EPIREFLECTIVE SUBCATEGORIES OF TOP, T_2UNIF, UNIF, CLOSED UNDER EPIMORPHIC IMAGES, OR BEING ALGEBRAIC

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Abstract. The epireflective subcategories of **Top**, that are closed under epimorphic (or bimorphic) images, are $\{X \mid |X| \leq 1\}$, $\{X \mid X \text{ is indiscrete}\}$ and **Top**. The epireflective subcategories of $\mathbf{T_2Unif}$, closed under epimorphic images, are: $\{X \mid |X| \leq 1\}$, $\{X \mid X \text{ is compact } T_2\}$, $\{X \mid \text{covering character of } X \text{ is } \leq \lambda_0\}$ (where λ_0 is an infinite cardinal), and $\mathbf{T_2Unif}$. The epireflective subcategories of **Unif**, closed under epimorphic (or bimorphic) images, are: $\{X \mid |X| \leq 1\}$, $\{X \mid X \text{ is indiscrete}\}$, $\{X \mid \text{covering character of } X \text{ is } \leq \lambda_0\}$ (where λ_0 is an infinite cardinal), and **Unif**. The epireflective subcategories of **Top**, that are algebraic categories, are $\{X \mid |X| \leq 1\}$, and $\{X \mid X \text{ is indiscrete}\}$. The subcategories of **Unif**, closed under products and closed subspaces and being varietal, are $\{X \mid |X| \leq 1\}$, $\{X \mid X \text{ is indiscrete}\}$, $\{X \mid X \text{ is compact } T_2\}$. The subcategories of **Unif**, closed under products and closed subspaces and being algebraic, are $\{X \mid X \text{ is indiscrete}\}$, and all epireflective subcategories of $\{X \mid X \text{ is compact } T_2\}$. Also we give a sharpened form of a theorem of Kannan-Soundararajan about classes of T_3 spaces, closed for products, closed subspaces and surjective images.

§1. Preliminaries

Birkhoff's theorem in universal algebra says that varieties are characterized, in a given type of universal algebras (i.e., given operations, with given arities), as those being closed under products, subalgebras and homomorphic images. These properties can be investigated also in other categories, yielding Birkhoff type theorems.

In topology it seems to have been Kannan [K] who initiated the investigation of simultaneously reflective and coreflective subcategories in certain categories. If we restrict our attention to simultaneously epireflective and monocoreflective subcategories, then under suitable hypotheses, these can be described as those closed under products, extremal subobjects, coproducts and extremal epi images. This poses the question if there are theorems characterizing subcategories of certain categories, closed under several of these operations. Birkhoff's theorem settles one of these questions.

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Herrlich [H81] §3.2 surveyed a large number of closure operations on subcategories of a given category, and the subcategories closed under some subsets of these closure operations, for categories ocurring in topology.

The category of compact T_2 topological spaces is characterized in several different and nice ways, cf., e.g., de Groot's famous characterization in [W], p. 51, Franklin-Lutzer-Thomas [FLT] Theorem 3.10, Richter [R82], Corollary 1.7, [R85a], Corollary 4.9, [R91b], Theorem 4.5, Corollary 4.7, [R92a], Remark 4.7. Also cf. [R91b], Corollary 4.7, for characterizing epireflective subcategories of compact T_2 topological spaces, and [R92a], Remarks 2.3 and 4.7, for characterizing reflective subcategories of compact T_2 topological spaces, containing the two-point discrete space.

The category of sets is denoted by **Set**. The categories of T_2 topological spaces, T_2 proximity spaces and T_2 uniform spaces are denoted by $\mathbf{T_2}$, $\mathbf{T_2Prox}$ and $\mathbf{T_2Unif}$, respectively. The categories of topological spaces, proximity spaces and uniform spaces (without the T_0, T_2, T_2 axiom) are denoted by **Top**, **Prox** and **Unif**, respectively. For any of these six categories (from $\mathbf{T_2}$ till **Unif**), U will denote their underlying set functors — for which of these categories, that will be given in our respective theorems. Also, if we have a subcategory $\mathcal{C}$ of them, U may denote also the underlying set functor of $\mathcal{C}$. It will be always clear, which one do we mean. The category of T_3 topological spaces is denoted by $\mathbf{T_3}$, and the notation U will be used for it in the above sense.

Subcategories are considered to be full and isomorphism closed, and will be identified with the classes of their objects. A subcategory of **Top**, **Prox** *or* **Unif** *is non-trivial* if it contains a space with at least two points.

