{u \in J(N_0/sN_0s^{-1})} uF_k(m_u)) = \\
= \sum_{w \in J(H_j/H_k)} \sum_{u \in J(N_0/sN_0s^{-1})} \varphi_k(f)(wuF_k(m_u)) = \\
= \sum_{\substack{u \in U \\ w \in J(H_j/H_k) \cap (sN_0s^{-1}u^{-1})}} f(\mathrm{Tr}_{H_{k,-}/H_k}((s^{-1}wus)m_u)) = \\
= f(\sum_{v := s^{-1}wu\overline{u}^{-1}s \in J(H_{j,-}/H_{k,-})} \mathrm{Tr}_{H_{k,-}/H_k} \sum_{u \in U} vs^{-1}\overline{u}sm_u) = \\
= \sum_{u \in U} f(s^{-1}\overline{u}s \, \mathrm{Tr}_{H_{j,-}/H_k}(m_u))
\end{aligned}$$

that equals $\varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k} m)$ by (8). Finally, let now $f \in M_k^\vee$ be arbitrary. Since $M_{k,*}$ is finite, there exists an integer $r \geq 0$ such that $X^r f$ vanishes on $M_{k,*}$. By the above discussion we have $\varphi_k(X^r f)(\mathrm{Tr}_{H_j/H_k} m) = \varphi_j(X^r f_*)(\mathrm{Tr}_{H_j/H_k} m)$. The statement follows noting that $\varphi(X^r)$ is invertible in the ring $\Lambda(N_0/H_j)/\varpi^h[1/X]$. $\square$ $\square$

So we may take the projective limit $M_\infty^\vee[1/X] := \varprojlim_k M_k^\vee[1/X]$ with respect to these quotient maps. The resulting object is an étale (φ,Γ)-module over the ring

$$\varprojlim_k \Lambda(N_0/H_k)/\varpi^h[1/X] \cong \Lambda_\ell(N_0)/\varpi^h \ .$$

Moreover, by taking the projective limit of (6) with respect to k we obtain a φ- and Γ-equivariant isomorphism $(M_\infty^\vee[1/X])_{H_j} \cong M_j^\vee[1/X]$. So we just proved

Corollary 2.9. *For any object $M \in \mathcal{M}(\pi^{H_0})$ the (φ,Γ)-module $M^\vee[1/X]$ over $o/\varpi^h((X))$ corresponds to $M_\infty^\vee[1/X]$ via the equivalence of categories in Theorem 8.20 in [10].*

Note that whenever $M \subset M'$ are two objects in $\mathcal{M}(\pi^{H_0})$ then we have a natural surjective map $M'^\vee_\infty[1/X] \twoheadrightarrow M_\infty^\vee[1/X]$. So in view of the above corollary we define

$$D^\vee_{\xi,\ell,\infty}(\pi) := \varprojlim_{k\geq 0, M\in\mathcal{M}(\pi^{H_0})} M_k^\vee[1/X] = \varprojlim_{M\in\mathcal{M}(\pi^{H_0})} M_\infty^\vee[1/X] \ .$$

We call two elements $M, M' \in \mathcal{M}(\pi^{H_0})$ equivalent ($M \sim M'$) if the inclusions $M \subseteq M+M'$ and $M' \subseteq M+M'$ induce isomorphisms $M^\vee[1/X] \cong (M+M')^\vee[1/X] \cong M'^\vee[1/X]$. This is equivalent to the condition that M equals M' upto finitely generated o-modules. In particular, this is an equivalence relation on the set $\mathcal{M}(\pi^{H_0})$. Similarly, we say that $M_k, M'_k \in \mathcal{M}_k(\pi^{H_k})$ are equivalent if the inclusions $M_k \subseteq M_k + M'_k$ and $M'_k \subseteq M_k + M'_k$ induce isomorphisms $M_k^\vee[1/X] \cong (M_k+M'_k)^\vee[1/X] \cong {M'_k}^\vee[1/X]$.

Proposition 2.10. *The maps*

$$\begin{array}{rcl} M & \mapsto & N_0 \operatorname{Tr}_{H_k/s^kH_0s^{-k}}(s^kM) \\ \operatorname{Tr}_{H_0/H_k}(M_k) & \leftarrow\!\shortmid & M_k \end{array}$$

induce a bijection between the sets $\mathcal{M}(\pi^{H_0})/\sim$ and $\mathcal{M}_k(\pi^{H_k})/\sim$. In particaular, we have

$$D^\vee_{\xi,\ell,\infty}(\pi) = \varprojlim_{k\geq 0} \varprojlim_{M_k\in\mathcal{M}_k(\pi^{H_k})} M_k^\vee[1/X] \ .$$

Proof. We have $\operatorname{Tr}_{H_0/H_k}(N_0 \operatorname{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)) = N_0 \operatorname{Tr}_{H_0/s^kH_0s^{-k}}(s^kM) = N_0F^k(M)$ which is equivalent to M. Conversely,

$$N_0 \operatorname{Tr}_{H_k/s^kH_0s^{-k}}(s^k \operatorname{Tr}_{H_0/H_k}(M_k)) = N_0 \operatorname{Tr}_{H_k/s^kH_ks^{-k}}(s^kM_k) = N_0F_k^k(M_k)$$

is equivalent to M_k as it is the image of the map

$$1\otimes F_k^k\colon \Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi^k,\Lambda(N_0/H_k)/\varpi^h} \to M_k$$

having finite cokernel. □ □

We equip the pseudocompact $\Lambda_\ell(N_0)$-module $D^\vee_{\xi,\ell,\infty}(\pi)$ with the weak topology, ie. with the projective limit topology of the weak topologies of $M_\infty^\vee[1/X]$. (The weak topology on $\Lambda_\ell(N_0)$ is defined in section 8 of [9].) Recall that the sets

$$O(M,l,l') := f_{M,l}^{-1}(\Lambda(N_0/H_l)\otimes_{u_\alpha} X^{l'}M^\vee[1/X]^{++}) \tag{9}$$

for $l,l' \geq 0$ and $M \in \mathcal{M}(\pi^{H_0})$ form a system of neighbourhoods of 0 in the weak topology of $D^\vee_{\xi,\ell,\infty}(\pi)$. Here $f_{M,l}$ is the natural projection map $f_{M,l}\colon D^\vee_{\xi,\ell,\infty}(\pi) \twoheadrightarrow M_l^\vee[1/X]$ and $M^\vee[1/X]^{++}$ denotes the set of elements $d \in M^\vee[1/X]$ with $\varphi^n(d) \to 0$ in the weak topology of $M^\vee[1/X]$ as $n\to\infty$.

2.2 A natural transformation from D_{SV} to $D^\vee_{\xi,\ell,\infty}$

In order to avoid confusion we denote by $D_{SV}(\pi)$ the $\Lambda(N_0)$-module with an action of B_+^{-1} associated to the smooth o-torsion representation π defined as $D(\pi)$ in [9] (note that in [9] the notation V is used for the o-torsion representation that we denote by π). For a brief review of this functor see section 1.2.

Lemma 2.11. *Let W be in $\mathcal{B}_+(\pi)$ and $M \in \mathcal{M}(\pi^{H_0})$. There exists a positive integer $k_0 > 0$ such that for all $k \geq k_0$ we have $s^k M \subseteq W$. In particular, both $M_k = N_0 \operatorname{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M)$ and $N_0 F^k(M)$ are contained in W for all $k \geq k_0$.*

Proof. By the assumption that M is finitely generated over $\Lambda(N_0/H_0)/\varpi^h[F]$ and W is a B_+-subrepresentation it suffices to find an integer s^{k_0} such that we have $s^{k_0} m_i$ lies in W for all the generators $m_1, \dots, m_r$ of M. This, however, follows from Lemma 2.1 in [9] noting that the powers of s are cofinal in T_+. □ □

In particular, we have a homomorphism $W^\vee \twoheadrightarrow M_k^\vee$ of $\Lambda(N_0)$-modules induced by this inclusion. We compose this with the localisation map $M_k^\vee \to M_k^\vee[1/X]$ and take projective limits with respect to k in order to obtain a $\Lambda(N_0)$-homomorphism

$$\operatorname{pr}_{W,M} \colon W^\vee \to M_\infty^\vee[1/X] \ .$$

Lemma 2.12. *The map $\operatorname{pr}_{W,M}$ is ψ_s- and Γ-equivariant.*

Proof. The Γ-equivariance is clear as it is given by the multiplication by elements of Γ on both sides. For the ψ_s-equivariance let $k > 0$ be large enough so that H_k is contained in $sH_0s^{-1} \leq sN_0s^{-1}$ (ie. $H_{k,-} = s^{-1}H_k s$) and M_k is contained in W. Let f be in $W^\vee = \operatorname{Hom}_o(W, o/\varpi^h)$ such that $f_{|N_0 s M_{k,*}} = 0$. By definition we have $\psi_s(f)(w) = f(sw)$ for any $w \in W$. Denote the restriction of f to M_k by $f_{|M_k}$ and choose an element $m \in M_k^* \leq M_k$ written in the form

$$m = \sum_{u \in J(N_0/sN_0s^{-1})} uF_k(m_u) = \sum_{u \in J(N_0/sN_0s^{-1})} us \operatorname{Tr}_{H_{k,-}/H_k}(m_u) \ .$$

Then we compute

$$\begin{aligned}
f_{|M_k}(m) = & \sum_{u \in J(N_0/sN_0s^{-1})} f(us \operatorname{Tr}_{H_{k,-}/H_k}(m_u)) = \\
= & \sum_{u \in J(N_0/sN_0s^{-1})} (u^{-1}f)(s \operatorname{Tr}_{H_{k,-}/H_k}(m_u)) = \\
= & \sum_{u \in J(N_0/sN_0s^{-1})} \psi_s(u^{-1}f)(\operatorname{Tr}_{H_{k,-}/H_k}(m_u)) = \\
\overset{(7)}{=} & \sum_{u \in J(N_0/sN_0s^{-1})} \varphi(\psi_s(u^{-1}f)_{|M_k})(F_k(m_u)) = \\
= & \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi_s(u^{-1}f)_{|M_k})(uF_k(m_u)) = \\
= & \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi_s(u^{-1}f)_{|M_k})(m)
\end{aligned}$$

as for distinct $u,v\in J(N_0/sN_0s^{-1})$ we have $u\varphi(f_0)(vF_k(m_v))=0$ for any $f_0\in(M_k^*)^\vee$. So by inverting X and taking projective limits with respect to k we obtain

$$\mathrm{pr}_{W,M}(f)=\sum_{u\in J(N_0/sN_0s^{-1})}u\varphi(\mathrm{pr}_{W,M}(\psi_s(u^{-1}f)))$$

as we have $(M_k^*)^\vee[1/X]\cong M_k^\vee[1/X]$. However, since $M_\infty^\vee[1/X]$ is an étale (φ,Γ)-module over $\Lambda_\ell(N_0)/\varpi^h$ we have a unique decomposition of $\mathrm{pr}_{W,M}(f)$ as

$$\mathrm{pr}_{W,M}(f)=\sum_{u\in J(N_0/sN_0s^{-1})}u\varphi(\psi(u^{-1}\,\mathrm{pr}_{W,M}(f)))$$

so we must have $\psi(\mathrm{pr}_{W,M}(f))=\mathrm{pr}_{W,M}(\psi_s(f))$. For general $f\in W^\vee$ note that $N_0sM_{k,*}$ is killed by $\varphi(X^r)$ for $r\geq 0$ big enough, so we have $X^r\psi(\mathrm{pr}_{W,M}(f))=\psi(\mathrm{pr}_{W,M}(\varphi(X^r)f))=\mathrm{pr}_{W,M}(\psi_s(\varphi(X^r)f))=X^r\,\mathrm{pr}_{W,M}(\psi_s(f))$. The statement follows since X^r is invertible in $\Lambda_\ell(N_0)$. $\square$ $\square$

By taking the projective limit with respect to $M\in\mathcal{M}(\pi^{H_0})$ and the injective limit with respect to $W\in\mathcal{B}_+(\pi)$ we obtain a ψ_s- and Γ-equivariant $\Lambda(N_0)$-homomorphism

$$\mathrm{pr}:=\varinjlim_W\varprojlim_M\mathrm{pr}_{W,M}\colon D_{SV}(\pi)\to D^\vee_{\xi,\ell,\infty}(\pi)\ .$$

Remarks.

1. Taking Pontryagin dual of the inclusion $M_k\leq\pi$ for all $M\in\mathcal{M}(\pi^{H_k})$ and $k\geq 0$ we obtain a composite map $\pi^\vee\twoheadrightarrow M_k^\vee\to M_k^\vee[1/X]$. These are compatible with the projective limit construction therefore induce natural maps $\pi^\vee\to D_\xi^\vee(\pi)$ and $\pi^\vee\to D^\vee_{\xi,\ell,\infty}(\pi)$. Both of these maps factor through the map $\pi^\vee\twoheadrightarrow D_{SV}(\pi)$ by Lemma 2.11.

2. The natural topology on D_{SV} obtained as the quotient topology from the compact topology on $\pi^\vee$ via the surjective map $\pi^\vee\twoheadrightarrow D_{SV}(\pi)$ is compact, but may not be Hausdorff in general. However, if $\mathcal{B}_+(\pi)$ contains a minimal element (as in the case of the principal series [7]) then it is also Hausdorff. However, the map pr factors through the maximal Hausdorff quotient of $D_{SV}(\pi)$, namely $\overline{D}_{SV}(\pi):=(\bigcap_{W\in\mathcal{B}_+(\pi)}W)^\vee$. Indeed, pr is continuous and $D^\vee_{\xi,\ell,\infty}(\pi)$ is Hausdorff, so the kernel of pr is closed in $D_{SV}(\pi)$ (and contains 0).

3. Assume that $h=1$, ie. π is a smooth representation in characteristic p. Then $D^\vee_{\xi,\ell,\infty}(\pi)$ has no nonzero $\Lambda(N_0)/\varpi$-torsion. Hence the $\Lambda(N_0)/\varpi$-torsion part of $D_{SV}(\pi)$ is contained in the kernel of pr.

4. If $D_{SV}(\pi)$ has finite rank and its torsion free part is étale over $\Lambda(N_0)$ then $\Lambda_\ell(N_0)\otimes_{\Lambda(N_0)}D_{SV}(\pi)$ is also étale and of finite rank r over $\Lambda_\ell(N_0)$. Moreover, the map $\Lambda_\ell(N_0)\otimes_{\Lambda(N_0)}$ pr : $\Lambda_\ell(N_0)\otimes_{\Lambda(N_0)}D_{SV}(\pi)\to D_{\xi,\ell,\infty}(\pi)$ has dense image by Lemma 2.11. Thus $D^\vee_{\xi,\ell,\infty}(\pi)$ has rank at most r over $\Lambda_\ell(N_0)$. In particular, for π being the principal series $D_{SV}(\pi)$ has rank 1 and its torsion free part is étale over $\Lambda(N_0)$ ([7]), hence we obtained that $D^\vee_{\xi,\ell,\infty}(\pi)$ has rank 1 over $\Lambda_\ell(N_0)$ (cf. Example 7.6 of [2]).

One can show the above Remark 2 algebraically, too. Let $M \in \mathcal{M}(\pi^{H_0})$ be arbitrary. Then the map $1 \otimes \mathrm{id}_{M^\vee} \colon M^\vee \to M^\vee[1/X]$ has finite kernel, so the image $(1 \otimes \mathrm{id}_{M^\vee})(M^\vee)$ is isomorphic to $M_0^\vee$ for some finite index submodule $M_0 \leq M$. Moreover, $M_0^\vee$ is a ψ- and Γ-invariant treillis in $D := M^\vee[1/X] = M_0^\vee[1/X]$. Therefore the map $(1 \otimes F)^\vee$ is injective on $M_0^\vee$ since it is injective after inverting X and $M_0^\vee$ has no X-torsion. This means that $1 \otimes F \colon o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]],\varphi} M_0 \to M_0$ is surjective, ie. we have $M_0 = N_0 F^k(M_0)$ for all $k \geq 0$. However, for any $W \in \mathcal{B}_+(\pi)$ and k large enough (depending a priori on W) we have $N_0 F^k(M_0) \subseteq W$, so we deduce $M_0 \subset \cap_{W \in \mathcal{B}_+} W$.

Corollary 2.13. *If $\pi = \mathrm{Ind}_{B_0}^B \pi_0$ is a compactly induced representation of B for some smooth o/ϖ^h-representation π_0 of B_0 then we have $D_\xi^\vee(\pi) = 0$. In particular, $D_\xi^\vee$ is not exact on the category of smooth o/ϖ^h-representations of B. (However, it may still be exact on a smaller subcategory with additional finiteness conditions.)*

Proof. By the 2nd remark above the map $\pi^\vee \to D_\xi^\vee(\pi)$ factors through the maximal Hausdorff quotient $\overline{D}_{SV}(\pi)$ of $D_{SV}(\pi)$. By Lemma 3.2 in [9], we have $\overline{D}_{SV}(\pi) = (\bigcap_\sigma W_\sigma)^\vee$ where the B_+-subrepresentations W_σ are indexed by order-preserving maps $\sigma \colon T_+/T_0 \to \mathrm{Sub}(\pi_0)$ where $\mathrm{Sub}(\pi_0)$ is the partially order set of B_0-subrepresentations of π_0. The explicit description of the B_+-subrepresentations W_σ (there denoted by M_σ) before Lemma 3.2 in [9] shows that we have in fact $\bigcap_\sigma W_\sigma = \{0\}$ whence the natural map $\pi^\vee \to D_\xi^\vee(\pi)$ is zero. However, by the construction of this map this can only be zero if $D_\xi^\vee(\pi) = 0$.

Since the principal series arises as a quotient of a compactly induced representation, the exactness of $D_\xi^\vee$ would imply the vanishing of $D_\xi^\vee$ on the principal series, too—which is not the case by Ex. 7.6 in [2]. □ □

Proposition 2.14. *Let D be an étale (φ, Γ)-module over $\Lambda_\ell(N_0)/\varpi^h$, and $f : D_{SV}(\pi) \to D$ be a continuous ψ_s and Γ-equivariant $\Lambda(N_0)$-homomorphism. Then f factors uniquely through* pr*, ie. there exists a unique ψ- and Γ-equi-variant $\Lambda(N_0)$-homomorphism $\hat{f} : D_{\xi,\ell,\infty}^\vee(\pi) \to D$ such that $f = \hat{f} \circ \mathrm{pr}$.*

Proof. For the uniqueness of $\hat{f}$ note that Lemma 2.11 implies the density of the image of $\Lambda_\ell(N_0) \otimes D_{SV}(\pi)$ in $D_{\xi,\ell,\infty}^\vee(\pi)$ as its composite with the projection onto $M_k^\vee[1/X]$ is surjective for k large enough and $M \in \mathcal{M}(\pi^{H_0})$ arbitrary. Therefore if $\hat{f}'$ is another lift then $\hat{f} - \hat{f}'$ vanishes on a dense subset whence it is zero by continuity.

At first we construct a homomorphism $\hat{f}_{H_0} : D_\xi^\vee = (D_{\xi,\ell,\infty}^\vee)_{H_0} \to D_{H_0}$ such that the following diagram commutes:

$$\begin{tikzcd}
D_{SV}(\pi) \arrow[r, "\mathrm{pr}"] \arrow[dr, "f"'] & D_{\xi,\ell,\infty}^\vee(\pi) \arrow[r, "(\cdot)_{H_0}"] & D_\xi^\vee(\pi) \arrow[d, "\hat{f}_{H_0}"] \\
& D \arrow[r, "(\cdot)_{H_0}"'] & D_{H_0}
\end{tikzcd}$$

Consider the composite map $f' : \pi^\vee \to D_{SV}(\pi) \xrightarrow{f} D \to D_{H_0}$. Note that f' is continuous and D_{H_0} is Hausdorff, so $\mathrm{Ker}(f')$ is closed in $\pi^\vee$. Therefore $M_0 = (\pi^\vee/\mathrm{Ker}(f'))^\vee$ is naturally a subspace in π. We claim that M_0 lies in $\mathcal{M}(\pi^{H_0})$. Indeed, $M_0^\vee$ is a quotient of $\pi_{H_0}^\vee$, hence $M_0 \leq \pi^{H_0}$ and it is Γ-invariant since f' is Γ-equivariant. M_0 is admissible because it is discrete, hence $M_0^\vee$ is compact, equivalently finitely generated over $o/\varpi^h[[X]]$, because $M_0^\vee$ can be identified with a $o/\varpi^h[[X]]$-submodule of D_{H_0} which is finitely generated over $o/\varpi^h((X))$. The last thing to verify is that M is finitely generated over $o/\varpi^h[[X]][F]$, which follows from the following

Lemma 2.15. *Let D be an étale (φ,Γ)-module over $o/\varpi^h((X))$ and $D_0 \subset D$ be a ψ and Γ-invariant compact (or, equivalently, finitely generated) $o/\varpi^h[[X]]$ submodule. Then $D_0^\vee$ is finitely generated as a module over $o/\varpi^h[[X]][F]$ where for any $m \in D_0^\vee = \mathrm{Hom}_o(D_0, o/\varpi^h)$ we put $F(m)(f) := m(\psi(f))$ (for all $f \in D_0$).*

Proof. As the extension of finitely generated modules over a ring is again finitely generated, we may assume without loss of generality that $h = 1$ and D is irreducible, ie. D has no nontrivial étale (φ,Γ)-submodule over $o/\varpi((X))$.