All the above six categories (from $\mathbf{T_2}$ till **Unif**) are complete, cocomplete, well-powered and co-well-powered. Thus, epireflective subcategories can be characterized in them, as those closed under products and extremal subobjects. Also, epimorphisms can be factorized as a composition of a bimorphism and an extremal epimorphism. In $\mathbf{T_2}$, $\mathbf{T_2Prox}$, $\mathbf{T_2Unif}$ *monomorphisms are the injections, epimorphisms are the dense maps, bimorphisms are the dense injections, extremal monomorphisms are the closed embeddings, and extremal epimorphisms are the quotient maps in the respective categories* (finest structures on surjective images making the surjective map a morphism). In **Top**, **Prox**, **Unif** *monomorphisms are the injections, epimorphisms are the surjections, bimorphisms are the bijections, extremal monomorphisms are the embeddings, and extremal epimorphisms are the quotient maps in the respective categories.*

For $\mathbf{M}$ a class of monomorphisms, or $\mathbf{E}$ a class of epimorphisms of some category, we say that a subcategory $\mathcal{C}$ is *closed under* $\mathbf{M}$*-subobjects*, or is *closed under* $\mathbf{E}$*-images* if $C \in \operatorname{Ob}\mathcal{C}$ and $\exists m \in \mathbf{M}$, $m : D \to C$, or $\exists e \in \mathbf{E}$, $e : C \to D$ imply $D \in \operatorname{Ob}\mathcal{C}$, respectively.

The *covering character* of a uniform space X, written as $\operatorname{cov\,char} X$, is the smallest infinite cardinal λ such that X admits a base of uniform coverings of cardinalities less than λ. Equivalently, it is the smallest infinite cardinal λ such that X has no uniformly discrete subspace of cardinality λ ([I]). The completion of a uniform space X is denoted by γX, and the precompact reflection of X is denoted by pX. Proximity spaces can be identified with precompact uniform spaces. For general information about uniform spaces, cf. the book of Isbell [I].

A concrete category $\langle \mathcal{C}, U : \mathcal{C} \to \mathbf{Set}\rangle$ with underlying set functor U is called *algebraic* if $\mathcal{C}$ has coequalizers, U has a left adjoint F, and U preserves and reflects regular epimorphisms. An algebraic category is *varietal* if additionally U reflects congruence relations. An algebraic category is the same as a *quasivariety*, i.e., all universal algebras of some given type (infinitary operations allowed, which may form a proper class), closed under products and subalgebras. A varietal category is the same as a *variety*, i.e., a quasivariety that is additionally closed under homomorphic images. Cf. [HS07], [AHS], or for a short description [R82]. We note that for concrete categories $\langle \mathcal{C}, U : \mathcal{C} \to \mathbf{Set}\rangle$ *varietal* is the same as *monadic* or *tripleable* (cf. [ML] and [AHS]).

For category theory, we refer to [ML], [H68], [HS07] and [AHS].

§2. Introduction

We begin with citing some theorems. The first one is a Birkhoff-type theorem for $\mathbf{T_2}$.

Theorem A. (D. Petz [P], Theorem). *Let $\mathcal{C}$ be a subcategory of $\mathbf{T}_2$. Then the following are equivalent:*

1) $\mathcal{C}$ is an epireflective subcategory of $\mathbf{T}_2$, closed under epi images.

2) Either $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{T_2} \mid |X| \le 1\}$, or $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{T_2} \mid X$ is compact $T_2\}$, or $\mathcal{C} = \mathbf{T_2}$.

If we write 1) as being closed under products, extremal subobjects, bimorphic images and extremal epi images, then none of these properties can be omitted, without invalidating the implication 1) $\Longrightarrow$ 2) of the theorem.

We remark that [P] did not decompose the hypotheses in the same way, therefore we have to give examples for the last two properties. The 1-st, 2-nd, 4-th properties are satisfied for T_2 spaces in which each at most countably infinite subset has a compact closure. The 1-st, 2-nd, 3-rd properties are satisfied for 0-dimensional compact T_2 spaces. (Both examples were given by [P]).

One of the main results of G. Richter [R89] was a generalization of Theorem A of Petz, with weaker (although more complicated) hypotheses, which we give here.