If $D_0 = \{0\}$ then there is nothing to prove. Otherwise D_0 contains the smallest ψ and Γ stable $o[[X]]$-submodule $D^\natural$ of D. So let $0 \neq m \in D_0^\vee$ be arbitrary such that the restriction of m to $D^\natural$ is nonzero and consider the $o/\varpi[[X]][F]$-submodule $M := o/\varpi[[X]][F]m$ of $D_0^\vee$ generated by m. We claim that M is not finitely generated over o. Suppose for contradiction that the elements $F^r m$ are not linearly independent over o/ϖ. Then we have a polynomial $P(x) = \sum_{i=0}^n a_i x^i \in o/\varpi[x]$ such that $0 = P(F)m(f) = m(\sum a_i \psi^i(f)) = m(P(\psi)f)$ for any $f \in D^\natural \subset D_0$. However, $P(\psi)\colon D^\natural \to D^\natural$ is surjective by Prop. II.5.15. in [3], so we obtain $m_{|D^\natural} = 0$ which is a contradiction. In particular, we obtain that $M^\vee[1/X] \neq 0$. However, note that $M^\vee[1/X]$ has the structure of an étale (φ,Γ)-module over $o/\varpi((X))$ by Lemma 2.6 in [2]. Indeed, M is admissible, Γ-invariant, and finitely generated over $o/\varpi[[X]][F]$ by construction. Moreover, we have a natural surjective homomorphism $D = D_0[1/X] = (D_0^\vee)^\vee[1/X] \to M^\vee[1/X]$ which is an isomorphism as D is assumed to be irreducible. Therefore we have $(D_0^\vee/M)^\vee[1/X] = 0$ showing that $D_0^\vee/M$ is finitely generated over o. In particular, both M and $D_0^\vee/M$ are finitely generated over $o/\varpi[[X]][F]$ therefore so is $D_0^\vee$. □ □

Now $D_0 = M_0^\vee$ is a ψ- and Γ-invariant $o/\varpi^h[[X]]$-submodule of D therefore we have an injection $f_0\colon M_0^\vee[1/X] \hookrightarrow D$ of étale (φ,Γ)-modules. The map $\hat{f}_{H_0}\colon D_\xi^\vee \to D_{H_0}$ is the composite map $D_\xi^\vee \twoheadrightarrow M_0^\vee[1/X] \hookrightarrow D$. It is well defined and makes the above diagram commutative, because the map

$$\pi^\vee \to D_{SV}(\pi) \overset{\mathrm{pr}}{\to} D^\vee_{\xi,\ell,\infty}(\pi) \overset{(\cdot)_{H_0}}{\to} D_\xi^\vee(\pi) \to M_0^\vee[1/X]$$

is the same as $\pi^\vee \to M_0^\vee \to M_0^\vee[1/X]$.

Finally, by Corollary 2.9 $M^\vee[1/X]$ (resp. D_{H_0}) corresponds to $M_\infty^\vee[1/X]$ (resp. to D) via the equivalence of categories in Theorem 8.20 in [10] therefore f_0 can uniquely be lifted to a φ- and Γ-equivariant $\Lambda_\ell(N_0)$-homomorphism $f_\infty\colon M_\infty^\vee[1/X] \hookrightarrow D$. The map $\hat{f}$ is defined as the composite $D^\vee_{\xi,\ell,\infty} \twoheadrightarrow M_\infty^\vee[1/X] \hookrightarrow D$. Now the image of $f - \hat{f}\circ\mathrm{pr}$ is a ψ_s-invariant $\Lambda(N_0)$-submodule in $(H_0-1)D$ therefore it is zero by Lemma 8.17 and the proof of Lemma 8.18 in [10]. Indeed, for any $x \in D_{SV}(\pi)$ and $k \geq 0$ we may write $(f - \hat{f}\circ\mathrm{pr})(x)$ in the form $\sum_{u\in J(N_0/s^kN_0s^{-k})} u\varphi^k((f-\hat{f}\circ\mathrm{pr})(\psi^k(u^{-1}x)))$ that lies in $(H_k-1)D$. □ □

2.3 Étale hull

In this section we construct the étale hull of $D_{SV}(\pi)$: an étale T_+-module $\widetilde{D_{SV}}(\pi)$ over $\Lambda(N_0)$ with an injection $\iota : D_{SV}(\pi) \to \widetilde{D_{SV}}(\pi)$ with the following universal property: For any étale (φ,Γ)-module D' over $\Lambda(N_0)$, and ψ_s and Γ-equivariant map $f: D_{SV}(\pi)\to D'$, f factors through $\widetilde{D_{SV}}(\pi)$, ie. there exists a unique ψ- and Γ-equivariant $\Lambda(N_0)$-homomorphism $\widetilde{f}:\widetilde{D_{SV}}(\pi)\to D'$ making the diagram

$$\begin{tikzcd} D_{SV}(\pi) \arrow[r, "\iota"] \arrow[d, "f"'] & \widetilde{D_{SV}}(\pi) \arrow[dl, "\widetilde{f}"] \\ D' & \end{tikzcd}$$

commutative. Moreover, if we assume further that D' is an étale T_+-module over $\Lambda(N_0)$ and the map f is ψ_t-equivariant for all $t\in T_+$ then the map $\widetilde{f}$ is T_+-equivariant.

Definition 2.16. *Let D be a $\Lambda(N_0)$-module and $T_*\leq T_+$ be a submonoid. Assume moreover that the monoid T_* (or in the case of ψ-actions the inverse monoid T_*^{-1}) acts o-linearly on D, as well.*

We call the action of T_* a φ-action (relative to the $\Lambda(N_0)$-action) and denote the action of t by $d\mapsto\varphi_t(d)$, if for any $\lambda\in\Lambda(N_0)$, $t\in T_*$ and $d\in D$ we have $\varphi_t(\lambda d)=\varphi_t(\lambda)\varphi_t(d)$. Moreover, we say that the φ-action is injective *if for all $t\in T_*$ the map φ_t is injective. The φ-action of T_* is* nondegenerate *if for all $t\in T_*$ we have*

$$D=\sum_{u\in J(N_0/tN_0t^{-1})}\mathrm{Im}(u\circ\varphi_t)=\sum_{u\in J(N_0/tN_0t^{-1})}u(\varphi_t(D))\ .$$

We call the action of T_*^{-1} a ψ-action of T_* (relative to the $\Lambda(N_0)$-action) and denote the action of $t^{-1}\in T_*^{-1}$ by $d\mapsto\psi_t(d)$, if for any $\lambda\in\Lambda(N_0)$, $t\in T_*$ and $d\in D$ we have $\psi_t(\varphi_t(\lambda)d)=\lambda\psi_t(d)$. Moreover, we say that the ψ-action of T_* is surjective *if for all $t\in T_*$ the map ψ_t is surjective. The ψ-action of T_* is* nondegenerate *if for all $t\in T_*$ we have*

$$\{0\}=\bigcap_{u\in J(N_0/tN_0t^{-1})}\mathrm{Ker}(\psi_t\circ u^{-1})\ .$$

The nondegeneracy is equivalent to the condition that for any $t\in T_*$ $\mathrm{Ker}(\psi_t)$ does not contain any nonzero $\Lambda(N_0)$-submodule of D.

We say that a φ- and a ψ-action of T_* are compatible *on D, if*

$(\varphi\psi)$ for any $t\in T_*$, $\lambda\in\Lambda(N_0)$, and $d\in D$ we have $\psi_t(\lambda\varphi_t(d))=\psi_t(\lambda)d$.

Note that with $\lambda=1$ we also have $\psi_t\circ\varphi_t=\mathrm{id}_D$ for any $t\in T_*$ assuming $(\varphi\psi)$.

We also consider φ- and ψ-actions of the monoid $\mathbb{Z}_p\setminus\{0\}$ on $\Lambda(N_0)$-modules via the embedding $\xi\colon\mathbb{Z}_p\setminus\{0\}\to T_+$. Modules with a φ-action (resp. ψ-action) of $\mathbb{Z}_p\setminus\{0\}$ are called (φ,Γ)-modules (resp. (ψ,Γ)-modules).

For example, the natural φ- and ψ-actions of T_+ on $\Lambda(N_0)$ are compatible.

Remarks. 1. Note that the ψ-action of the monoid T_* is in fact an action of the inverse monoid T_*^{-1}. However, we assume T_+ to be commutative so it may also be viewed as an action of T_*.

2. Pontryagin duality provides an equivalence of categories between compact $\Lambda(N_0)$-modules with a continuous ψ-action of T_* and discrete $\Lambda(N_0)$-modules with a continuous φ-action of T_*. The surjectivity of the ψ-action corresponds to the injectivity of φ-action. Moreover, the ψ-action is nondegenerate if and only if so is the corresponding φ-action on the Pontryagin dual.

If D is a $\Lambda(N_0)$-module with a φ-action of T_* then there exists a homomorphism

$$\Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} D \to D, \lambda \otimes d \mapsto \lambda\varphi_t(d) \tag{10}$$

of $\Lambda(N_0)$-modules. We say that the T_*-action on D is *étale* if the above map is an isomorphism. The φ-action of T_* on D is étale if and only if it is injective and for any $t \in T_*$ we have

$$D = \bigoplus_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(D) \ . \tag{11}$$

Similarly, we call a $\Lambda(N_0)$-module together with a φ-action of the monoid $\mathbb{Z}_p \setminus \{0\}$ an étale (φ, Γ)-module over $\Lambda(N_0)$ if the action of $\varphi = \varphi_s$ is étale.

If D is an étale T_*-module over $\Lambda(N_0)$ then there exists a ψ-action of T_* compatible with the étale φ-action (see [9] Section 6).

Dually, if D is a $\Lambda(N_0)$-module with a ψ-action of T_* then there exists a map

$$\begin{aligned} \iota_t \colon D &\to \Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} D \\ d &\mapsto \sum_{u \in J(N_0/tN_0t^{-1})} u \otimes \psi_t(u^{-1}d) \ . \end{aligned}$$

Lemma 2.17. *Fix $t \in T_*$. For any $\lambda \in \Lambda(N_0)$ and $u, v \in N_0$ we put $\lambda_{u,v} := \psi_t(u^{-1}\lambda v)$. For any fixed $v \in N_0$ we have*

$$\lambda v = \sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(\lambda_{u,v})$$

and for any fixed $u \in N_0$ we have

$$u^{-1}\lambda = \sum_{v \in J(N_0/tN_0t^{-1})} \varphi_t(\lambda_{u,v})v^{-1} \ .$$

Proof. The above formulae follow from the usual identities

$$\sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}\mu)) = \mu = \sum_{v \in J(N_0/tN_0t^{-1})} \varphi_t(\psi_t(\mu v))v^{-1}$$

for $\mu \in \Lambda(N_0)$ as the inverses of elements of $J(N_0/tN_0t^{-1})$ form a set of representatives of the right cosets of tN_0t^{-1}. □ □

Lemma 2.18. *For any $t \in T_*$ the map ι_t is a homomorphism of $\Lambda(N_0)$-modules. It is injective for all $t \in T_*$ if and only if the ψ-action of T_* on D is nondegenerate.*

Proof. Using Lemma 2.17 we compute

$$\begin{aligned}
\iota_t(\lambda x) &= \sum_{u\in J(N_0/tN_0t^{-1})} u\otimes\psi_t(u^{-1}\lambda x) = \\
&= \sum_{u,v\in J(N_0/tN_0t^{-1})} u\otimes\psi_t(\varphi_t(\lambda_{u,v})v^{-1}x) = \\
&= \sum_{u,v\in J(N_0/tN_0t^{-1})} u\otimes\lambda_{u,v}\psi_t(v^{-1}x) = \\
&= \sum_{u,v\in J(N_0/tN_0t^{-1})} u\varphi_t(\lambda_{u,v})\otimes\psi_t(v^{-1}x) = \\
&= \sum_{v\in J(N_0/tN_0t^{-1})} \lambda v\otimes\psi_t(v^{-1}x) = \lambda\iota_t(x)\ .
\end{aligned}$$

The second statement follows from noting that $\Lambda(N_0)$ is a free right module over itself via the map φ_t with free generators $u\in J(N_0/tN_0t^{-1})$. □ □

Lemma 2.19. *Let D be a $\Lambda(N_0)$-module with a ψ-action of T_* and $t\in T_*$. Then there exists a ψ-action of T_* on $\varphi_t^*D := \Lambda(N_0)\otimes_{\Lambda(N_0),\varphi_t} D$ making the homomorphism ι_t ψ-equivariant. Moreover, if we assume in addition that the ψ-action on D is nondegenerate then so is the ψ-action on φ_t^*D.*

Proof. Let $t'\in T_*$ be arbitrary and define the action of $\psi_{t'}$ on φ_t^*D by putting

$$\psi_{t'}(\lambda\otimes d) := \sum_{u'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\otimes\psi_{t'}(u'^{-1}d) \text{ for } \lambda\in\Lambda(N_0), d\in D\ ,$$

and extending $\psi_{t'}$ to φ_t^*D o-linearly. Note that we have

$$\begin{aligned}
&\psi_{t'}(\varphi_{t'}(\mu)\lambda\otimes d) = \\
&= \sum_{u'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\varphi_{t'}(\mu)\lambda\varphi_t(u'))\otimes\psi_{t'}(u'^{-1}d) = \mu\psi_{t'}(\lambda\otimes d)\ .
\end{aligned}$$

Moreover, the map $\psi_{t'}$ is well-defined since we have

$$\begin{aligned}
\psi_{t'}(\lambda\varphi_t(\mu)\otimes d) &= \sum_{v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(\mu)\varphi_t(v'))\otimes\psi_{t'}(v'^{-1}d) = \\
&= \sum_{v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(\mu v'))\otimes\psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'\varphi_{t'}(\mu_{u',v'})))\otimes\psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\varphi_t(\mu_{u',v'})\otimes\psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\otimes\mu_{u',v'}\psi_{t'}(v'^{-1}d) = \\
&= \sum_{u',v'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\otimes\psi_{t'}(\varphi_{t'}(\mu_{u',v'})v'^{-1}d) = \\
&= \sum_{u'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\otimes\psi_{t'}(u'^{-1}\mu d) = \psi_{t'}(\lambda\otimes\mu d)\ ,
\end{aligned}$$

using Lemma 2.17 where $\mu_{u',v'}=\psi_{t'}(u'^{-1}\mu v')$. Introducing the notation $J':=J(N_0/t'N_0t'^{-1})$ and $J'':=J(N_0/t''N_0t''^{-1})$ we further compute

$$\begin{aligned}
\psi_{t''}(\psi_{t'}(\lambda\otimes d)) &= \psi_{t''}(\sum_{u'\in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\otimes\psi_{t'}(u'^{-1}d)) = \\
&= \sum_{u''\in J''}\sum_{u'\in J'} \psi_{t''}(\psi_{t'}(\lambda\varphi_t(u'))\varphi_t(u''))\otimes\psi_{t''}(u''^{-1}\psi_{t'}(u'^{-1}d)) = \\
&= \sum_{u''\in J''}\sum_{u'\in J'} \psi_{t''}(\psi_{t'}(\lambda\varphi_t(u'\varphi_{t'}(u''))))\otimes\psi_{t''}(\psi_{t'}(\varphi_{t'}(u'')^{-1}u'^{-1}d)) = \\
&= \psi_{t''t'}(\lambda\otimes d)
\end{aligned}$$

showing that it is indeed a ψ-action of the monoid T_*.

For the second statement of the Lemma we compute

$$\begin{aligned}
&\psi_{t'}(\iota_t(x)) = \\
&= \sum_{u'\in J(N_0/t'N_0t'^{-1})}\sum_{u\in J(N_0/tN_0t^{-1})} \psi_{t'}(u\varphi_t(u'))\otimes\psi_{t'}(u'^{-1}\psi_t(u^{-1}x)) = \\
&= \sum_{u'\in J(N_0/t'N_0t'^{-1})}\sum_{u\in J(N_0/tN_0t^{-1})} \psi_{t'}(u\varphi_t(u'))\otimes\psi_{t'}(\psi_t(\varphi_t(u')^{-1}u^{-1}x))\ .
\end{aligned}$$

Note that in the above sum $u\varphi_t(u')$ runs through a set of representatives for the cosets $N_0/tt'N_0t'^{-1}t^{-1}$. Moreover, $v:=\psi_{t'}(u\varphi_t(u'))$ is nonzero if and only if $u\varphi_t(u')$ lies in $t'N_0t'^{-1}$ and the nonzero values of v run through a set $J'(N_0/tN_0t^{-1})$ of representatives of the cosets N_0/tN_0t^{-1}. In case $v\neq 0$ we have $\varphi_{t'}(v)^{-1}=(u\varphi_t(u'))^{-1}=\varphi_t(u')^{-1}u^{-1}$. So we continue computing by replacing $\psi_{t'}(u\varphi_t(u'))$ by v and omitting the terms with $v=0$

$$\psi_{t'}(\iota_t(x)) =$$
$$= \sum_{u'\in J(N_0/t'N_0t'^{-1})} \sum_{u\in J(N_0/tN_0t^{-1})} \psi_{t'}(u\varphi_t(u')) \otimes \psi_{t'}(\psi_t(\varphi_t(u')^{-1}u^{-1}x)) =$$
$$= \sum_{v\in J'(N_0/tN_0t^{-1})} v \otimes \psi_t(\psi_{t'}(\varphi_{t'}(v^{-1})x)) =$$
$$= \sum_{v\in J'(N_0/tN_0t^{-1})} v \otimes \psi_t(v^{-1}\psi_{t'}(x)) = \iota_t(\psi_{t'}(x)) \ .$$

Assume now that the ψ-action of T_* on D is nondegenerate. Any element in $x \in \varphi_t^*D$ can be uniquely written in the form $\sum_{u\in J(N_0/tN_0t^{-1})} u \otimes x_u$. Assume that for a fixed $t' \in T_*$ we have $\psi_{t'}(u_0'^{-1}x) = 0$ for all $u_0' \in N_0$. Then we compute

$$0 = \psi_{t'}(u_0'^{-1}x) =$$
$$= \sum_{u'\in J(N_0/t'N_0t'^{-1})} \sum_{u\in J(N_0/tN_0t^{-1})} \psi_{t'}(u_0'^{-1}u\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}x_u) \ .$$

Put $y = u_0'^{-1}u\varphi_t(u')$. For any fixed u_0' the set $\{y \mid u \in J(N_0/tN_0t^{-1}), u' \in J(N_0/t'N_0t'^{-1})\}$ forms a set of representatives of $N_0/tt'N_0(tt')^{-1}$, and we have $\psi_{t'}(y) \neq 0$ if and only if y lies in $t'N_0t'^{-1}$ in which case we have $\psi_{t'}(y) = t'^{-1}yt'$. So the nonzero values of $\psi_{t'}(y)$ run through a set of representatives of N_0/tN_0t^{-1}. Since we have the direct sum decomposition $\varphi_t^*D = \bigoplus_{v\in J(N_0/tN_0t^{-1})} v \otimes D$ we obtain $\psi_{t'}(u'^{-1}x_u) = 0$ for all $u' \in J(N_0/t'N_0t'^{-1})$ and $u \in J(N_0/tN_0t^{-1})$ such that $y = u_0'^{-1}u\varphi_t(u')$ is in $t'N_0t'^{-1}$. However, for any choice of u' and u there exists such a u_0', so we deduce $x = 0$. $\square$ $\square$

Proposition 2.20. *Let D be a $\Lambda(N_0)$-module with a ψ-action of T_*. The following are equivalent:*

1. *There exists a unique φ-action on D, which is compatible with ψ and which makes D an étale T_*-module.*

2. *The ψ-action is surjective and for any $t \in T_*$ we have*
$$D = \bigoplus_{u_0\in J(N_0/tN_0t^{-1})} \bigcap_{\substack{u\in J(N_0/tN_0t^{-1})\\ u\neq u_0}} \mathrm{Ker}(\psi_t \circ u^{-1}) \ . \tag{12}$$
In particular, the action of ψ is nondegenerate.

3. *The map ι_t is bijective for all $t \in T_*$.*

Proof. $1 \Longrightarrow 3$ In this case the map ι_t is the inverse of the isomorphism (10) so it is bijective by the étale property.

$3 \Longrightarrow 2$: The injectivity of ι_t shows the nondegeneracy of the ψ-action. Further if $1 \otimes d = \iota_t(x)$ then we have $\psi_t(x) = d$ so the ψ-action is surjective. Moreover, $\iota_t^{-1}(u_0 \otimes D)$ equals $\bigcap_{u_0\neq u\in J(N_0/tN_0t^{-1})} \mathrm{Ker}(\psi_t \circ u^{-1})$ therefore D can be written as a direct sum (12).

$2 \Longrightarrow 1$: In order to define the φ-action of T_* on D we fix $t \in T_*$. For any $d \in D$ we have to choose $\varphi_t(d)$ such that $\psi_t(\varphi_t(d)) = d$. By the surjectivity of ψ_t we can choose $x \in D$ such that $\psi_t(x) = d$. Using the assumption we can write $x = \sum_{u_0 \in J(N_0/tN_0t^{-1})} x_{u_0}$, with

$$x_{u_0} \in \bigcap_{\substack{u \in J(N_0/tN_0t^{-1}) \\ u \neq u_0}} \mathrm{Ker}(\psi_t \circ u^{-1}) \ .$$

By the compatibility $(\varphi\psi)$ we should have

$$\varphi_t(d) \in \bigcap_{\substack{u \in J(N_0/tN_0t^{-1}) \\ u \neq 1}} \mathrm{Ker}(\psi_t \circ u^{-1})$$

as we have $\psi_t(u) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$.

A convenient choice is $\varphi_t(d) = x_1$, and there exists exactly one such element in D: if x' would be an other, then

$$x_1 - x' \in \bigcap_{u \in J(N_0/tN_0t^{-1})} \mathrm{Ker}(\psi_t \circ u^{-1}) = \{0\} \ .$$

This shows the uniqueness of the φ-action. Further, $x_1 = \varphi_t(d) = 0$ would mean that x lies in $\mathrm{Ker}(\psi_t)$ whence $d = \psi_t(x) = 0$—therefore the injectivity. Similarly, by definition we also have $x_{u_0} = u_0\varphi_t \circ \psi_t(u_0^{-1}x)$ for all $u_0 \in J(N_0/sN_0s^{-1})$. By the surjectivity of the ψ-action any element in D can be written of the form $\psi_t(u_0^{-1}x)$ for any fixed $u_0 \in J(N_0/tN_0t^{-1})$ so we obtain

$$u_0\varphi_t(D) = \bigcap_{u_0 \neq u \in J(N_0/tN_0t^{-1})} \mathrm{Ker}(\psi_t \circ u^{-1}) \ .$$

The étale property (11) follows from this using our assumption 2. Moreover, this also shows $\psi_t(u\varphi_t(d)) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$ which implies $(\varphi\psi)$ using that $\psi_t \circ \varphi_t = \mathrm{id}_D$ by construction. Finally, $\varphi_t(\lambda)\varphi_t(d) - \varphi_t(\lambda d)$ lies in the kernel of $\psi_t \circ u_0^{-1}$ for any $u_0 \in J(N_0/tN_0t^{-1})$, $\lambda \in \Lambda(N_0)$ and $d \in D$, so it is zero. □ □

From now on if we have an étale T_*-module over $\Lambda(N_0)$ we a priori equip it with the compatible ψ-action, and if we have a $\Lambda(N_0)$-module with a ψ-action, which satisfies the above property 2, we equip it with the compatible φ-action, which makes it étale. The construction of the étale hull and its universal property is given in the following

Proposition 2.21. *For any $\Lambda(N_0)$-module D, with a ψ-action of T_* there exists an étale T_*-module $\widetilde{D}$ over $\Lambda(N_0)$ and a ψ-equivariant $\Lambda(N_0)$-homomor-phism $\iota\colon D \to \widetilde{D}$ with the following universal property: For any ψ-equivariant $\Lambda(N_0)$-homomorphism $f : D \to D'$ into an étale T_*-module D' we have a unique morphism $\widetilde{f} : \widetilde{D} \to D'$ of étale T_*-modules over $\Lambda(N_0)$ making the diagram*

$$\begin{tikzcd} D \arrow[r, "\iota"] \arrow[d, "f"'] & \widetilde{D} \arrow[dl, "\widetilde{f}"] \\ D' & \end{tikzcd}$$

commutative. $\widetilde{D}$ is unique upto a unique isomorphism. If we assume the ψ-action on D to be nondegenerate then ι is injective.

Proof. We will construct $\widetilde{D}$ as the injective limit of φ_t^*D for $t \in T_*$. Consider the following partial order on the set T_*: we put $t_1 \leq t_2$ whenever we have $t_2t_1^{-1} \in T_*$. Note that by Lemma 2.19 we obtain a ψ-equivariant isomorphism $\varphi_{t_2t_1^{-1}}^*\varphi_{t_1}^*D \cong \varphi_{t_2}^*D$ for any pair $t_1 \leq t_2$ in T_*. In particular, we obtain a ψ-equivariant map $\iota_{t_1,t_2}\colon \varphi_{t_1}^*D \to \varphi_{t_2}^*D$. Applying this observation to $\varphi_{t_1}^*D$ for a sequence $t_1 \leq t_2 \leq t_3$ we see that the $\Lambda(N_0)$-modules φ_t^*D ($t \in T_*$) with the ψ-action of T_* form a direct system with respect to the connecting maps ι_{t_1,t_2}. We put

$$\widetilde{D} := \varinjlim_{t \in T_*} \varphi_t^*D$$

as a $\Lambda(N_0)$-module with a ψ-action of T_*. For any fixed $t' \in T_*$ we have

$$\varphi_{t'}^*\widetilde{D} = \Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_{t'}} \varinjlim_{t \in T_*} \varphi_t^*D \cong$$
$$\cong \varinjlim_{t \in T_*} \Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_{t'}} \varphi_t^*D \cong \varinjlim_{t't \in T_*} \varphi_{t't}^*D \cong \widetilde{D}$$

showing that there exists a unique φ-action of T_* on $\widetilde{D}$ making $\widetilde{D}$ an étale T_*-module over $\Lambda(N_0)$ by Proposition 2.20.

For the universal property, let $f : D \to D'$ be an ψ-equivariant map into an étale T_*-module D' over $\Lambda(N_0)$. By construction of the map φ_t on $\widetilde{D}$ ($t \in T_*$) we have $\varphi_t(\iota(x)) = (1 \otimes x)_t$ where $(1 \otimes x)_t$ denotes the image of $1 \otimes x \in \varphi_t^*D$ in $\widetilde{D}$. So we put

$$\widetilde{f}((\lambda \otimes x)_t) := \lambda\varphi_t(f(x)) \in D'$$

and extend it o-linearly to $\widetilde{D}$. Note right away that $\widetilde{f}$ is unique as it is φ_t-equivariant. The map $\widetilde{f}\colon \widetilde{D} \to D'$ is well-defined as we have

$$\widetilde{f}(\iota_{t,tt'}(1 \otimes_t x)) = \widetilde{f}(\sum_{u' \in N_0/t'N_0t'^{-1}} u' \otimes_{t'} \psi_{t'}(u'^{-1} \otimes_t x)) =$$
$$= \sum_{u',v' \in N_0/t'N_0t'^{-1}} \widetilde{f}(u' \otimes_{t'} \psi_{t'}(u'^{-1}\varphi_t(v')) \otimes_t \psi_{t'}(v'^{-1}x)) =$$
$$= \sum_{u',v' \in N_0/t'N_0t'^{-1}} \widetilde{f}(u'\varphi_{t'} \circ \psi_{t'}(u'^{-1}\varphi_t(v')) \otimes_{tt'} \psi_{t'}(v'^{-1}x)) =$$
$$= \sum_{v' \in N_0/t'N_0t'^{-1}} \widetilde{f}(\varphi_t(v') \otimes_{tt'} \psi_{t'}(v'^{-1}x)) =$$
$$= \sum_{v' \in N_0/t'N_0t'^{-1}} \varphi_t(v')\varphi_{tt'}(f(\psi_{t'}(v'^{-1}x))) =$$
$$= \sum_{v' \in N_0/t'N_0t'^{-1}} \varphi_t(v'\varphi_{t'} \circ \psi_{t'}(v'^{-1}f(x))) = \varphi_t(f(x)) = \widetilde{f}(1 \otimes_t x)$$

noting that $\iota_{t,tt'}$ is a $\Lambda(N_0)$-homomorphism. Here the notation $\otimes_t$ indicates that the tensor product is via the map φ_t. By construction $\widetilde{f}$ is a homomorphism of étale T_*-modules over $\Lambda(N_0)$ satisfying $\widetilde{f} \circ \iota = f$.

The injectivity of ι in case the ψ-action on D is nondegenerate follows from Lemmata 2.18 and 2.19. $\square$ $\square$

Example 2.22. *If D itself is étale then we have $\widetilde{D} = D$.*

Corollary 2.23. *The functor $D \mapsto \widetilde{D}$ from the category of $\Lambda(N_0)$-modules with a ψ-action of T_* to the category of étale T_*-modules over $\Lambda(N_0)$ is exact.*

Proof. $\Lambda(N_0)$ is a free $\varphi_t(\Lambda(N_0))$-module, so $\Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} -$ is exact, and so is the direct limit functor. □ □

Corollary 2.24. *Assume that D is a $\Lambda(N_0)$-module with a nondegenerate ψ-action of T_* and $f\colon D \to D'$ is an injective ψ-equivariant $\Lambda(N_0)$-homomor-phism into the étale T_*-module D' over $\Lambda(N_0)$. Then $\widetilde{f}$ is also injective.*

Proof. Since D is nondegenerate we may identify φ_t^*D with a $\Lambda(N_0)$-submodule of $\widetilde{D}$. Assume that $x = \sum_{u\in J(N_0/tN_0t^{-1})} u \otimes_t x_u \in \varphi_t^*D$ lies in the kernel of $\widetilde{f}$. Then $x_u = \psi_t(u^{-1}x) \in D \subseteq \varphi_t^*D \subseteq \widetilde{D}$ ($u \in J(N_0/tN_0t^{-1})$) also lies in the kernel of $\widetilde{f}$. However, we have $\widetilde{f}(x_u) = f(x_u)$ showing that $x_u = 0$ for all $u \in J(N_0/tN_0t^{-1})$ as f is injective. □ □

Example 2.25. *Let D be a (classical) irreducible étale (φ, Γ)-module over $k((X))$ and $D_0 \subset D$ a ψ- and Γ-invariant treillis in D. Then we have $\widetilde{D_0} \cong D$ unless D is 1-dimensional and $D_0 = D^\natural$ in which case we have $\widetilde{D_0} = D_0$.*

Proof. If D is 1-dimensional then $D^\natural = D^+$ is an étale (φ, Γ)-module over $k[[X]]$ (Prop. II.5.14 in [3]) therefore it is equal to its étale hull. If $\dim D > 1$ then we have $D^\natural = D^\# \subseteq D_0$ by Cor. II.5.12 and II.5.21 in [3]. By Corollary 2.24 $\widetilde{D^\#} \subseteq \widetilde{D_0}$ injects into D and it is φ- and ψ-invariant. Since $D^\#$ is not φ-invariant (Prop. II.5.14 in [3]) and it is the maximal compact $o[[X]]$-submodule of D on which ψ acts surjectively (Prop. II.4.2 in [3]) we obtain that $\widetilde{D_0}$ is not compact. In particular, its X-divisible part is nonzero therefore equals D as the X-divisible part of $\widetilde{D_0}$ is an étale (φ, Γ)-submodule of the irreducible D. □ □

Proposition 2.26. *The T_+^{-1} action on $D_{SV}(\pi)$ is a surjective nondegenerate ψ-action of T_+.*

Proof. Let $d \in D_{SV}(\pi)$ and $t \in T_+$. Since the action of both t and $\Lambda(N_0)$ on $D_{SV}(\pi)$ comes from that on $\pi^\vee$ we have $t^{-1}\varphi_t(\lambda)d = t^{-1}t\lambda t^{-1}d = \lambda t^{-1}d$, so this is indeed a ψ-action. The surjectivity of each ψ_t follows from the injectivity of the multiplication by t on each $W \in \mathcal{B}_+(\pi)$ and the exactness of $\varinjlim$ and $(\cdot)^\vee$. Finally, if W is in $\mathcal{B}_+(\pi)$ then so is $t^*W := \sum_{u\in J(N_0/tN_0t^{-1})} utW$ for any $t \in T_+$. Take an element $d \in D_{SV}(\pi)$ lying in the kernel of $\psi_t(u^{-1}\cdot)$ for all $u \in J(N_0/tN_0t^{-1})$. Now $D_{SV}(\pi)$ is by definition the direct limit of $W^\vee$ for all $W \in \mathcal{B}_+(\pi)$, so $\psi_t(u^{-1}d) = 0$ means that $t^{-1}u^{-1}d$ vanishes on some $W \in \mathcal{B}_+(\pi)$ (depending a priori on u). Since the set $J(N_0/tN_0t^{-1})$ is finite, we may even choose a common W for all u (taking the intersection and using Lemma 2.2 in [9]). Then the restriction of d to t^*W is zero showing that d is zero in $D_{SV}(\pi)$ therefore the nondegeneracy. Alternatively, the nondegeneracy of the ψ-action also follows from the existence of a ψ-equivariant injective map $D_{SV}(\pi) \hookrightarrow D^0_{SV}(\pi)$ into an étale T_+-module $D^0_{SV}(\pi)$ ([9] Proposition 3.5 and Remark 6.1). □ □

Question 1. *Let $D^{(0)}_{SV}(\pi)$ as in [9]. We have that $D^{(0)}_{SV}(\pi)$ is an étale T_*-module over $\Lambda(N_0)$ ([9] Proposition 3.5) and $f : D_{SV}(\pi) \hookrightarrow D^{(0)}_{SV}(\pi)$ is a ψ-equivariant map ([9] Remark 6.1). By the universal property of the étale hull and Corollary 2.24 $\widetilde{D_{SV}}(\pi)$ also injects into $D^{(0)}_{SV}(\pi)$. Whether or not this injection is always an isomorphism is an open question. In case of the Steinberg representation this is true by Proposition 11 in [12].*

We call the submonoid $T'_* \leq T_* \leq T_+$ cofinal in T_* if for any $t \in T_*$ there exists a $t' \in T'_*$ such that $t \leq t'$. For example $\xi(\mathbb{Z}_p \setminus \{0\})$ is cofinal in T_+.

Corollary 2.27. *Let D be a $\Lambda(N_0)$-module with a ψ-action of T_* and denote by $\widetilde{D}$ (resp. by $\widetilde{D}'$) the étale hull of D for the ψ-action of T_* (resp. of T'_*). Then we have a natural isomorphism $\widetilde{D}' \xrightarrow{\sim} \widetilde{D}$ of étale T'_*-modules over $\Lambda(N_0)$. More precisely, if $f\colon D \to D_1$ is a ψ-equivariant $\Lambda(N_0)$-homomorphism into an étale T'_*-module D_1 then f factors uniquely through $\iota\colon D \to \widetilde{D}$.*

Proof. Since $T'_* \leq T_*$ is cofinal in T_* we have $\varinjlim_{t' \in T'_*} \varphi^*_{t'} D \cong \varinjlim_{t \in T_*} \varphi^*_t D = \widetilde{D}$. □ □

By Corollary 2.27 there exists a homomorphism $\widetilde{\mathrm{pr}} : \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ of étale (φ,Γ)-modules over $\Lambda(N_0)$ such that $\mathrm{pr} = \widetilde{\mathrm{pr}} \circ \iota$. Our main result in this section is the following

Theorem 2.28. *$D^\vee_{\xi,\ell,\infty}(\pi)$ is the pseudocompact completion of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ in the category of étale (φ,Γ)-modules over $\Lambda_\ell(N_0)$, ie. we have*

$$D^\vee_{\xi,\ell,\infty}(\pi) \cong \varprojlim_D D$$

where D runs through the finitely generated étale (φ,Γ)-modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ by a closed submodule. This holds in any *topology on $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ making both the maps $1 \otimes \iota\colon D_{SV}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$, $d \mapsto 1 \otimes \iota(d)$ and $1 \otimes \widetilde{\mathrm{pr}}\colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ continuous.*

Remark. Since the map $\mathrm{pr}\colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ is continuous, there exists such a topology on $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$. For instance we could take either the final topology of the map $D_{SV}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ or the initial topology of the map $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$.

Proof. The homomorphism $\widetilde{\mathrm{pr}}$ factors through the map $1 \otimes \mathrm{id}\colon \widetilde{D_{SV}}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ since $D^\vee_{\xi,\ell,\infty}(\pi)$ is a module over $\Lambda_\ell(N_0)$, so we obtain a homomorphism

$$1 \otimes \widetilde{\mathrm{pr}}\colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$$

of étale (φ,Γ)-modules over $\Lambda_\ell(N_0)$. At first we claim that $1 \otimes \widetilde{\mathrm{pr}}$ has dense image. Let $M \in \mathcal{M}(\pi^{H_0})$ and $W \in \mathcal{B}_+(\pi)$ be arbitrary. Then by Lemma 2.11 the map $\mathrm{pr}_{W,M,k}\colon W^\vee \to M_k^\vee$ is surjective for $k \geq 0$ large enough. This shows that the natural map

$$1 \otimes \mathrm{pr}_{W,M,k}\colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} W^\vee \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} M_k^\vee \cong M_k^\vee[1/X]$$

is surjective. However, $1 \otimes \mathrm{pr}_{W,M,k}$ factors through $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi)$ by the Remarks after Lemma 2.12. In particular, the natural map

$$1 \otimes \mathrm{pr}_{M,k}\colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \to M_k^\vee[1/X]$$

is surjective for all $M \in \mathcal{M}(\pi^{H_0})$ and $k \geq 0$ large enough (whence in fact for all $k \geq 0$). This shows that the image of the map

$$1 \otimes \mathrm{pr}\colon \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$$

is dense whence so is the image of $1 \otimes \widetilde{\mathrm{pr}}$. By the assumption that $1 \otimes \widetilde{\mathrm{pr}}$ is continuous we obtain a surjective homomorphism

$$\widehat{1 \otimes \widetilde{\mathrm{pr}}} \colon \varprojlim_{D} D \to D^{\vee}_{\xi,\ell,\infty}(\pi)$$

of pseudocompact (φ, Γ)-modules over $\Lambda_\ell(N_0)$ where D runs through the finitely generated étale (φ, Γ)-modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$.

Let $0 \neq (x_D)_D$ be in the kernel of $\widehat{1 \otimes \widetilde{\mathrm{pr}}}$. Then there exists a finitely generated étale (φ, Γ)-module D over $\Lambda_\ell(N_0)$ with a surjective continuous homomorphism $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \twoheadrightarrow D$ such that $x_D \neq 0$. By Proposition 2.14 this map factors through $D^{\vee}_{\xi,\ell,\infty}(\pi)$ contradicting to the assumption $\widehat{1 \otimes \widetilde{\mathrm{pr}}}((x_D)_D) = 0$. □ □

Remark. Breuil's functor $D^{\vee}_{\xi}$ can therefore be computed from D_{SV} the following way: For a smooth o/ϖ^h-representation π we have $D^{\vee}_{\xi}(\pi) \cong (\varprojlim_D D)_{H_0} \cong \varprojlim_D D_{H_0}$ where D runs through the finitely generated étale (φ, Γ)-modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ by a closed submodule.

3 Nongeneric ℓ

Assume from now on that $\ell = \ell_\alpha$ is a nongeneric Whittaker functional defined by the projection of N_0 onto $N_{\alpha,0} \cong \mathbb{Z}_p$ for some simple root $\alpha \in \Delta$.

Remark. In [2] the Whittaker functional ℓ is assumed to be generic. However, even if ℓ is not generic, the functor $D^{\vee}_{\xi}$ (hence also $D^{\vee}_{\xi,\ell,\infty}$) is right exact even though the restriction of $D^{\vee}_{\xi}$ to the category SP_{o/ϖ^h} may not be exact in general.

3.1 Compatibility with parabolic induction

Let $P = L_P N_P$ be a parabolic subgroup of G containing B with Levi component L_P and unipotent radical N_P and let π_P be a smooth o/ϖ^h-representation of L_P that we view as a representation of P^- via the quotient map $P^- \twoheadrightarrow L_P$ where $P^- = L_P N_{P^-}$ is the parabolic subgroup opposite to P. Since T is contained in L_P, we may consider the same cocharacter $\xi \colon \mathbb{Q}_p^\times \to T$ for the group L_P instead of G. Further, we put $N_{L_P} := N \cap L_P$ and $N_{L_P,0} := N_0 \cap L_P$.

As in [2] denote by $W := N_G(T)/T$ (resp. by $W_P := (N_G(T) \cap L_P)/T$) the Weyl group of G (resp. of L_P) and by $w_0 \in W$ the element of maximal length. We have a canonical system

$$K_P := \{w \in W \mid w^{-1}(\Phi^+_P) \subseteq \Phi^+\}$$

of representatives (the Kostant representatives) of the right cosets $W_P \backslash W$ where Φ^+_P denotes the set of positivie roots of L_P with respect to the Borel subgroup $L_P \cap B$. We have a generalized Bruhat decomposition

$$G = \coprod_{w \in K_P} P^- w B = \coprod_{w \in K_P} P^- w N \ .$$

Now let π_P be a smooth representation of L_P over A. We regard π_P as a representation of P^- via the quotient map $P^- \twoheadrightarrow L_P$. Then the parabolically induced representation $\mathrm{Ind}_{P^-}^G \pi_P$ admits [11] (see also [6] §4.3) a filtration by B-subrepresentations whose graded pieces are contained in

$$\mathcal{C}_w(\pi_P) := c - \mathrm{Ind}_{P^-}^{P^-wN} \pi_P$$

for $w \in K_P$ where $c - \mathrm{Ind}_{P^-}^*$ stands for the space of locally constant functions on $* \supseteq P^-$ with compact support modulo P^-. B acts on $\mathcal{C}_w(\pi_P)$ by right translations. Moreover, the first graded piece equals $\mathcal{C}_1(\pi_P)$.

Lemma 3.1. *Let $\pi' \leq \mathcal{C}_w(\pi_P)$ be any B-subrepresentation for some $w \in K_P \setminus \{1\}$. Then we have $D_\xi^\vee(\pi') = 0$.*

Proof. By the right exactness of $D_\xi^\vee$ (Prop. 2.7(*ii*) in [2]) it suffices to treat the case $\pi' = \mathcal{C}_w(\pi_P)$. For this the same argument works as in Prop. 6.2 [2] with the following modification:

The particular shape of ℓ is only used in Lemma 6.5 in [2] (note that the subgroup $H_0 = \mathrm{Ker}(\ell\colon N_0 \to \mathbb{Z}_p)$ is denoted by N_1 therein). For an element $w \neq 1$ in the Weyl group we have $(w^{-1}N_{P^-}w \cap N_0)\backslash N_0/H_0 = \{1\}$ if and only if H_0 does not contain $w^{-1}N_{P^-}w \cap N_0$. Whenever $w^{-1}N_{P^-}w \cap N_0 \not\subseteq H_0$, the statement of Lemma 6.5 in [2] is true and there is nothing to prove.

In case we have $\{1\} \neq w^{-1}N_{P^-}w \cap N_0 \subseteq H_0$, the statement of Lemma 6.5 is not true for $\ell = \ell_\alpha$. However, the argument using it in the proof of Prop. 6.2 can be replaced by the following: the operator F acts on the space $\mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{H_0}$ nilpotently. Indeed, the trace map $\mathrm{Tr}_{H_0/sH_0s^{-1}}$

$$\mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{sH_0s^{-1}} \to \mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{H_0}$$

is zero as each double coset $(w^{-1}N_{P^-}w \cap H_0)\backslash H_0/sH_0s^{-1}$ has size divisible by p and any function in $\mathcal{C}((w^{-1}N_{P^-}w \cap N_0)\backslash N_0, \pi_P^w)^{sH_0s^{-1}}$ is constant on these double cosets. The statement follows from Prop. 2.7(*iii*) in [2]. □ □

In order to extend Thm. 6.1 in [2] (the compatibility with parabolic induction) to our situation ($\ell = \ell_\alpha$) we need to distinguish two cases: whether the root subgroup N_α is contained in L_P or in N_P. Similarly to [6] we define the $s^{\mathbb{Z}}N_{L_P}$-ordinary part $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$ of a smooth representation π_P of L_P as follows. We equip $\pi_P^{N_{L_P,0}}$ with the Hecke action $F_P := \mathrm{Tr}_{N_{L_P,0}/sN_{L_P,0}s^{-1}} \circ (s\cdot)$ of s making $\pi_P^{N_{L_P,0}}$ a module over the polynomial ring $o/\varpi^h[F_P]$ and put

$$\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P) := \mathrm{Hom}_{o/\varpi^h[F_p]}(o/\varpi^h[F_P, F_P^{-1}], \pi_P^{N_{L_P,0}})_{F_P-fin}$$

where $F_P - fin$ stands for those elements in the Hom-space whose orbit under the action of F_P is finite. By Lemmata 3.1.5 and 3.1.6 in [6] we may identify $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$ with an $o/\varpi^h[F_P]$-submodule in $\pi_P^{N_{L_P,0}}$ by sending a map $f \in \mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$ to its value $f(1) \in \pi_P^{N_{L_P,0}}$ at $1 \in o/\varpi^h[F_P, F_P^{-1}]$.

Proposition 3.2. *Let π_P be a smooth locally admissible representation of L_P over A which we view by inflation as a representation of P^-. We have an isomorphism*

$$D_\xi^\vee\left(\mathrm{Ind}_{P^-}^G \pi_P\right) \cong \begin{cases} D_\xi^\vee(\pi_P) & \text{if } N_\alpha \subseteq L_P \\ o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^\vee & \text{if } N_\alpha \subseteq N_P \end{cases}$$

as étale (φ,Γ)-modules. In particular, for $P=B$ we have $D^\vee_\xi(\mathrm{Ind}_{B^-}\pi_B)\cong o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}\pi_B^\vee$, ie. the value of $D^\vee_\xi$ at the principal series is the same (φ,Γ)-module of rank 1 regardless of the choice of ℓ (generic or not).

Proof. By Lemma 3.1 and the right exactness of $D^\vee_\xi$ (Prop. 2.7(*ii*) in [2]) it suffices to show that $D^\vee_\xi(\mathcal{C}_1(\pi_P))\cong D^\vee_\xi(\pi_P)$. Moreover, the proof of Prop. 6.7 in [2] goes through without modification so we have an isomorphism $D^\vee_\xi(\mathcal{C}_1(\pi_P))\cong D^\vee((\mathrm{Ind}^{N_0}_{P^-\cap N_0}\pi_P)^{H_0})$. Hence we are reduced to computing $D^\vee((\mathrm{Ind}^{N_0}_{P^-\cap N_0}\pi_P)^{H_0})$ in terms of π_P. We further have an identification

$$\mathrm{Ind}^{N_0}_{P^-\cap N_0}\pi_P\cong\mathcal{C}(N_{P,0},\pi_P)\cong\mathcal{C}(N_{P,0},o/\varpi^h)\otimes_{o/\varpi^h}\pi_P$$

by equation (40) in [2]. We need to distinguish two cases.

Case 1: $N_\alpha\subseteq L_P$. In this case we have $N_{P,0}\subseteq H_0$. Hence we deduce $(\mathcal{C}(N_{P,0},o/\varpi^h)\otimes_{o/\varpi^h}\pi_P)^{H_0}=\pi_P^{H_0/N_{P,0}}=\pi_P^{H_{P,0}}$. So we have

$$D^\vee_\xi\left(\mathrm{Ind}^G_{P^-}\pi_P\right)\cong D^\vee(\mathrm{Ind}^{N_0}_{P^-\cap N_0}\pi_P)^{H_0})\cong D^\vee(\pi_P^{H_{P,0}})\cong D^\vee_\xi(\pi_P)$$

in this case as claimed.

Case 2: $N_\alpha\subseteq N_P$. In this case we have $N_{L_P,0}\subseteq H_0$ and $N_{P,0}/(N_{P,0}\cap H_0)\cong\mathbb{Z}_p$. So we have an identification

$$\mathcal{C}(N_{P,0},\pi_P)^{H_0}\cong\mathcal{C}(N_{P,0}/(N_{P,0}\cap H_0),\pi_P^{N_{L_P,0}})\cong\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}})\ .$$

Here the Hecke action $F=F_G=\mathrm{Tr}_{H_0/sH_0s^{-1}}\circ(s\cdot)$ of s on the right hand side is given by the formula

$$F_G(f)(a)=\begin{cases}F_P(f(a/p)) & \text{if } a\in p\mathbb{Z}_p\\ 0 & \text{if } a\in\mathbb{Z}_p\setminus p\mathbb{Z}_p\end{cases},$$

where $F_P=\mathrm{Tr}_{N_{L_P,0}/sN_{L_P,0}s^{-1}}\circ(s\cdot)$ denotes the Hecke action of s on $\pi_P^{N_{L_P,0}}$.

Now let M be a finitely generated $o/\varpi^h[[X]][F]$ submodule of $\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}})$ that is stable under the action of Γ and is admissible as a representation of $\mathbb{Z}_p$. By possibly passing to a finite index submodule of M we may assume without loss of generality that the natural map $M^\vee\to M^\vee[1/X]$ is injective whence the map $\mathrm{id}\otimes F\colon o/\varpi^h[[X]]\otimes_{o/\varpi^h[[X]],F}M\to M$ is surjective. Let $f\in M$ be arbitrary. By continuity of f there exists an integer $n\geq 0$ such that f is constant on the cosets of $p^n\mathbb{Z}_p$. Writing $f=\sum_{i=0}^{p^n-1}[i]\cdot F^n(f_i)$ (where $[i]\cdot$ denotes the multiplication by the group element $i\in\mathbb{Z}_p$) by the surjectivity of $\mathrm{id}\otimes F$ we find that each f_i is necessarily constant as a function on $\mathbb{Z}_p$ satisfying $F_P^n(f_0(0))=f(0)$. Put $M_*:=\{f(0)\mid f\in M\}\subseteq\pi_P^{N_{L_P,0}}$. By the previous discussion F_P acts surjectively on M_* and is generated by the values of elements in $M^{\mathbb{Z}_p}$ (ie. constant functions) as a module over $A[F_P]$. By the admissibility of M we deduce that $M^{\mathbb{Z}_p}$ hence M_* is finite (or, equivalently, finitely generated over o/ϖ^h). We deduce that in fact we have $M=\mathcal{C}(\mathbb{Z}_p,M_*)$, ie. $M^\vee\cong o/\varpi^h[[X]]\otimes_{o/\varpi^h}M_*^\vee$. Conversely, whenever we have a $o/\varpi^h[F_P]$-submodule $M'\leq\pi_P^{N_{L_P,0}}$ that is finitely generated over o/ϖ^h and on which F_P acts surjectively (hence bijectively as the cardinality of o/ϖ^h is finite) then for $M:=\mathcal{C}(\mathbb{Z}_p,M')$ we have $M'=M_*$, $M\in\mathcal{M}(\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}}))$, and $M^\vee\cong o/\varpi^h[[X]]\otimes_{o/\varpi^h}(M')^\vee$ is X-torsion free. In particular, we compute

$$\begin{aligned} D_\xi^\vee(\mathcal{C}_1(\pi_P)) &\cong \varprojlim_{M\in\mathcal{M}(\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}}))} M^\vee[1/X] \cong \\ &\cong \varprojlim_{\substack{M\in\mathcal{M}(\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}})),\\ M^\vee\hookrightarrow M^\vee[1/X]}} o/\varpi^h((X))\otimes_{o/\varpi^h} M_*^\vee \cong \\ & o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}(\varinjlim_{\substack{M\in\mathcal{M}(\mathcal{C}(\mathbb{Z}_p,\pi_P^{N_{L_P,0}})),\\ M^\vee\hookrightarrow M^\vee[1/X]}} M_*)^\vee = \\ &= o/\varpi^h((X))\widehat{\otimes}_{o/\varpi^h}\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^\vee \end{aligned}$$

as claimed. $\square$ $\square$

Corollary 3.3. *Assume $L_P\cong \mathrm{GL}_2(\mathbb{Q}_p)\times T'$ where T' is a torus and let $\pi_P\cong\pi_2\otimes_k\chi$ be the twist of a supercuspidal modulo p representation π_2 of $\mathrm{GL}_2(\mathbb{Q}_p)$ by a character χ of the torus. Then we have*

$$\dim_{k((X))} D_\xi^\vee\left(\mathrm{Ind}_{P^-}^G\pi_P\right)=\begin{cases}0 & \text{if } N_\alpha\not\subseteq L_P\\ 2 & \text{if } N_\alpha\subseteq L_P\end{cases} .$$

Proof. Let the superscript $^{(2)}$ denote the analogous construction of the subgroups B,T,N,T_0 and element s of G in case $G=\mathrm{GL}_2(\mathbb{Q}_p)$. Note that the torus $T^{(2)}$ is generated by $s^{(2)}$ and $T_0^{(2)}$. So in this case we have an isomorphism $\mathrm{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)\cong(\mathrm{Ord}_{B^{(2)}}(\pi_2)\otimes\chi)_{|k[F_P]}=0$ by the adjunction formula of Emerton's ordinary parts (Thm. 4.4.6 in [6]). In the other case we apply Thm. 0.10 in [4]. $\square$ $\square$

3.2 The action of T_+

Our goal in this section is to define a φ-action of T_+ on $D_{\xi,\ell,\infty}^\vee(\pi)$ or, equivalently, on $D_\xi^\vee(\pi)$ extending the action of $\xi(\mathbb{Z}_p\setminus\{0\})\leq T_+$ and making $D_{\xi,\ell,\infty}^\vee(\pi)$ an étale T_+-module over $\Lambda_\ell(N_0)$. Let $t\in T_+$ be arbitrary. Note that by the choice of this ℓ we have $tH_0t^{-1}\subseteq H_0$. In particular, T_+ acts via conjugation on the ring $\Lambda(N_0/H_0)\cong o[[X]]$; we denote the action of $t\in T_+$ by φ_t. This action is via the character α mapping T_+ onto $\mathbb{Z}_p\setminus\{0\}$. In particular, $o[[X]]$ is a free module of finite rank over itself via φ_t. Moreover, we define the Hecke action of $t\in T_+$ on π^{H_0} by the formula $F_t(m):=\mathrm{Tr}_{H_0/tH_0t^{-1}}(tm)$ for any $m\in\pi^{H_0}$. For $t,t'\in T_+$ we have

$$\begin{aligned} F_{t'}\circ F_t &= \mathrm{Tr}_{H_0/t'H_0t'^{-1}}\circ(t'\cdot)\circ\mathrm{Tr}_{H_0/tH_0t^{-1}}\circ(t\cdot)= \\ &= \mathrm{Tr}_{H_0/t'H_0t'^{-1}}\circ\mathrm{Tr}_{t'H_0t'^{-1}/t'tH_0t^{-1}t'^{-1}}\circ(t't\cdot)=F_{t't} . \end{aligned}$$

For any $M\in\mathcal{M}(\pi^{H_0})$ we put $F_t^*M:=N_0F_t(M)$.

Lemma 3.4. *For any $M\in\mathcal{M}(\pi^{H_0})$ we have $F_t^*M\in\mathcal{M}(\pi^{H_0})$.*

Proof. We have

$$\begin{aligned} F(F_t^*M) &= F(N_0F_t(M))\subset N_0FF_t(M)= \\ &= N_0F_{st}(M)=N_0F_t(F(M))\subseteq F_t^*M . \end{aligned}$$

So F_t^*M is a module over $\Lambda(N_0/H_0)/\varpi^h[F]$. Moreover, if $m_1,\dots m_r$ generates M, then the elements $F_t(m_i)$ $(1\leq i\leq r)$ generate F_t^*M, so it is finitely generated. The admissibility is clear as $F_t^*M=\sum_{u\in J(N_0/tN_0t^{-1})}uF_t(M)$ is the sum of finitely many admissible submodules. Finally, F_t^*M is stable under the action of Γ as F_t commutes with the action of Γ. □ □

By the definition of F_t^*M we have a surjective $o/\varpi^h[[X]]$-homomorphism

$$1\otimes F_t\colon o/\varpi^h[[X]]\otimes_{o/\varpi^h[[X]],\varphi_t}M\twoheadrightarrow F_t^*M$$

which gives rise to an injective $o/\varpi^h((X))$-homomorphism

$$(1\otimes F_t)^\vee[1/X]\colon (F_t^*M)^\vee[1/X]\hookrightarrow o/\varpi^h((X))\otimes_{o/\varpi^h((X)),\varphi_t}M^\vee[1/X]\ . \tag{13}$$

Moreover, there is a structure of an $o/\varpi^h[[X]][F]$-module on

$$o/\varpi^h[[X]]\otimes_{o/\varpi^h[[X]],\varphi_t}M$$

by putting $F(\lambda\otimes m):=\varphi_t(\lambda)\otimes F(m)$. Similarly, the group Γ also acts on $o/\varpi^h[[X]]\otimes_{o/\varpi^h[[X]],\varphi_t}M$ semilinearly. The map $1\otimes F_t$ is F and Γ-equivariant as F_t, F, and the action of Γ all commute. We deduce that $(1\otimes F_t)^\vee[1/X]$ is a φ- and Γ-equivariant map of étlae (φ,Γ)-modules.

Note that for any $t\in T_+$ there exists a positive integer $k\geq 0$ such that $t\leq s^k$, ie. $t':=t^{-1}s^k$ lies in T_+. So we have $F_t^*(F_{t'}^*M)=F_{s^k}^*M=N_0F^k(M)\subseteq M$. So we obtain an isomorphism $M^\vee[1/X]\cong(F_{s^k}^*M)^\vee[1/X]=(F_t^*(F_{t'}^*M))^\vee[1/X]$ as $M/N_0F^k(M)$ is finitely generated over o.

Lemma 3.5. *The map* (13) *is an isomorphism of étale (φ,Γ)-modules for any $M\in\mathcal{M}(\pi^{H_0})$ and $t\in T_+$.*

Proof. The composite $(1\otimes F_{t'})^\vee[1/X]\circ(1\otimes F_t)^\vee[1/X]=(1\otimes F^k)^\vee[1/X]$ is an isomorphism by Lemma 2.6 in [2]. So $(1\otimes F_t)^\vee[1/X]$ is also an isomorphism as both $(1\otimes F_t)^\vee[1/X]$ and $(1\otimes F_{t'})^\vee[1/X]$ are injective. □ □

Now taking projective limits we obtain an isomorphism of pseudocompact étale (φ,Γ)-modules

$$\begin{aligned}(1\otimes F_t)^\vee[1/X]\colon D_\xi^\vee(\pi)&\to\varprojlim_{M\in\mathcal{M}(\pi^{H_0})}(o/\varpi^h((X))\otimes_{o/\varpi^h((X)),\varphi_t}M^\vee[1/X])\\ (m)_{(F_t^*M)^\vee[1/X]}&\mapsto((1\otimes F_t)^\vee[1/X](m))_{M^\vee[1/X]}\ .\end{aligned}$$

Moreover, since $o((X))$ is finite free over itself via φ_t, we have an identification

$$\begin{aligned}\varprojlim_{M\in\mathcal{M}(\pi^{H_0})}&(o/\varpi^h((X))\otimes_{o/\varpi^h((X)),\varphi_t}M^\vee[1/X])\cong\\ &\cong o/\varpi^h((X))\otimes_{o/\varpi^h((X)),\varphi_t}D_\xi^\vee(\pi)\ .\end{aligned}$$

Using the maps $(1\otimes F_t)^\vee[1/X]$ we define a φ-action of T_+ on $D_\xi^\vee(\pi)$ by putting $\varphi_t(d):=((1\otimes F_t)^\vee[1/X])^{-1}(1\otimes d)$ for $d\in D_\xi^\vee(\pi)$.

Proposition 3.6. *The above action of T_+ extends the action of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$ and makes $D_\xi^\vee(\pi)$ into an étale T_+-module over $o/\varpi^h[[X]]$.*

Proof. By the definition of the T_+-action it is indeed an extension of the action of the monoid $\mathbb{Z}_p \setminus \{0\}$. For $t, t' \in T_+$ we compute

$$\begin{aligned}\varphi_{t'} \circ \varphi_t(d) = ((1 \otimes F_{t'})^\vee[1/X])^{-1} \circ ((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d) = \\ = ((1 \otimes F_t)^\vee[1/X] \circ (1 \otimes F_{t'})^\vee[1/X])^{-1}(1 \otimes d) = \\ = ((1 \otimes F_{tt'})^\vee[1/X])^{-1}(1 \otimes d) = \varphi_{tt'}(d) = \varphi_{t't}(d) \ .\end{aligned}$$

Further, we have

$$\begin{aligned}\varphi_t(\lambda d) = ((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes \lambda d) = ((1 \otimes F_t)^\vee[1/X])^{-1}(\varphi_t(\lambda) \otimes d) = \\ = \varphi_t(\lambda)((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d) = \varphi_t(\lambda)\varphi_t(d)\end{aligned}$$

showing that this is indeed a φ-action of T_+. The étale property follows from the fact that $(1 \otimes F_t)^\vee[1/X]$ is an isomorphism for each $t \in T_+$. □ □

The inclusion $u_\alpha \colon \mathbb{Z}_p \to N_{\alpha,0} \leq N_0$ induces an injective ring homomor-phism—still denoted by u_α by a certain abuse of notation—$u_\alpha \colon \widehat{o((X))}^p \hookrightarrow \Lambda_\ell(N_0)$ where $\widehat{o((X))}^p$ denotes the p-adic completion of the Laurent-series ring $o((X))$. For each $t \in T_+$ this gives rise to a commutative diagram

$$\begin{tikzcd} \widehat{o((X))}^p \arrow[r, "u_\alpha"] \arrow[d, "\varphi_t"'] & \Lambda_\ell(N_0) \arrow[d, "\varphi_t"] \\ \widehat{o((X))}^p \arrow[r, "u_\alpha"] & \Lambda_\ell(N_0) \end{tikzcd}$$

with injective ring homomorphisms. On the other hand, by the equivalence of categories in Thm. 8.20 in [10] we have a φ- and Γ-equivariant identification $M_\infty^\vee[1/X] \cong \Lambda_\ell(N_0) \otimes_{\widehat{o((X))}^p, u_\alpha} M^\vee[1/X]$. Therefore tensoring the isomorphism (13) with $\Lambda_\ell(N_0)$ via u_α we obtain an isomorphism

$$\begin{aligned}(1 \otimes F_t)_\infty^\vee[1/X] \colon (F_t^* M)_\infty^\vee[1/X] \cong \Lambda_\ell(N_0) \otimes_{u_\alpha} (F_t^* M)^\vee[1/X] \to \\ \to \Lambda_\ell(N_0) \otimes_{u_\alpha} o/\varpi^h((X)) \otimes_{o/\varpi^h((X)),\varphi_t} M^\vee[1/X] \cong \\ \cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\varphi_t} \Lambda_\ell(N_0) \otimes_{u_\alpha} M^\vee[1/X] \cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\varphi_t} M_\infty^\vee[1/X] \ . \qquad (14)\end{aligned}$$

Taking projective limits again we deduce an isomorphism

$$\begin{aligned}(1 \otimes F_t)_\infty^\vee[1/X] \colon D_{\xi,\ell,\infty}^\vee(\pi) &\to \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\varphi_t} D_{\xi,\ell,\infty}^\vee(\pi) \\ (m)_{(F_t^* M)_\infty^\vee[1/X]} &\mapsto ((1 \otimes F_t)_\infty^\vee[1/X](m))_{M_\infty^\vee[1/X]}\end{aligned}$$

for all $t \in T_+$ using the identification

$$\varprojlim_{M \in \mathcal{M}(\pi^{H_0})} (\Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\varphi_t} M_\infty^\vee[1/X]) \cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\varphi_t} D_{\xi,\ell,\infty}^\vee(\pi) \ .$$

Using the maps $(1 \otimes F_t)_\infty^\vee[1/X]$ we define a φ-action of T_+ on $D_{\xi,\ell,\infty}^\vee(\pi)$ by putting $\varphi_t(d) := ((1 \otimes F_t)_\infty^\vee[1/X])^{-1}(1 \otimes d)$ for $d \in D_{\xi,\ell,\infty}^\vee(\pi)$.

Corollary 3.7. *The above action of T_+ extends the action of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$ and makes $D^\vee_{\xi,\ell,\infty}(\pi)$ into an étale T_+-module over $\Lambda_\ell(N_0)$. The reduction map $D^\vee_{\xi,\ell,\infty}(\pi) \to D^\vee_\xi(\pi)$ is T_+-equivariant for the φ-action.*

We can view this φ-action of T_+ in a different way: Let us define $F_{t,k} := \mathrm{Tr}_{H_k/tH_kt^{-1}} \circ (t\cdot)$. Then we have a map

$$1 \otimes F_{t,k} \colon \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h,\varphi_t} M_k \to F^*_{t,k} M_k := N_0 F_{t,k}(M_k) \ , \tag{15}$$

where we have $F^*_{t,k}M \in \mathcal{M}_k(\pi^{H_k})$. Let k be large enough such that we have $tH_0t^{-1} \geq H_k$. After taking Pontryagin duals, inverting X, taking projective limit and using the remark after Lemma 2.5 we obtain a homomorphism of étale (φ,Γ)-modules

$$\varprojlim_k \mathrm{Tr}^{-1}_{t^{-1}H_kt} \circ (1 \otimes F_{t,k})^\vee[1/X] \colon (F^*_t M)^\vee_\infty[1/X] \to \Lambda_\ell(N_0) \otimes_{\varphi_t} M^\vee_\infty[1/X] \ . \tag{16}$$

This map is indeed Γ- and φ-equivariant because we compute

$$\begin{aligned} F_k \circ F_{t,k} = \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot) \circ \mathrm{Tr}_{H_k/tH_kt^{-1}} \circ (t\cdot) &= \\ = \mathrm{Tr}_{H_k/s^ktH_kt^{-1}s^{-k}} \circ (s^k t\cdot) &= \\ = \mathrm{Tr}_{H_k/tH_kt^{-1}} \circ (t\cdot) \circ \mathrm{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot) = F_{t,k} \circ F_k \ . \end{aligned}$$

Now we have two maps (14) and (16) between $(F^*_t M)^\vee_\infty[1/X]$ and $\Lambda_\ell(N_0) \otimes_{\varphi_t} M^\vee_\infty[1/X]$ that agree after taking H_0-coinvariants by definition. Hence they are equal by the equivalence of categories in Thm. 8.20 in [10].

We obtain in particular that the map (15) has finite kernel and cokernel as it becomes an isomorphism after taking Pontryagin duals and inverting X. Hence there exists a finite $\Lambda(N_0/H_k)/\varpi^h$-submodule $M_{t,k,*}$ of M_k such that the kernel of $1 \otimes F_{t,k}$ is contained in the image of $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_{t,k,*}$ in $\Lambda(N_0/H_k)/\varpi^h \otimes_\varphi M_k$. We denote by $M^*_{t,k} \leq F^*_{t,k} M_k$ the image of $1 \otimes F_{t,k}$. We conclude that as in Proposition 2.6, we can describe the φ_t-action in the following way:

$$\begin{aligned} \varphi_t \colon M^\vee_k[1/X] &\to (F^*_{t,k} M_k)^\vee[1/X] \\ f &\mapsto (\mathrm{Tr}^{-1}_{t^{-1}H_kt/H_k} \circ (1 \otimes F_{t,k})^\vee[1/X])^{-1}(1 \otimes f) \end{aligned} \tag{17}$$

Being an étale T_+-module over $\Lambda_\ell(N_0)$ we equip $D^\vee_{\xi,\ell,\infty}(\pi)$ with the ψ-action of T_+: ψ_t is the canonical left inverse of φ_t for all $t \in T_+$.

Proposition 3.8. *The map* $\mathrm{pr} \colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ *is ψ-equivariant for the ψ-actions of T_+ on both sides.*

Proof. We proceed as in the proofs of Proposition 2.8 and Lemma 2.12. We fix $t \in T_+$, $W \in \mathcal{B}_+(\pi)$ and $M \in \mathcal{M}(\pi^{H_0})$ and show that $\mathrm{pr}_{W,M}$ is ψ_t-equivariant. Fix k such that $F^*_{t,k} M_k \leq W$ and $tH_0t^{-1} \geq H_k$.

At first we compute the formula analogous to (7). Let f be in $M^\vee_k$ such that its restriction to $M_{t,k,*}$ is zero and $m \in M^*_{t,k} \leq F^*_{t,k} M_k$ be in the form

$$m = \sum_{u \in J(N_0/tN_0t^{-1})} u F_{t,k}(m_u)$$

with elements $m_u \in M_k$ for $u \in J(N_0/tN_0t^{-1})$. $M^*_{t,k}$ is a finite index submodule of $F^*_{t,k}M_k$. Note that the elements m_u are unique upto $M_{t,k,*} + \mathrm{Ker}(F_{t,k})$. Therefore $\varphi_t(f) \in (M^*_{t,k})^\vee$ is well-defined by our assumption that $f_{|M_{t,k,*}} = 0$ noting that the kernel of $F_{t,k}$ equals the kernel of $\mathrm{Tr}_{t^{-1}H_kt/H_k}$ since the multiplication by t is injective and we have $F_{t,k} = t \circ \mathrm{Tr}_{t^{-1}H_kt/H_k}$. So we compute

$$\begin{aligned}\varphi_t(f)(m) = ((1\otimes F_{t,k})^\vee)^{-1}(\mathrm{Tr}_{t^{-1}H_kt/H_k}(1\otimes f))(m) =\\ = ((1\otimes F_{t,k})^\vee)^{-1}(1\otimes \mathrm{Tr}_{t^{-1}H_kt/H_k}(f))(\sum_{u\in J((N_0/H_k)/t(N_0/H_k)t^{-1})} uF_{t,k}(m_u)) =\\ = \mathrm{Tr}_{t^{-1}H_kt/H_k}(f)(F_{t,k}^{-1}(u_0F_{t,k}(m_{u_0}))) = f(\mathrm{Tr}_{t^{-1}H_kt/H_k}((t^{-1}u_0t)m_{u_0}))\end{aligned} \tag{18}$$

where u_0 is the single element in $J(N_0/tN_0t^{-1})$ corresponding to the coset of 1.

Now let f be in $W^\vee$ such that the restriction $f_{|N_0tM_{t,k,*}} = 0$. By definition we have $\psi_t(f)(w) = f(tw)$ for any $w \in W$. Choose an element $m \in M^*_{t,k} \in F^*_{t,k}M_k$ written in the form

$$m = \sum_{u\in J(N_0/tN_0t^{-1})} uF_{t,k}(m_u) = \sum_{u\in J(N_0/tN_0t^{-1})} ut\,\mathrm{Tr}_{t^{-1}H_kt/H_k}(m_u)\ .$$

Then we compute

$$\begin{aligned} f_{|F^*_{t,k}M_k}(m) &= \sum_{u\in J(N_0/tN_0t^{-1})} f(ut\,\mathrm{Tr}_{t^{-1}H_kt/H_k}(m_u)) =\\ &= \sum_{u\in J(N_0/tN_0t^{-1})} \psi_t(u^{-1}f)(\mathrm{Tr}_{t^{-1}H_kt/H_k}(m_u)) =\\ &\overset{(18)}{=} \sum_{u\in J(N_0/tN_0t^{-1})} \varphi_t(\psi_t(u^{-1}f)_{|F^*_{t,k}M_k})(F_{t,k}(m_u)) =\\ &= \sum_{u\in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}f)_{|M_k})(uF_{t,k}(m_u)) =\\ &= \sum_{u\in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}f)_{|M_k})(m)\end{aligned}$$

as for distinct $u, v \in J(N_0/tN_0t^{-1})$ we have $u\varphi_t(f_0)(vF_{t,k}(m_v)) = 0$ for any $f_0 \in (M^*_{t,k})^\vee$. So by inverting X and taking projective limits with respect to k we obtain

$$\mathrm{pr}_{W,F_t^*M}(f) = \sum_{u\in J(N_0/tN_0t^{-1})} u\varphi_t(\mathrm{pr}_{W,M}(\psi_t(u^{-1}f)))$$

as we have $(M^*_{t,k})^\vee[1/X] \cong (F^*_{t,k}M)^\vee[1/X]$. Since the map (14) is an isomorphism we may decompose $\mathrm{pr}_{W,F_t^*M}(f)$ uniquely as

$$\mathrm{pr}_{W,F_t^*M}(f) = \sum_{u\in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}\,\mathrm{pr}_{W,F_t^*M}(f)))$$

so we must have $\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) = \mathrm{pr}_{W,M}(\psi_t(f))$. For general $f \in W^\vee$ note that $N_0sM_{t,k,*}$ is killed by $\varphi_t(X^r)$ for $r \geq 0$ big enough, so we have

$$\begin{aligned} X^r\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) = \psi_t(\mathrm{pr}_{W,F_t^*M}(\varphi_t(X^r)f)) =\\ = \mathrm{pr}_{W,M}(\psi_t(\varphi_t(X^r)f)) = X^r\,\mathrm{pr}_{W,M}(\psi_t(f))\ .\end{aligned}$$

Since X^r is invertible in $\Lambda_\ell(N_0)$, we obtain

$$\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) = \mathrm{pr}_{W,M}(\psi_t(f))$$

for any $f \in W^\vee$. The statement follows taking the projective limit with respect to $M \in \mathcal{M}(\pi^{H_0})$ and the inductive limit with respect to $W \in \mathcal{B}_+(\pi)$. $\square$ $\square$

We end this section by proving a Lemma that will be needed several times later on.

Lemma 3.9. *For any $M \in \mathcal{M}(\pi^{H_0})$ there exists an open subgroup $T' = T'(M) \leq T$ such that M is T'-stable.*

Proof. Choose $m_1, \ldots, m_a \in M$ $(a \geq 1)$ generating M as a module over $o/\varpi^h[[X]][F]$. Since π is smooth, there exists an open subgroup $T' \leq T_0$ stabilizing all $m_1, \ldots, m_a$. Now T' normalizes N_0 and all the elements $t \in T'$ commute with F we deduce that T' acts on M. $\square$ $\square$

4 Compatibility with a reverse functor

Assume $\ell = \ell_\alpha$ for some simple root $\alpha \in \Delta$ so we may apply the results of section 3.

4.1 A G-equivariant sheaf $\mathfrak{Y}$ on G/B attached to $D^\vee_{\xi,\ell,\infty}(\pi)$

Let D be an étale (φ, Γ)-module over the ring $\Lambda_\ell(N_0)/\varpi^h$. Recall that the $\Lambda(N_0)$-submodule D^{bd} of bounded elements in D is defined [10] as

$$D^{bd} = \{x \in D \mid \{\ell_D(\psi_s^k(u^{-1}x)) \mid k \geq 0, u \in N_0\} \subseteq D_{H_0} \text{ is bounded}\} .$$

where ℓ_D denotes the natural map $D \to D_{H_0}$. Note that D_{H_0} is an étale (φ, Γ)-module over $o/\varpi^h((X))$, so the bounded subsets of D_{H_0} are exactly those contained in a compact $o/\varpi^h[[X]]$-submodule of D_{H_0}.

Lemma 4.1. *Assume that D is a finitely generated étale (φ, Γ)-module over $\Lambda_\ell(N_0)/\varpi^h$. Then $d \in D$ lies in D^{bd} if and only if d is contained in a compact ψ_s-invariant $\Lambda(N_0)$-submodule of D.*

Proof. If d is in D^{bd} then it is contained in

$$D^{bd}(D_0) = \{x \in D \mid \ell_D(\psi_s^k(u^{-1}x)) \subseteq D_0\}$$

for some treillis $D_0 \subset D_{H_0}$ where $D^{bd}(D_0)$ is a compact ψ_s-stable $\Lambda(N_0)$-submodule of D by Prop. 9.10 in [10]. On the other hand if $x \in D_1$ for some compact ψ_s-invariant $\Lambda(N_0)$-submodule $D_1 \subset D$ then we have

$$\{\ell_D(\psi_s^k(u^{-1}x)) \mid k \geq 0, u \in N_0\} \subseteq \ell_D(D_1)$$

where $\ell_D(D_1)$ is bounded as D_1 is compact and ℓ_D is continuous. $\square$ $\square$

We call a pseudocompact $\Lambda_\ell(N_0)$-module together with a φ-action of the monoid T_+ (resp. $\mathbb{Z}_p \setminus \{0\}$) a pseudocompact étale T_+-module (resp. (φ,Γ)-module) over $\Lambda_\ell(N_0)$ if it is a topologically étale $o[B_+]$-module in the sense of section 4.1 in [10]. Recall that a pseudocompact module over the pseudocompact ring $\Lambda_\ell(N_0)$ is the projective limit of finitely generated $\Lambda_\ell(N_0)$-modules. As for $D = D^\vee_{\xi,\ell,\infty}(\pi)$ in section 2.1 we equip the pseudocompact $\Lambda_\ell(N_0)$-modules D with the weak topology, ie. with the projective limit topology of the weak topologies of these finitely generated quotients of D. Recall from section 4.1 in [10] that the condition for D to be topologically étale means in this case that the map

$$\begin{array}{rcl} B_+ \times D & \to & D \\ (b,x) & \mapsto & \varphi_b(x) \end{array} \tag{19}$$

is continuous and $\psi = \psi_s \colon D \to D$ is continuous (Lemma 4.1 in [10]).

Lemma 4.2. *$D^\vee_{\xi,\ell,\infty}(\pi)$ is a pseudocompact étale T_+-module over $\Lambda_\ell(N_0)$.*

Proof. At first we show that the map (19) is continuous in the weak topology of $D = D^\vee_{\xi,\ell,\infty}(\pi)$. Let $b = ut \in B_+$ ($u \in N_0$, $t \in T_+$), $x, y \in D^\vee_{\xi,\ell,\infty}(\pi)$ be such that $u\varphi_t(y) = x$ and let $M \in \mathcal{M}(\pi^{H_0})$, $l, l' \geq 0$ be arbitrary. Recall from (9) that the sets

$$O(M,l,l') := f^{-1}_{M,l}(\Lambda(N_0/H_l) \otimes_{u_\alpha} X^{l'} M^\vee[1/X]^{++})$$

form a system of neighbourhoods of 0 in the weak topology of $D^\vee_{\xi,\ell,\infty}(\pi)$. We need to verify that the preimage of $x + O(M,l,l')$ under (19) contains a neighbourhood of (b,y). By Lemma 3.9 there exists an open subgroup $T' \leq T_0 \leq T$ acting on M therefore also on $M_l^\vee[1/X]$ as T_0 normalizes H_l for all $l \geq 0$ by the assumption $\ell = \ell_\alpha$. Moreover, this action is continuous in the weak topology of $M_l^\vee[1/X]$, so there exists an open subgroup $T_1 \leq T'$ such that we have $(T_1 - 1)x \subset O(M,l,l')$. Moreover, since we have $D^\vee_{\xi,\ell,\infty}(\pi)/O(M,l,l') \cong M_l^\vee[1/X]/(\Lambda(N_0/H_l) \otimes_{u_\alpha} X^{l'} M^\vee[1/X]^{++})$ is a smooth representation of N_0, we have an open subgroup $N_1 \leq N_0$ with $(N_1 - 1)x \subset O(M,l,l')$. Moreover, we may assume that T_1 normalizes N_1 so that $B_1 := N_1T_1$ is an open subgroup in $B_0 \leq B_+$ for which we have $(B_1 - 1)x \subset O(M,l,l')$ as $O(M,l,l')$ is N_0-invariant. Choose an elemet $t' \in T_+$ such that $tt' = s^r$ for some $r \geq 0$. Note that the composite map $D^\vee_{\xi,\ell,\infty}(\pi) \overset{\varphi_t}{\to} D^\vee_{\xi,\ell,\infty} \twoheadrightarrow M^\vee[1/X]$ factors through the φ_s-equivariant map

$$((1 \otimes F_t)^\vee[1/X])^{-1} \colon (F^*_{t'}M)^\vee[1/X] \to M^\vee[1/X]$$

mapping $X^{l'}(F^*_{t'}M)^\vee[1/X]^{++}$ into $X^{l'}M^\vee[1/X]^{++}$. Since $X^{l'}M^\vee[1/X]^{++}$ is B_1-invariant (as each φ_{t_1} for $t_1 \in T_1$ commutes with φ_s), so is $O(M,l,l')$. We deduce that

$$B_1 b \times (y + O(F^*_{t'}M,l,l')) \subset B_+ \times D^\vee_{\xi,\ell,\infty}(\pi)$$

maps into $x + O(M,l,l')$ via (19).

The continuity of ψ_s follows from Proposition 8.22 in [10] since ψ_s on $D^\vee_{\xi,\ell,\infty}(\pi)$ is the projective limit of the maps $\psi_s \colon M^\vee_\infty[1/X] \to M^\vee_\infty[1/X]$ for $M \in \mathcal{M}(\pi^{H_0})$. $\square$ $\square$

In view of the above Lemmata we define D^{bd} for a pseudocompact étale (φ,Γ)-module D over $\Lambda_\ell(N_0)$ as

$$D^{bd} = \bigcup_{D_c \in \mathfrak{C}_0(D)} D_c$$

where we denote the set of ψ_s-invariant compact $\Lambda(N_0)$-submodules $D_c \subset D$ by $\mathfrak{C}_0 = \mathfrak{C}_0(D)$.

The following is a generalization of Prop. 9.5 in [10].

Proposition 4.3. *Let D be a pseudocompact étale (φ, Γ)-module over $\Lambda_\ell(N_0)$. Then D^{bd} is an étale (φ, Γ)-module over $\Lambda(N_0)$. If we assume in addition that D is an étale T_+-module over $\Lambda_\ell(N_0)$ (for a φ-action of the monoid T_+ extending that of $\xi(\mathbb{Z}_p \setminus \{0\})$) then D^{bd} is an étale T_+-module over $\Lambda(N_0)$ (with respect to the action of T_+ restricted from D).*

Proof. We prove the second statement assuming that D is an étale T_+-module. The first statement follows easily the same way.

At first note that D^{bd} is ψ_t-invariant for all $t \in T_+$ as for $D_c \in \mathfrak{C}_0$ we also have $\psi_t(D_c) \in \mathfrak{C}_0$. So it suffices to show that it is also stable under the φ-action of T_+ since these two actions are clearly compatible (as they are compatible on D). At first we show that we have $\varphi_s(D^{bd}) \subset D^{bd}$. Let $D_c \in \mathfrak{C}_0$ be arbitrary. Then the ψ-action of the monoid $p^{\mathbb{Z}}$ (ie. the action of ψ_s) is nondegenerate on D_c as D_c is a ψ_s-invariant submodule of an étale module D. So by the remark after Proposition 2.21 and by Corollary 2.24 we obtain an injective ψ_s and φ_s-equivariant homomorphism $i\colon \widetilde{D}_c \hookrightarrow D$. However, each $\varphi^*_{s^k} D_c \subseteq \widetilde{D}_c$ is compact and ψ-equivariant therefore the image of $\widetilde{D}_c$ is contained in D^{bd} showing that $\varphi_s(D_c) \subset N_0\varphi_s(D_c) = i(\varphi^*_s D_c) \subseteq D^{bd}$. However, for each $t \in T_+$ there exists a $t' \in T_+$ with $tt' = s^k$ for some $k \geq 0$, so $\varphi_t(D_c) = \psi_{t'}(\varphi_{s^k}(D_c)) \subseteq D^{bd}$ showing that D^{bd} is φ_t-invariant for all $t \in T_+$. □ □

Corollary 4.4. *The image of the map $\widetilde{\mathrm{pr}}\colon \widetilde{D_{SV}}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ is contained in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$.*

Proof. By Propositions 2.21 and 4.3 it suffices to show that the image of $\mathrm{pr}\colon D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ lies in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$. However, this is clear since $\mathrm{pr}(D_{SV}(\pi))$ is a ψ_s-invariant compact $\Lambda(N_0)$-submodule of $D^\vee_{\xi,\ell,\infty}(\pi)$. □ □

Let $\mathfrak{C}$ be the set of all compact subsets C of $D^\vee_{\xi,\ell,\infty}(\pi)$ contained in one of the compact subsets $D_c \in \mathfrak{C}_0 = \mathfrak{C}_0(D^\vee_{\xi,\ell,\infty}(\pi))$. Recall from Definition 6.1 in [10] that the family $\mathfrak{C}$ is said to be special if it satisfies the following axioms:

$\mathfrak{C}(1)$ Any compact subset of a compact set in $\mathfrak{C}$ also lies in $\mathfrak{C}$.

$\mathfrak{C}(2)$ If $C_1, C_2, \ldots, C_n \in \mathfrak{C}$ then $\bigcup_{i=1}^n C_i$ is in $\mathfrak{C}$, as well.

$\mathfrak{C}(3)$ For all $C \in \mathfrak{C}$ we have $N_0C \in \mathfrak{C}$.

$\mathfrak{C}(4)$ $D(\mathfrak{C}) := \bigcup_{C\in\mathfrak{C}} C$ is an étale T_+-submodule of D.

Lemma 4.5. *The set $\mathfrak{C}$ is a special family of compact sets in $D^\vee_{\xi,\ell,\infty}(\pi)$ in the sense of Definition 6.1 in [10].*

Proof. $\mathfrak{C}(1)$ is satisfied by construction. So is $\mathfrak{C}(3)$ by noting that any $C \in \mathfrak{C}$ is contained in a $D_c \in \mathfrak{C}_0$ which is N_0-stable. For $\mathfrak{C}(2)$ note that for any $D_{c,1}, \ldots, D_{c,r} \in \mathfrak{C}_0$ we have $\sum_{i=1}^r D_{c,i} \in \mathfrak{C}_0$. Finally, $\mathfrak{C}(4)$ is just Proposition 4.3. □ □

Our next goal is to construct a G-equivariant sheaf $\mathfrak{Y} = \mathfrak{Y}_{\alpha,\pi}$ on G/B in [10] with sections $\mathfrak{Y}(\mathcal{C}_0)$ on $\mathcal{C}_0 := N_0w_0B/B$ isomorphic to $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ as a B_+-module. Here $w_0 \in N_G(T)$ is a representative of an element in the Weyl group $N_G(T)/C_G(T)$ of *maximal length.* For this we identify $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ with the global sections of a B_+-equivariant sheaf on N_0 as in [10]. The restriction maps $\mathrm{res}^{N_0}_{us^kN_0s^{-k}}$ are defined as $u \circ \varphi^k_s \circ \psi^k_s \circ u^{-1}$. The open sets $us^kN_0s^{-k}$ form a basis of the topology on N_0, so it suffices to give these restriction maps. Indeed, any open compact subset $\mathcal{U} \subseteq N_0$ is the disjoint union of cosets of the form $us^kN_0s^{-k}$ for $k \geq k'(\mathcal{U})$ large enough. For a fixed $k \geq k'(\mathcal{U})$ we put

$$\mathrm{res}_{\mathcal{U}} = \mathrm{res}_{\mathcal{U}}^{N_0} := \sum_{u \in J(N_0/s^kN_0s^{-k}) \cap \mathcal{U}} u\varphi_{s^k} \circ \psi_s^k \circ (u^{-1}\cdot) \ .$$

This is independent of the choice of $k \geq k'(\mathcal{U})$ by Prop. 3.16 in [10]. Note that the map

$$u \mapsto x_u := uw_0B/B \in \mathcal{C}_0$$

is a B_+-equivariant homeomorphism from N_0 to $\mathcal{C}_0$ therefore we may view $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ as the global sections of a sheaf on $\mathcal{C}_0$. For an open subset $U \subseteq N_0$ we denote the image of U by $x_U \subseteq \mathcal{C}_0$ under the above map $u \mapsto x_u$. Moreover, we regard res as an $\mathrm{End}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi))$-valued measure on $\mathcal{C}_0$, ie. a ring homomorphism $\mathrm{res} : C^\infty(\mathcal{C}_0, o) \to \mathrm{End}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi))$. We restrict res to a map $\mathrm{res}\colon C^\infty(\mathcal{C}_0, o) \to \mathrm{Hom}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}, D^\vee_{\xi,\ell,\infty}(\pi))$. Put $\mathcal{C} := Nw_0B/B \supset \mathcal{C}_0$. By the discussion in section 5 of [10] in order to construct a G-equivariant sheaf on G/B with the required properties we need to integrate the map

$$\begin{aligned} \alpha_g\colon \mathcal{C}_0 &\to \mathrm{Hom}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}, D^\vee_{\xi,\ell,\infty}(\pi)) \\ x_u &\mapsto \alpha(g,u) \circ \mathrm{res}(1_{\alpha(g,u)^{-1}\mathcal{C}_0 \cap \mathcal{C}_0}) \end{aligned}$$

with respect to the measure res where for $x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0 \subset g^{-1}\mathcal{C} \cap \mathcal{C}$ we take $\alpha(g,u)$ to be the unique element in B with the property

$$guw_0N = \alpha(g,u)uw_0N \ .$$

Note that since x_u lies in $g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ we also have $x_u \in \alpha(g,u)^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ so the latter set is nonempty and open in G/B. Recall from section 6.1 in [10] that a map $F\colon \mathcal{C}_0 \to \mathrm{Hom}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}, D^\vee_{\xi,\ell,\infty}(\pi))$ is called integrable with respect to (s, res, $\mathfrak{C}$) if the limit

$$\int_{\mathcal{C}_0} F d\,\mathrm{res} := \lim_{k\to\infty} \sum_{u \in J(N_0/s^kNs^{-k})} F(x_u) \circ \mathrm{res}(1_{x_{us^kNs^{-k}}})$$

exists in $\mathrm{Hom}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}, D^\vee_{\xi,\ell,\infty}(\pi))$ and does not depend on the choice of the sets of representatives $J(N_0/s^kNs^{-k})$.

Proposition 4.6. *The map α_g is $(s, \mathrm{res}, \mathfrak{C})$-integrable for any $g \in G$.*

Proof. By Proposition 6.8 in [10] it suffices to show that $\mathfrak{C}$ satisfies:

$\mathfrak{C}(5)$ For any $C \in \mathfrak{C}$ the compact subset $\psi_s(C) \subseteq D^\vee_{\xi,\ell,\infty}(\pi)$ also lies in $\mathfrak{C}$.

$\mathfrak{T}(1)$ For any $C \in \mathfrak{C}$ such that $C = N_0C$, any open $o[N_0]$-submodule $\mathcal{D}$ of $D^\vee_{\xi,\ell,\infty}(\pi)$, and any compact subset $C_+ \subseteq T_+$ there exists a compact open subgroup $B_1 = B_1(C, \mathcal{D}, C_+) \subseteq B_0$ and an integer $k(C, \mathcal{D}, C_+) \geq 0$ such that

$$\varphi_s^k \circ (1 - B_1)C_+\psi_s^k(C) \subseteq \mathcal{D} \qquad \text{for any } k \geq k(C, \mathcal{D}, C_+) \ .$$

Here the multiplication by C_+ is via the φ-action of T_+ on $D^\vee_{\xi,\ell,\infty}(\pi)$.

The condition $\mathfrak{C}(5)$ is clearly satisfied as for any $D_c \in \mathfrak{C}_0$ we have $\psi_s(D_c) \in \mathfrak{C}_0$, as well. For the condition $\mathfrak{T}(1)$ choose a $C \in \mathfrak{C}$ with $C = N_0 C$, a compact subset $C_+ \subset T_+$, and an open $o[N_0]$-submodule $\mathcal{D} \subseteq D^\vee_{\xi,\ell,\infty}(\pi)$. As $D^\vee_{\xi,\ell,\infty}(\pi)$ is the topological projective limit $\varprojlim_{M\in\mathcal{M}(\pi^{H_0}),n\geq 0} M_n^\vee[1/X]$ we may assume without loss of generality that $\mathcal{D}$ is the preimage of a compact $\Lambda(N_0)$-submodule $D_n \leq M_n^\vee[1/X]$ with $D_n[1/X] = M_n^\vee[1/X]$ under the natural surjective map $f_{M,n}\colon D^\vee_{\xi,\ell,\infty}(\pi) \twoheadrightarrow M_n^\vee[1/X]$ for some $M \in \mathcal{M}(\pi^{H_0})$ and $n \geq 0$. Moreover, since $B_0 = T_0 N_0$ is compact and normailizes H_0, the T_0-orbit of any element $m \in M \leq \pi^{H_0}$ is finite and contained in π^{H_0}. Therefore we also have $B_0 M = T_0 M \in \mathcal{M}(\pi^{H_0})$. So we may assume without loss of generality that M is B_0-invariant whence we have an action of B_0 on $M_n^\vee[1/X]$. Choose a $D_c \in \mathfrak{C}_0$ with $C \subseteq D_c$. Since D_c is ψ_s-invariant, we have $C_+\psi_s^k(C) \subseteq C_+\psi_s^k(D_c) \subseteq C_+ D_c$. Moreover, $C_+ D_c$ is compact as both C_+ and D_c are compact, so $f_{M,n}(C_+\psi_s^k(C)) \subset M_n^\vee[1/X]$ is bounded. In particular, we have a compact $\Lambda(N_0)$-submodule D' of $M_n^\vee[1/X]$ containing $f_{M,n}(C_+\psi_s^k(C))$. So by the continuity of the action of B_0 on $M_n^\vee[1/X]$ there exists an open subgroup $B_1 \leq B_0$ such that we have

$$\begin{aligned}(1-B_1)f_{M,n}(C_+\psi_s^k(C)) \subset \Lambda(N_0/H_n)\otimes_{\Lambda(N_{\alpha,0})}(M^\vee[1/X]^{++}) \leq \\ \leq \Lambda(N_0/H_n)\otimes_{\Lambda(N_{\alpha,0})} M^\vee[1/X] \cong M_n^\vee[1/X]\end{aligned}$$

for any $k \geq 0$. Here $M^\vee[1/X]^{++}$ denotes the treillis in $M^\vee[1/X]$ consisting of those elements $d \in M^\vee[1/X]$ such that $\varphi_s^n(d) \to 0$ in $M^\vee[1/X]$ as $n \to \infty$ (cf. section I.3.2 in [4]). Finally, since D_n is open and $M^\vee[1/X]^{++}$ is finitely generated over $\Lambda(N_{\alpha,0}) \cong o[[X]]$ there exists an integer $k_1 \geq 0$ such that $\varphi_s^k(\Lambda(N_0/H_n)\otimes_{\Lambda(N_{\alpha,0})}(M^\vee[1/X]^{++}))$ is contained in D_n for all $k \geq k_1$. In particular, we have

$$\begin{aligned}f_{M,n}(\varphi_s^k\circ(1-B_1)C_+\psi_s^k(C)) = \varphi_s^k\circ(1-B_1)(f_{M,n}(C_+\psi_s^k(C))) \subseteq \\ \subseteq \varphi_s^k\circ(1-B_1)(M^\vee[1/X]^{++}) \subseteq D_n\end{aligned}$$

showing that $\varphi_s^k\circ(1-B_1)C_+\psi_s^k(C)$ is contained in $\mathcal{D}$. □ □

For all $g \in G$ we denote by $\mathcal{H}_g \in \mathrm{Hom}_o^{cont}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}, D^\vee_{\xi,\ell,\infty}(\pi))$ the integral

$$\mathcal{H}_g := \int_{\mathcal{C}_0}\alpha_g d\,\mathrm{res} = \lim_{k\to\infty}\sum_{u\in J(N_0/s^kNs^{-k})}\alpha_g(x_u)u\circ\varphi_s^k\circ\psi_s^k\circ u^{-1}$$

we have just proven to converge. We denote the kth term of the above sequence by

$$\mathcal{H}_g^{(k)} = \mathcal{H}_{g,J(N_0/s^kN_0s^{-k})} := \sum_{u\in J(N_0/s^kNs^{-k})}\alpha_g(x_u)u\circ\varphi_s^k\circ\psi_s^k\circ u^{-1}\ . \tag{20}$$

Our main result in this section is the following

Proposition 4.7. *The image of the map $\mathcal{H}_g\colon D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \to D^\vee_{\xi,\ell,\infty}(\pi)$ is contained in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$. There exists a G-equivariant sheaf $\mathfrak{Y} = \mathfrak{Y}_{\alpha,\pi}$ on G/B with sections $\mathfrak{Y}(\mathcal{C}_0)$ on $\mathcal{C}_0$ isomorphic B_+-equivariantly to $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ such that we have $\mathcal{H}_g = \mathrm{res}^{G/B}_{\mathcal{C}_0}\circ(g\cdot)\circ\mathrm{res}^{G/B}_{\mathcal{C}_0}$ as maps on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd} = \mathfrak{Y}(\mathcal{C}_0)$.*

Proof. By Prop. 5.14 and 6.9 in [10] it suffices to check the following conditions:

$\mathfrak{C}(6)$ For any $C \in \mathfrak{C}$ the compact subset $\varphi_s(C) \subseteq M$ also lies in $\mathfrak{C}$.

$\mathfrak{T}(2)$ Given a set $J(N_0/s^k N_0 s^{-k}) \subset N_0$ of representatives for all $k \geq 1$, for any $x \in D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ and $g \in G$ there exists a compact ψ_s-invariant $\Lambda(N_0)$-submodule $D_{x,g} \in \mathfrak{C}$ and a positive integer $k_{x,g}$ such that $\mathcal{H}_g^{(k)}(x) \subseteq D_{x,g}$ for any $k \geq k_{x,g}$.

The condition $\mathfrak{C}(6)$ follows from (the proof of) Prop. 4.3 as for $C \subseteq D_c \in \mathfrak{C}_0$ we have $\varphi_s(C) \subseteq \varphi_s(D_c) \subseteq i(\varphi_s^* D_c) \in \mathfrak{C}_0$.

The proof of $\mathfrak{T}(2)$ is very similar to the proof of Corollary 9.15 in [10]. However, it is not a direct consequence of that as $D^\vee_{\xi,\ell,\infty}(\pi)$ is not necessarily finitely generated over $\Lambda_\ell(N_0)$, so we recall the details. For any x in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$, the element $\mathcal{H}_g^{(k)}(x)$ also lies in $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ for any fixed k since the set of bounded elements form an étale T_+-submodule (by axiom $\mathfrak{C}(4)$) whence they are closed under the operations (φ-, ψ-, and N_0-actions) defining the map $\mathcal{H}_g^{(k)}$. So by axiom $\mathfrak{C}(2)$ we only need to show that for k large enough the difference

$$s_g^{(k)}(x) := \mathcal{H}_g^{(k)}(x) - \mathcal{H}_g^{(k+1)}(x)$$

lies in a compact submodule $D_{x,g} \leq D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ in $\mathfrak{C}_0$ independent of k. In order to do so we proceed in four steps. In steps 1, 2, and 3 the goal is to show that for a fixed choice $M \in \mathcal{M}(\pi^{H_0})$ the image of $s_g^{(k)}(x)$ lies in a compact ψ-invariant $\Lambda(N_0)$-submodule of $M_\infty^\vee[1/X]$ under the projection map $D^\vee_{\xi,\ell,\infty}(\pi) \twoheadrightarrow M_\infty^\vee[1/X]$ for k large enough *not depending* on M. This compact submodule of $M_\infty^\vee[1/X]$ will be of the form

$$\{m \in M_\infty^\vee[1/X] \mid \ell_M(\psi_s^r(u^{-1}m)) \text{ is in } D_0 \text{ for all } r \geq 0, u \in N_0\}$$

for some treillis $D_0 \subset M^\vee[1/X]$ where $\ell_M \colon M_\infty^\vee[1/X] \to M^\vee[1/X]$ is the natural projection map. Step 1 is devoted to showing this for smaller r (compared to k) with some choice of a treillis and in Step 2 we take care of all larger r (using a different treillis in $M^\vee[1/X]$). In both of these steps $k \geq k(M)$ is large enough *depending* on M. In Step 3 we eliminate this dependence on M of the lower bound for k by choosing a third treillis so that the sum D_0 of these three different choices of a treillis will do. In Step 4 we take the projective limit of these compact sets for all possible choices of M to obtain a compact subset of $D^\vee_{\xi,\ell,\infty}(\pi)$.

Step 1. Equation (43) in [10] shows that for any compact open subgroup $B_1 \leq B_0$ there exist integers $0 \leq k_g^{(1)} \leq k_g^{(2)}(B_1)$ and a compact subset $\Lambda_g \subset T_+$ such that for $k \geq k_g^{(2)}(B_1)$ we have

$$s_g^{(k)} \in \langle N_0 s^{k-k_g^{(1)}}(1-B_1)\Lambda_g s \psi_s^{k+1} N_0\rangle_o \ , \tag{21}$$

where we denote by $\langle\cdot\rangle_o$ the generated o-submodule. Here $k_g^{(1)}$ is chosen so that $\{\alpha(g,u)us^{k_g^{(1)}} \mid x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0\}$ is contained in $B_+ = N_0T_+$. There exists such an integer $k_g^{(1)}$ since $\{\alpha(g,u)u \mid x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0\}$ is a compact subset in N_0T. Choose a compact ψ_s-invariant $\Lambda(N_0)$-submodule $D_c \in \mathfrak{C}_0$ containing the element $x \in D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ and pick an M in $\mathcal{M}(\pi^{H_0})$. Applying $\mathfrak{T}(1)$ in the situation $C = D_c$, $C_+ = \Lambda_g s$, and $\mathcal{D} = f_{M,0}^{-1}(M^\vee[1/X]^{++})$ we find an integer $k_1 \geq 0$ and a compact open subgroup $B_1 \leq B_0$ such that $\varphi_s^k \circ (1-B_1)\Lambda_g s D_c \subseteq \mathcal{D}$ for all $k \geq k_1$. Noting that D_c is ψ_s-stable and $\mathcal{D}$ is a $\Lambda(N_0)$-submodule we obtain $s_g^{(k)}(D_c) \subseteq N_0\varphi_s^r(\mathcal{D})$ for $k \geq r + k_1 + k_g^{(2)}(B_1)$. Applying ψ_s^r to this using (21) and putting $k_g(M) := k_1 + k_g^{(2)}(B_1)$ we deduce

$$\psi_s^r(\Lambda(N_0)s_g^{(k)}(D_c)) \subseteq \mathcal{D} \qquad \text{for all } k \geq k_g(M) \text{ and } r \leq k - k_g(M) \ . \tag{22}$$

Note that the subgroup B_1 depends on M therefore so do $k_g^{(2)}(B_1)$ and $k_g(M)$, but not $k_g^{(1)}$.

Step 2. We are going to find another treillis $D_1 \leq M^\vee[1/X]$ such that for all $k \geq k_g(M)$ and $r \geq k - k_g(M)$ we have

$$\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)) \subseteq \mathcal{D}_1 := f_{M,0}^{-1}(D_1) \ . \tag{23}$$

For $x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ write $\alpha(g,u)u$ in the form $\alpha(g,u)u = n(g,u)t(g,u)$ with $n(g,u) \in N_0$ and $t(g,u) \in T$. Since $g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ is compact, $t(g,\cdot)$ is continuous, and $k_g(M) \geq k_g^{(1)}$ the set $C'_+ := \{t(g,u)s^{k_g(M)} \mid x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0\} \subset T$ is compact and contained in T_+. So we compute

$$\begin{aligned}\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)) &= \\ = \psi_s^r(\Lambda(N_0) \sum_{u \in J(N_0/s^k N_0 s^{-k})} n(g,u)\varphi_{t(g,u)s^k} \circ \psi_s^k(u^{-1}D_c)) &\subseteq \\ \subseteq \psi_s^r(\Lambda(N_0)\varphi_s^{k-k_g(M)} \circ \varphi_{t(g,u)s^{k_g(M)}}(D_c)) \subseteq \psi_s^{r-k+k_g(M)}(\Lambda(N_0)C'_+(D_c)) \ . \end{aligned}$$

Since $C'_+ \subset T_+$ is compact , there exists an integer $k(C'_+)$ such that $s^k t^{-1}$ lies in T_+ for all $t \in C'_+$. So we have $C'_+(D_c) \subseteq i(\varphi^*_{s^{k(C'_+)}} D^\vee_{\xi,\ell,\infty}(\pi)^{bd}) \in \mathfrak{C}_0$ showing that

$$D_1 := f_{M,0}(i(\varphi^*_{s^{k(C'_+)}} D^\vee_{\xi,\ell,\infty}(\pi)^{bd}))$$

is a good choice as $i(\varphi^*_{s^{k(C'_+)}} D^\vee_{\xi,\ell,\infty}(\pi)^{bd})$ is a ψ_s-stable $\Lambda(N_0)$ submodule.

Step 3. For each fixed $k \geq k_g^{(1)}$ there exists a compact ψ_s-invariant $\Lambda(N_0)$-submodule $D_{c,k} \in \mathfrak{C}_0$ containing $\mathcal{H}_g^{(k)}(D_c)$. In particular, we may choose a treillis $D_2 \leq M^\vee[1/X]$ containing

$$f_{M,0}(\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)))$$

for all $k_g^{(1)} \leq k \leq k_g(M)$ and $r \geq 0$. Putting $\mathcal{D}_2 := f_{M,0}^{-1}(D_2)$ and combining this with (22) and (23) we obtain

$$\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)) \subseteq \mathcal{D} + \mathcal{D}_1 + \mathcal{D}_2 \tag{24}$$

for all $k \geq k_{x,g} := k_g^{(1)}$ and $r \geq 0$. Denote by $f_{M,\infty}$ the natural surjective map $f_{M,\infty}\colon D^\vee_{\xi,\ell,\infty} \to M^\vee_\infty[1/X]$. Note that $f_{M,0}$ factors through $f_{M,\infty}$. The equation (24) implies (in fact, is equivalent to) that

$$f_{M,\infty}\left(\bigcup_{k \geq k_{x,g}} \mathcal{H}_g^{(k)}(D_c)\right) \subseteq M^\vee_\infty[1/X]^{bd}(D_0)$$

where

$$\begin{aligned} M^\vee_\infty[1/X]^{bd}(D_0) = \{m \in M^\vee_\infty[1/X] \mid \ell_M(\psi_s^r(u^{-1}m)) \text{ is in} \\ D_0 := M^\vee[1/X]^{++} + D_1 + D_2 \text{ for all } r \geq 0, u \in N_0\} \end{aligned}$$

is a compact ψ_s-invariant $\Lambda(N_0)$-submodule in $M^\vee_\infty[1/X]$ (Prop. 9.10 in [10]).

Step 4. We put $D_{x,g}(M) := \bigcap \mathfrak{D}$ where $\mathfrak{D}$ runs through all the ψ_s-invariant compact $\Lambda(N_0)$-submodules of $M^\vee_\infty[1/X]$ containing $f_{M,\infty}(\bigcup_{k \geq k_{x,g}} \mathcal{H}_g^{(k)}(D_c))$. Therefore

$$D_{x,g} := \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} D_{x,g}(M)$$

is a ψ_s-invariant compact $\Lambda(N_0)$-submodule of $D^\vee_{\xi,\ell,\infty}(\pi)$ (ie. we have $D_{x,g}\in\mathfrak{C}_0$) containing $\bigcup_{k\geq k_{x,g}}\mathcal{H}_g^{(k)}(D_c)$. □ □

We end this section by putting a natural topology (called the weak topology) on the global sections $\mathfrak{Y}(G/B)$ that will be needed in the next section. At first we equip $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ with the inductive limit topology of the compact topologies of each $D_c\in\mathfrak{C}_0$. This makes sense as the inclusion maps $D_c\hookrightarrow D'_c$ for $D_c\subseteq D'_c\in\mathfrak{C}_0$ are continuous as these compact topologies are obtained as the subspace topologies in the weak topology of $D^\vee_{\xi,\ell,\infty}(\pi)$. We call this topology the weak topology on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$.

Lemma 4.8. *The operators $\mathcal{H}_g$ and $\mathrm{res}_{\mathcal{U}}$ on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ are continuous in the weak topology of $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ for all $g\in G$ and $\mathcal{U}\subseteq N_0$ compact open. In particular, $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ is the topological direct sum of $\mathrm{res}_{\mathcal{U}}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd})$ and $\mathrm{res}_{N_0\setminus\mathcal{U}}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd})$.*

Proof. By the property $\mathfrak{T}(2)$ the restriction of $\mathcal{H}_g^{(k)}$ to a compact subset D_c in $\mathfrak{C}_0$ has image in a compact set $D_{c,g}\in\mathfrak{C}_0$ for all large enough k. Moreover, each $\mathcal{H}_g^{(k)}$ is continuous by Lemma 4.2. On the other hand, the limit $\mathcal{H}_g=\lim_{k\to\infty}\mathcal{H}_g^{(k)}$ is uniform on each compact subset $D_c\in\mathfrak{C}_0$ by Proposition 6.3 in [10], so the limit $\mathcal{H}_g\colon D_c\to D_{c,g}$ is also continuous. Taking the inductive limit on both sides we deduce that $\mathcal{H}_g\colon D^\vee_{\xi,\ell,\infty}(\pi)\to D^\vee_{\xi,\ell,\infty}(\pi)$ is also continuous. The continuity of $\mathrm{res}_{\mathcal{U}}$ follows in a similar but easier way. □ □

So far we have put a topology on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}=\mathfrak{Y}(\mathcal{C}_0)$. The multiplication by an element $g\in G$ gives an o-linear bijection $g\colon\mathfrak{Y}(\mathcal{C}_0)\to\mathfrak{Y}(g\mathcal{C}_0)$. We define the weak topology on $\mathfrak{Y}(g\mathcal{C}_0)$ so that this is a homeomorphism. Now we equip $\mathfrak{Y}(G/B)$ with the coarsest topology such that the restriction maps $\mathrm{res}^{G/B}_{g\mathcal{C}_0}\colon\mathfrak{Y}(G/B)\to\mathfrak{Y}(g\mathcal{C}_0)$ are continuous for all $g\in G$. We call this the weak topology on $\mathfrak{Y}(G/B)$ making $\mathfrak{Y}(G/B)$ a linear-topological o-module.

Lemma 4.9. *a) The multiplication by g on $\mathfrak{Y}(G/B)$ is continuous (in fact a homeomorphism) for each $g\in G$.*

b) The weak topology on $\mathfrak{Y}(G/B)$ is Hausdorff.

Proof. For a) we need to check that the composite of the function

$$(g\cdot)_{G/B}\colon\mathfrak{Y}(G/B)\to\mathfrak{Y}(G/B)$$

with the projections $\mathrm{res}^{G/B}_{h\mathcal{C}_0}$ is continuous for all $h\in G$. However, $\mathrm{res}^{G/B}_{h\mathcal{C}_0}\circ(g\cdot)_{G/B}=(g\cdot)_{g^{-1}h\mathcal{C}_0}\circ\mathrm{res}^{G/B}_{g^{-1}h\mathcal{C}_0}$ is the composite of two continuous maps hence also continuous.

For b) note that the weak topology on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ is finer than the subspace topology inherited from $D^\vee_{\xi,\ell,\infty}(\pi)$ therefore it is Hausdorff. To see this we need to show that the inclusion $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}\hookrightarrow D^\vee_{\xi,\ell,\infty}(\pi)$ is continuous. As the weak topology on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}$ is defined as a direct limit, it suffices to check this on the defining compact sets $D_c\in\mathfrak{C}_0$. However, on these compact sets the inclusion map is even a homeomorphism by definition.

So the topology on $\mathfrak{Y}(G/B)$ is also Hausdorff as for any two different global sections $x\neq y\in\mathfrak{Y}(G/B)$ there exists an element $g\in G$ such that $\mathrm{res}^{G/B}_{g\mathcal{C}_0}(x)\neq\mathrm{res}^{G/B}_{g\mathcal{C}_0}(y)$. □ □

4.2 A G-equivariant map $\pi^\vee \to \mathfrak{Y}(G/B)$

Here we generalize Thm. IV.4.7 in [4] to $\mathbb{Q}_p$-split reductive groups G over $\mathbb{Q}_p$ with connected centre. Assume in this section that π is an *admissible* smooth o/ϖ^h-representation of G of *finite length.*

By Corollary 4.4 we have the composite maps

$$\beta_{g\mathcal{C}_0}\colon \pi^\vee \xrightarrow{g^{-1}\cdot} \pi^\vee \xrightarrow{\mathrm{pr}_{SV}} D_{SV}(\pi) \xrightarrow{\mathrm{pr}} D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \xrightarrow{\sim} \mathfrak{Y}(\mathcal{C}_0) \xrightarrow{g\cdot} \mathfrak{Y}(g\mathcal{C}_0)$$

for each $g \in G$. By definition we have $\beta_{g\mathcal{C}_0}(\mu) = g\beta_{\mathcal{C}_0}(g^{-1}\mu)$ for all $\mu \in \pi^\vee$ and $g \in G$. Our goal is to show that these maps glue together to a G-equivariant map $\beta_{G/B}\colon \pi^\vee \to \mathfrak{Y}(G/B)$.

Let $n_0 = n_0(G) \in \mathbb{N}$ be the maximum of the degrees of the algebraic characters $\beta\circ\xi\colon \mathbb{G}_m \to \mathbb{G}_m$ for all β in Φ^+ and put $U^{(k)} := \mathrm{Ker}(G_0 \twoheadrightarrow G(\mathbb{Z}_p/p^k\mathbb{Z}_p))$ where $G_0 = \mathbf{G}(\mathbb{Z}_p)$.

Lemma 4.10. *For any fixed $r_0 \geq 1$ we have $t^{-1}U^{(k)}t \leq U^{(k-n_0r_0)}$ for all $t \leq s^{r_0}$ in T_+ and $k \geq r_0n_0$.*

Proof. The condition $t \leq s^{r_0}$ implies that $v_p(\beta(t)) \leq v_p(\beta(s^{r_0})) = v_p(\beta\circ\xi(p^{r_0})) \leq r_0n_0$ for all $\beta \in \Phi^+$ by the maximality of n_0. On the other hand, by the Iwahori factorization we have $U^{(k)} = (U^{(k)} \cap \overline{N})(U^{(k)} \cap T)(U^{(k)} \cap N)$. Since t is in T_+ we deduce

$$\begin{aligned}
t^{-1}(U^{(k)} \cap \overline{N})t \leq (U^{(k)} \cap \overline{N}) &\leq (U^{(k-r_0n_0)} \cap \overline{N})\\
t^{-1}(U^{(k)} \cap T)t = (U^{(k)} \cap T) &\leq (U^{(k-r_0n_0)} \cap T)\\
t^{-1}(U^{(k)} \cap N)t = &\\
\prod_{\beta\in\Phi^+} t^{-1}(U^{(k)} \cap N_\beta)t &\leq \prod_{\beta\in\Phi^+} (U^{(k-r_0n_0)} \cap N_\beta)\\
&= (U^{(k-r_0n_0)} \cap N)\ . \quad \square
\end{aligned}$$

$\square$

Lemma 4.11. *Assume that π is an admissible representation of G of finite length. Then there exists a finitely generated o-submodule $W_0 \leq \pi$ such that $\pi = BW_0$.*

Proof. Since π has finite length, by induction we may assume it is irreducible (hence killed by ϖ). In this case we may take $W_0 = \pi^{U^{(1)}}$ which is G_0-stable as $U^{(1)}$ is normal in G_0. It is nonzero since π is smooth, and finitely generated over o as π is admissible. By the Iwasawa decomposition we have $\pi = GW_0 = BG_0W_0 = BW_0$. $\square$ $\square$

Let W_0 be as in Lemma 4.11 and put $W := B_+W_0$, $W_r := \bigcup_{t\leq s^r} N_0tW_0$ so we have

$$W = \varinjlim_r W_r = \bigcup_{r\geq 0} W_r \tag{25}$$

where W_r is finitely generated over o for all $r \geq 0$. By construction W is a generating B_+-subrepresentation of π. So the map pr_{SV} factors through the natural projection map $\mathrm{pr}_W\colon \pi^\vee \to W^\vee$. Here the Pontryagin dual $W^\vee$ is a compact $\Lambda(N_0)$-module with a ψ-action of T_+ coming from the multiplication by T_+ on W. By Proposition 2.21 we may form the étale hull $\widetilde{W^\vee}$ of $W^\vee$ which is an étale T_+-module over $\Lambda(N_0)$. Since $D^\vee_{\xi,\ell,\infty}(\pi)$ is an étale

T_+-module over $\Lambda(N_0)$ and the composite map $W^\vee \to D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)$ is ψ-equivariant, it factors through $\widetilde{W^\vee}$. All in all we have factored the map $\mathrm{pr} \circ \mathrm{pr}_{SV}$ as

$$\mathrm{pr} \circ \mathrm{pr}_{SV} \colon \pi^\vee \xrightarrow{\widetilde{\mathrm{pr}_W}} \widetilde{W^\vee} \xrightarrow{\mathrm{pr}_D^{\widetilde{W^\vee}}} D^\vee_{\xi,\ell,\infty}(\pi)\ .$$

The advantage of considering $\widetilde{W^\vee}$ is that the operators $\mathcal{H}_g^{(k)}$ make sense as maps $\widetilde{W^\vee} \to \widetilde{W^\vee}$ and the map $\widetilde{W^\vee} \to D^\vee_{\xi,\ell,\infty}(\pi)$ is $\mathcal{H}_g^{(k)}$-equivariant as it is a morphism of étale T_+-modules over $\Lambda(N_0)$. More precisely, let g be in G and put $\mathcal{U}_g := \{u \in N_0 \mid x_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0\}$, $\mathcal{U}_g^{(k)} := J(N_0/s^kN_0s^{-k}) \cap \mathcal{U}_g$. For any $u \in \mathcal{U}_g$ we write gu in the form $gu = n(g,u)t(g,u)\overline{n}(g,u)$ for some unique $n(g,u) \in N_0$, $t(g,u) \in T$, $\overline{n}(g,u) \in \overline{N}$.

Lemma 4.12. *There exists an integer $k_0 = k_0(g)$ such that for all $k \geq k_0$ and $u \in \mathcal{U}_g$ we have $us^kN_0s^{-k} \subseteq \mathcal{U}_g$, $s^kt(g,u) \in T_+$, and $s^{-k}\overline{n}(g,u)s^k \in \overline{N}_0 = G_0 \cap \overline{N}$. In particular, for any set $J(N_0/s^kN_0s^{-k})$ of representatives of the cosets in $N_0/s^kN_0s^{-k}$ we have $\mathcal{U}_g = \bigcup_{u \in \mathcal{U}_g^{(k)}} us^kN_0s^{-k}$.*

Proof. Since $\mathcal{U}_g$ is compact and open in N_0, it is a union of finitely many cosets of the form $us^kN_0s^{-k}$ for k large enough. Moreover, the maps $t(g,\cdot)$ and $\overline{n}(g,\cdot)$ are continuous in the p-adic topology. So the image of $t(g,\cdot)$ is contained in finitely many cosets of T/T_0 as T_0 is open. For the statement regarding $\overline{n}(g,u)$ note that we have $\overline{N} = \bigcup_{k \geq 0} s^k\overline{N}_0s^{-k}$. □ □

For $k \geq k_0 = k_0(g)$ let $J(N_0/s^kN_0s^{-k}) \subset N_0$ be an arbitrary set of representatives of $N_0/s^kN_0s^{-k}$. Recall from the proof of Prop. 4.7 Step 2 (see also [10]) that for fixed $g \in G$ and all $u \in N_0$ we may write $\alpha_g(x_u)u$ in the form $n(g,u)t(g,U)$ for some $n(g,u) \in N_0$ and $t(g,U) \in s^{-k_0}T_+$. In particular the equation (20) defining $\mathcal{H}_g^{(k)}$ reads

$$\mathcal{H}_g^{(k)} = \mathcal{H}_{g,J(N_0/s^kN_0s^{-k})} := \sum_{u \in \mathcal{U}_g^{(k)}} n(g,u)\varphi_{t(g,u)s^k} \circ \psi_s^k \circ (u^{-1}\cdot)$$

where $t(g,u)s^k$ lies in T_+. Further, any open compact subset $\mathcal{U} \subseteq N_0$ is the disjoint union of cosets of the form $us^kN_0s^{-k}$ for $k \geq k'(\mathcal{U})$ large enough. For a fixed $k \geq k'(\mathcal{U})$ we put

$$\mathrm{res}_{\mathcal{U}} := \sum_{u \in J(N_0/s^kN_0s^{-k}) \cap \mathcal{U}} u\varphi_{s^k} \circ \psi_s^k \circ (u^{-1}\cdot)\ .$$

The operators $\mathcal{H}_g^{(k)}$ and $\mathrm{res}_{\mathcal{U}}$ make sense in any étale T_+-module over $\Lambda(N_0)$, in particular also in $\widetilde{W^\vee}$ and $D^\vee_{\xi,\ell,\infty}(\pi)$. Moreover, $\mathrm{res}_{\mathcal{U}}$ is independent of the choice of $k \geq k'(\mathcal{U})$. Further, any morphism between étale T_+-modules over $\Lambda(N_0)$ is $\mathcal{H}_g^{(k)}$- and $\mathrm{res}_{\mathcal{U}}$-equivariant.

Lemma 4.13. *Let g be in G, u be in $\mathcal{U}_g$, and $k \geq k_0 + 1$ be an integer. Then the map*

$$n(g,\cdot)\colon us^kN_0s^{-k} \to n(g,u)t(g,u)s^kN_0s^{-k}t(g,u)^{-1} \tag{26}$$

is a bijection. In particular, for any set $J(N_0/s^kN_0s^{-k})$ of representatives of the cosets in $N_0/s^kN_0s^{-k}$ the set $\mathcal{U}_{g^{-1}}$ is the disjoint union of the cosets $n(g,u)t(g,u)s^kN_0s^{-k}t(g,u)^{-1}$ for $u \in \mathcal{U}_g^{(k)}$.

Proof. By our assumption $k \geq k_0+1$, $s^{-k}\overline{n}(g,u)s^k$ lies in $s^{-1}\overline{N}_0 s \subseteq U^{(1)}$. So for any $v \in N_0$ we have $s^{-k}\overline{n}(g,u)s^k v = vv_1$ for some v_1 in $v^{-1}U^{(1)}v = U^{(1)}$. Further, by the Iwahori factorization we have $U^{(1)} = (N \cap U^{(1)})(T \cap U^{(1)})(\overline{N} \cap U^{(1)})$. So we obtain that $s^{-k}\overline{n}(g,u)s^k v w_0 B \subset \mathcal{C}_0$ for all $v \in N_0$, whence we deduce $s^{-k}\overline{n}(g,u)s^k\mathcal{C}_0 \subseteq \mathcal{C}_0$. Similarly we have $s^{-k}\overline{n}(g,u)^{-1}s^k\mathcal{C}_0 \subseteq \mathcal{C}_0$ showing that in fact $s^{-k}\overline{n}(g,u)s^k\mathcal{C}_0 = \mathcal{C}_0$. We compute

$$\begin{aligned} g(us^kN_0s^{-k})w_0B = gus^kN_0w_0B &= n(g,u)t(g,u)s^k(s^{-k}\overline{n}(g,u)s^k)\mathcal{C}_0 = \\ &= n(g,u)t(g,u)s^k\mathcal{C}_0 = n(g,u)(t(g,u)s^kN_0s^{-k}t(g,u)^{-1})w_0B \ . \end{aligned}$$

Since the map $n(g,\cdot)$ is induced by the multiplication by g on $g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ (identified with $\mathcal{U}_g$), we deduce that the map (26) is a bijection. The second statement follows as $n(g,\cdot)\colon \mathcal{U}_g \to \mathcal{U}_{g^{-1}}$ is a bijection and we have a partition of $\mathcal{U}_g$ into cosets $us^kN_0s^{-k}$ for $u \in \mathcal{U}_g^{(k)}$ by Lemma 4.12. □ □

Lemma 4.14. *Let M be arbitrary in $\mathcal{M}(\pi^{H_0})$ and $l, l' \geq 0$ be integers. There exists an integer $k_1 = k_1(M, W_0, l, l') \geq 0$ such that for all $r \geq k_1$ the image of the natural composite map*

$$(W/W_r)^\vee \hookrightarrow W^\vee \to D^\vee_{\xi,\ell,\infty}(\pi) \overset{f_{M,l}}{\twoheadrightarrow} M_l^\vee[1/X]$$

lies in $\Lambda(N_0/H_l) \otimes_{u_\alpha} X^{l'} M^\vee[1/X]^{++} \subset \Lambda(N_0/H_l) \otimes_{u_\alpha} M^\vee[1/X] \cong M_l^\vee[1/X]$. Here $M^\vee[1/X]^{++}$ denotes the $o/\varpi^h[[X]]$-submodule of $M^\vee[1/X]$ consisting of elements $d \in M^\vee[1/X]$ with $\varphi_s^n(d) \to 0$ as $n \to \infty$.

Proof. By (25) the $\Lambda(N_0)$-submodules $(W/W_r)^\vee$ form a system of neighbourhoods of 0 in $W^\vee$. On the other hand, $X^{l'}M^\vee[1/X]^{++}$ being a treillis in $M^\vee[1/X]$ (Prop. II.2.2 in [3]), $\Lambda(N_0/H_l) \otimes_{u_\alpha} X^{l'}M^\vee[1/X]^{++}$ is open in the weak topology of $M_l^\vee[1/X]$. Therefore its preimage in $W^\vee$ contains $(W/W_r)^\vee$ for r large enough. □ □

Since $t(g,\cdot)$ is continuous and $\mathcal{U}_g$ is compact, there exists an integer $c \geq 0$ such that for all $u \in \mathcal{U}_g$ there is an element $t'(g,u) \in T_+$ such that $t(g,u)s^{k_0}t'(g,u) = s^c$.

Lemma 4.15. *For any fixed $M \in \mathcal{M}(\pi^{H_0})$ there are finitely many different values of $F^*_{t'(g,u)}M$ where $g \in G$ is fixed and u runs on $\mathcal{U}_g$.*

Proof. By Lemma 3.9 there exists an open subgroup $T' \leq T$ acting on M. In particular, $F^*_{t'(g,u)}M$ only depends on the coset $t'(g,u)T'$. Now $t'(g,\cdot) = s^{c-k_0}t(g,\cdot)^{-1}$ is continuous and $\mathcal{U}_g$ is compact therefore there are only finitely many cosets of the form $t'(g,u)T'$. □ □

Our key proposition is the following:

Proposition 4.16. *For all $g \in G$ we have $\mathrm{res}^{\mathcal{C}_0}_{g\mathcal{C}_0\cap\mathcal{C}_0} \circ \beta_{\mathcal{C}_0} = \mathrm{res}^{g\mathcal{C}_0}_{g\mathcal{C}_0\cap\mathcal{C}_0} \circ \beta_{g\mathcal{C}_0}$.*

Proof. Note that since G/B is totally disconnected in the p-adic topology, in particular $g\mathcal{C}_0 \cap \mathcal{C}_0$ is both open and closed in $\mathcal{C}_0$, we have $\mathfrak{Y}(\mathcal{C}_0) = \mathfrak{Y}(g\mathcal{C}_0 \cap \mathcal{C}_0) \oplus \mathfrak{Y}(\mathcal{C}_0 \setminus g\mathcal{C}_0)$. By Prop. 4.7 $\mathcal{H}_g$ is the composite map

$$D^\vee_{\xi,\ell,\infty}(\pi)^{bd} = \mathfrak{Y}(\mathcal{C}_0) \overset{g\cdot}{\to} \mathfrak{Y}(g\mathcal{C}_0) \overset{\mathrm{res}^{g\mathcal{C}_0}_{g\mathcal{C}_0\cap\mathcal{C}_0}}{\twoheadrightarrow} \mathfrak{Y}(g\mathcal{C}_0 \cap \mathcal{C}_0) \hookrightarrow \mathfrak{Y}(\mathcal{C}_0) = D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \ ,$$

ie. we obtain $\mathrm{res}^{g\mathcal{C}_0}_{g\mathcal{C}_0\cap\mathcal{C}_0}\circ(g\cdot)=\mathcal{H}_g$ as maps on $\mathfrak{Y}(\mathcal{C}_0)$ once we identify $\mathfrak{Y}(g\mathcal{C}_0\cap\mathcal{C}_0)$ with a subspace in $\mathfrak{Y}(\mathcal{C}_0)$ via the above direct sum decomposition. On the other hand, by definition $\beta_{\mathcal{C}_0}=\mathrm{pr}\circ\mathrm{pr}_{SV}\colon\pi^\vee\to D^\vee_{\xi,\ell,\infty}(\pi)^{bd}\subset D^\vee_{\xi,\ell,\infty}(\pi)$ is the natural map and we have $\beta_{g\mathcal{C}_0}(\mu)=g\beta_{\mathcal{C}_0}(g^{-1}\mu)$ for any $g\in G$ and $\mu\in\pi^\vee$. Further, as maps on $D^\vee_{\xi,\ell,\infty}(\pi)^{bd}=\mathfrak{Y}(\mathcal{C}_0)$ we have $\mathrm{res}_{\mathcal{U}_{g^{-1}}}=\mathrm{res}^{\mathcal{C}_0}_{g\mathcal{C}_0\cap\mathcal{C}_0}$. Putting these together our equation to show reads

$$\mathrm{res}_{\mathcal{U}_{g^{-1}}}\circ\mathrm{pr}\circ\mathrm{pr}_{SV}(\mu)=\mathcal{H}_g(\mathrm{pr}\circ\mathrm{pr}_{SV}(g^{-1}\mu))\ .$$

We want to write $\mathcal{H}_g$ as the limit of the maps $\mathcal{H}_g^{(k)}$, so we set $\mathcal{U}_g^{(k)}:=\{u\in J(N_0/s^kN_0s^{-k})\mid x_u\in g^{-1}\mathcal{C}_0\cap\mathcal{C}_0\}$ and compute

$$\begin{aligned}
&\mathcal{H}_g^{(k)}\circ\widetilde{\mathrm{pr}_W}(g^{-1}\mu)=\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}n(g,u)\varphi_{t(g,u)s^k}\circ\psi_s^k(u^{-1}\widetilde{\mathrm{pr}_W}(g^{-1}\mu))=\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}n(g,u)\varphi_{t(g,u)s^k}\circ\widetilde{\mathrm{pr}_W}(s^{-k}u^{-1}g^{-1}\mu)=\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}\iota_{t(g,u)s^k,\infty}(n(g,u)\otimes_{s^k}\mathrm{pr}_W(s^{-k}u^{-1}g^{-1}\mu))=\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}\iota_{t(g,u)s^k,\infty}(n(g,u)\otimes_{s^k}\mathrm{pr}_W(s^{-k}\overline{n}(g,u)^{-1}t(g,u)^{-1}n(g,u)^{-1}\mu))\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}\iota_{t(g,u)s^k,\infty}(n(g,u)\otimes_{s^k}\mathrm{pr}_W((s^{-k}\overline{n}(g,u)^{-1}s^k)t(g,u)^{-1}s^{-k}n(g,u)^{-1}\mu))
\end{aligned}\tag{27}$$

where $\iota_{t(g,u)s^k,\infty}\colon\varphi^*_{t(g,u)s^k}W^\vee\to\varinjlim_t\varphi_t^*W^\vee=\widetilde{W^\vee}$ is the natural map. By Lemma 4.12 we have

$$s^{-k}\overline{n}(g,u)^{-1}s^k\in s^{-k+k_0}(G_0\cap\overline{N})s^{k-k_0}\leq U^{(k-k_0)}\ .$$

As π is a smooth representation of G and W_0 is finite, there exists an integer $k_2=k_2(W_0)$ such that for all $k'\geq k_2$ the subgroup $U^{(k')}$ acts trivially on W_0. By Lemma 4.10 we deduce

$$\mathrm{pr}_W(s^{-k}\overline{n}(g,u)^{-1}t(g,u)^{-1}n(g,u)^{-1}\mu)\mid_{W_r}=\mathrm{pr}_W(s^{-k}t(g,u)^{-1}n(g,u)^{-1}\mu)\mid_{W_r}$$

for all $r\leq\frac{k-k_2-k_0}{n_0}$ since N_0 normalizes $U^{(k-k_0)}$. Therefore by Lemma 4.13 and (27) we obtain

$$\begin{aligned}
&\mathcal{H}_g^{(k)}\circ\widetilde{\mathrm{pr}_W}(g^{-1}\mu)-\mathrm{res}_{\mathcal{U}_{g^{-1}}}\circ\widetilde{\mathrm{pr}_W}(\mu)=\\
&=\mathcal{H}_g^{(k)}\circ\widetilde{\mathrm{pr}_W}(g^{-1}\mu)-\sum_{u\in\mathcal{U}_g^{(k)}}n(g,u)\varphi_{t(g,u)s^k}\circ\psi_{t(g,u)s^k}(n(g,u)^{-1}\widetilde{\mathrm{pr}_W}(\mu))=\\
&=\sum_{u\in\mathcal{U}_g^{(k)}}\iota(n(g,u)\otimes\mathrm{pr}_W((s^{-k}\overline{n}(g,u)^{-1}s^k-1)s^{-k}t(g,u)^{-1}n(g,u)^{-1}\mu))\\
&\in\sum_{u\in\mathcal{U}_g^{(k)}}\iota(\Lambda(N_0)\otimes_{\Lambda(N_0),\varphi_{t(g,u)s^k}}(W/W_r)^\vee)
\end{aligned}$$

where $\iota = \iota_{t(g,u)s^k,\infty}$.

Finally, the sets $O(M,l,l') \subset D^\vee_{\xi,\ell,\infty}(\pi)$ in (9) form a system of open neighbourhoods of 0 in $D^\vee_{\xi,\ell,\infty}(\pi)$. Moreover, for any fixed choice $l,l' \geq 0$ and $M \in \mathcal{M}(\pi^{H_0})$ there exists an integer $k_1 \geq 0$ such that for all $r \geq k_1$ and $u \in \mathcal{U}_g$ we have

$$\mathrm{pr}_{W,F^*_{t'(g,u)}M_l}((W/W_r)^\vee) \subseteq \Lambda(N_0/H_l) \otimes_{u_\alpha} X^{l'}(F^*_{t'(g,u)}M)^\vee[1/X]^{++}$$

(see Lemmata 4.14 and 4.15). Note that the composite map $D^\vee_{\xi,\ell,\infty}(\pi) \overset{\varphi_{t(g,u)s^k}}{\to} D^\vee_{\xi,\ell,\infty}(\pi) \overset{f_{M,0}}{\twoheadrightarrow} M^\vee[1/X]$ factors through the φ_s-equivariant map

$$((1 \otimes F_{t(g,u)s^k})^\vee[1/X])^{-1} \colon (F^*_{t'(g,u)}M)^\vee[1/X] \to M^\vee[1/X]$$

mapping $X^{l'}(F^*_{t'(g,u)}M)^\vee[1/X]^{++}$ into $X^{l'}M^\vee[1/X]^{++}$. So we deduce that

$$\mathcal{H}_g^{(k)} \circ \mathrm{pr} \circ \mathrm{pr}_{SV}(g^{-1}\mu) - \mathrm{res}_{\mathcal{U}_{g^{-1}}} \circ \mathrm{pr} \circ \mathrm{pr}_{SV}(\mu)$$

lies in $O(M,l,l')$ for all $k \geq k_0 + k_2 + n_0k_1$ and any choice of $J(N_0/s^kN_0s^{-k})$. The result follows by taking the limit $\mathcal{H}_g = \lim_{k\to\infty} \mathcal{H}_g^{(k)}$. □ □

Now for any fixed $\mu \in \pi^\vee$ consider the the elements $\beta_{g\mathcal{C}_0}(\mu) \in \mathfrak{Y}(g\mathcal{C}_0)$ for $g \in G$. By Proposition 4.16 we also deduce

$$\begin{aligned}
\mathrm{res}^{g\mathcal{C}_0}_{g\mathcal{C}_0\cap h\mathcal{C}_0} \circ \beta_{g\mathcal{C}_0}(\mu) &= \mathrm{res}^{g\mathcal{C}_0}_{g\mathcal{C}_0\cap h\mathcal{C}_0}(g\beta_{\mathcal{C}_0}(g^{-1}\mu)) = \\
= g\,\mathrm{res}^{\mathcal{C}_0}_{\mathcal{C}_0\cap g^{-1}h\mathcal{C}_0} \circ \beta_{\mathcal{C}_0}(g^{-1}\mu) &\overset{4.16}{=} g\,\mathrm{res}^{g^{-1}h\mathcal{C}_0}_{\mathcal{C}_0\cap g^{-1}h\mathcal{C}_0} \circ \beta_{g^{-1}h\mathcal{C}_0}(g^{-1}\mu) = \\
= \mathrm{res}^{h\mathcal{C}_0}_{g\mathcal{C}_0\cap h\mathcal{C}_0}(g(g^{-1}h)\beta_{\mathcal{C}_0}((g^{-1}h)^{-1}g^{-1}\mu)) &= \mathrm{res}^{h\mathcal{C}_0}_{g\mathcal{C}_0\cap h\mathcal{C}_0}(h\beta_{\mathcal{C}_0}(h^{-1}\mu)) = \\
&= \mathrm{res}^{h\mathcal{C}_0}_{g\mathcal{C}_0\cap h\mathcal{C}_0} \circ \beta_{h\mathcal{C}_0}(\mu)
\end{aligned}$$

for all $g,h \in G$. Since $\mathfrak{Y}$ is a sheaf and we have $\bigcup_{g\in G} g\mathcal{C}_0 = G/B$, there exists a unique element $\beta_{G/B}(\mu)$ in the global sections $\mathfrak{Y}(G/B)$ with

$$\mathrm{res}^{G/B}_{g\mathcal{C}_0}(\beta_{G/B}(\mu)) = \beta_{g\mathcal{C}_0}(\mu)$$

for all $g \in G_0$. So we obtained a map $\beta_{G/B} \colon \pi^\vee \to \mathfrak{Y}(G/B)$. Our main result in this section is the following

Theorem 4.17. *The family of morphisms $\beta_{G/B,\pi}$ for smooth, admissible o-torsion representations π of G of finite length form a natural transformation between the functors $(\cdot)^\vee$ and $\mathfrak{Y}_{\alpha,\cdot}(G/B)$. Whenever $D^\vee_{\xi,\ell}(\pi)$ is nonzero, the map $\beta_{G/B,\pi}$ is nonzero either. In particular, if we further assume that π is irreducible then $\beta_{G/B}$ is injective.*

Proof. At first we need to check that $\beta_{G/B,\pi} \colon \pi^\vee \to \mathfrak{Y}_{\alpha,\pi}(G/B)$ is G-equivari-ant and continuous for all π. For $g,h \in G$ and $\mu \in \pi^\vee$ we compute

$$\begin{aligned}
\mathrm{res}^{G/B}_{g\mathcal{C}_0}(\beta_{G/B}(h\mu)) &= \beta_{g\mathcal{C}_0}(h\mu) = g\beta_{\mathcal{C}_0}(g^{-1}h\mu) = \\
= h\beta_{h^{-1}g\mathcal{C}_0}(\mu) &= h\,\mathrm{res}^{G/B}_{h^{-1}g\mathcal{C}_0} \circ \beta_{G/B}(\mu) = \mathrm{res}^{G/B}_{g\mathcal{C}_0}(h\beta_{G/B}(\mu))
\end{aligned}$$

showing that $\beta_{G/B}(h\mu)$ and $h\beta_{G/B}(\mu)$ are equal locally everywhere, so they are equal globally, too. The continuity follows from the fact that $\beta_{g\mathcal{C}_0}$ is continuous for each $g \in G$.

By Thm. 9.24 in [10] the assignment $\pi \mapsto \mathfrak{Y}_{\alpha,\pi}$ is functorial. Moreover, by definition we have $\beta_{g\mathcal{C}_0,\pi} = (g\cdot)\circ\beta_{\mathcal{C}_0,\pi}\circ(g^{-1}\cdot)$ so we are reduced to showing the naturality of $\beta_{\mathcal{C}_0,\cdot}$. This follows from the fact that for any morphism $f \colon \pi \to \pi'$ of smooth, admissible o-torsion representations of G of finite length and $M_k \in \mathcal{M}_k(\pi^{H_k})$ for any $k \geq 0$ we have $f(M_k) \in \mathcal{M}_k(\pi'^{H_k})$. □ □

Acknowledgements

Our debt to the works of Christophe Breuil [2], Pierre Colmez [3] [4], Peter Schneider, and Marie-France Vigneras [9] [10] will be obvious to the reader. We would especially like to thank Breuil for discussions on the exactness properties of his functor and its dependence on the choice of ℓ. We would also like to thank P. Schneider for discussions on the topic.

References

[1] Breuil Ch., The emerging p-adic Langlands programme, in: *Proceedings of the International Congress of Mathematicians* Volume II, Hindustan Book Agency, New Delhi (2010), 203–230.

[2] Breuil Ch., Induction parabolique et (φ, Γ)-modules, *Algebra & Number Theory* **9**(10) (2015), 2241–2291.

[3] Colmez P., (φ, Γ)-modules et représentations du mirabolique de $GL_2(\mathbb{Q}_p)$, *Astérisque* **330** (2010), 61–153.

[4] Colmez P., Représentations de $GL_2(\mathbb{Q}_p)$ et (φ, Γ)-modules, *Astérisque* **330** (2010), 281–509.

[5] Emerton M., On a class of coherent rings with applications to the smooth representation theory of $GL_2(\mathbb{Q}_p)$ in characteristic p, preprint (2008)

[6] Emerton M., Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties, *Astérisque* **331** (2010), 355–402.

[7] Erdélyi M., The Schneider–Vigneras functor for principal series, *J. of Number Theory* **162** (2016), 68–85.

[8] Fontaine J.-M., Représentations p-adiques des corps locaux, in "The Grothendieck Festschrift", vol 2, Prog. in Math. 87, 249–309, Birkhäuser 1991.

[9] Schneider P., Vigneras M.-F., A functor from smooth o-torsion representations to (φ, Γ)-modules, Volume in honour of F. Shahidi, Clay Mathematics Proceedings Volume **13** (2011), 525–601.

[10] Schneider P., Vigneras M.-F., Zábrádi G., From étale P_+-representations to G-equivariant sheaves on G/P, in: *Automorphic forms and Galois representations* (Volume 2), LMS Lecture Note Series **415** (2014), Cambridge Univ. Press, 248–366.

[11] Vigneras M.-F., Série principale modulo p de groupes réductifs p-adiques, *Geom. and Funct. Analysis* **17** (2008), 2090–2112.

[12] Zábrádi G., Exactness of the reduction on étale modules, *Journal of Algebra* **331** (2011), 400–415.

[13] Zábrádi G., (φ, Γ)-modules over noncommutative overconvergent and Robba rings, *Algebra & Number Theory* **8**(1) (2014), 191–242.