Theorem B. (Richter [R89], Corollary 3.3) *Let $\operatorname{Ob}\mathcal{C} \subset \operatorname{Ob}\mathbf{T_2}$. Then the implication 1) $\Longrightarrow$ 2) of Theorem A remains true under the following weaker hypotheses.*
(P1'): the underlying set functor U of $\mathcal{C}$ has a left adjoint F with pointwise dense unit $\eta : id_{\mathbf{Set}} \to UF$ (i.e., for all $X \in \operatorname{Ob}\mathbf{Set}$ $\eta_X(X)$ is dense in FX);
(P2): $\mathcal{C}$ is closed under surjective images;
(P3'): if $\eta_X : X \to UFX$ is a C^*-embedding (i.e., it underlies a C^*-embedding $X_d \to FX$, where X_d is the discrete topological space on the set X) and $i : FX \to C$ is an open, dense C^*-extension, such that in C disjoint zero-sets can be separated by a clopen set, then C belongs to the maximal subcategory $\hat{\mathcal{C}}$ of $\mathbf{T}_2$, containing $\mathcal{C}$, for whose underlying set functor $\hat{U}$ we have that F and η still serve as left adjoint and unit of adjunction.

Here (P1') is strictly weaker than epireflectivity, and (P3') is strictly weaker than closedness under dense extensions. For more details we refer to [R89] (in particular to its Theorem 3.1).

H. Herrlich-G. R. Strecker [HS71] initiated the investigation of algebraically behaving subcategories of $\mathbf{T}_2$.

Theorem C. (H. Herrlich-G. R. Strecker [HS71], Theorem, G. Richter [R82], Corollary 1.6, [R85a], Corollary 3.4) *Let $\mathcal{C}$ be a subcategory of $\mathbf{T}_2$. Then the following are equivalent:*

1) $\mathcal{C}$ is an epireflective subcategory of $\mathbf{T_2}$, that is varietal.

2) Either $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{T_2} \mid |X| \leq 1\}$, *or* $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{T_2} \mid X \textit{ is compact } T_2\}$.

If we write 1) as being closed under products, extremal subobjects, and being varietal, then none of these properties can be omitted, without invalidating the implication 1) $\Longrightarrow$ 2) *of the theorem.*

For algebraic subcategories, G. Richter proved an analogous result.

Theorem D. (G. Richter [R82], Corollaries 1.5, 1.6) *Let $\mathcal{C}$ be a subcategory of $\mathbf{T_2}$. Then the following are equivalent:*

1) $\mathcal{C}$ is an epireflective subcategory of $\mathbf{T_2}$ and is algebraic.

2) $\mathcal{C}$ is an epireflective subcategory of $\{X \in \operatorname{Ob}\mathbf{T_2} \mid X \textit{ is compact } T_2\}$.

If we write 1) as being closed under products, extremal subobjects, and being algebraic, then none of these properties can be omitted, without invalidating the implication 1) $\Longrightarrow$ 2) of the theorem. All, but the 1-st, 2-nd or 3-rd property are satisfied by examples 1) and 2) of [HS71] and by 3) $\mathbf{T_2}$ (observe that its underlying set functor does not reflect isomorphisms therefore it is not algebraic). Example 1) is discrete topological spaces, which form even a varietal category [HS71]. Example 2) is the powers of a compact T_2 topological space, consisting of more than one point, that is strongly rigid — i.e., whose only continuous self-maps are the identity and the constant maps (such spaces exist, cf. [HS71]) — together with the empty space, which category is even varietal, cf. [R92a], p. 368.

Theorem C raises the analogous question for **Top**. Both Theorem C, and its word for word analogue for **Top**, rather than $\mathbf{T_2}$, follow from the following theorem of Richter [R91a].

Theorem E. (G. Richter [R91a], Corollary 4.4). *Let $\mathcal{C}$ be a subcategory of* **Top**. *Then the following are equivalent:*

1) $\mathcal{C}$ is closed under products, closed subspaces and is varietal.

2) Either $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Top} \mid |X| \leq 1\}$, *or* $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Top} \mid X \textit{ is indiscrete}\}$, *or* $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Top} \mid X \textit{ is compact } T_2\}$.

Here none of the properties in 1) can be omitted, without invalidating the implication 1) $\Longrightarrow$ 2) of the theorem, as follows from Theorem C and from the fact that $\mathbf{T_2}$ is closed under products and closed subspaces.

Earlier, Richter [R85a], Corollary 3.4 proved a weaker result. Namely, he added to the hypotheses of Theorem E that the two-point discrete space belongs to $\operatorname{Ob}\mathcal{C}$, cf. [R85a], p. 80 (then, of course, in 2) only the third case is possible.)

One could obtain a common generalization (roughly) of Theorems A and 1 (cf. §3, dealing with epireflective subcategories of **Top**, closed under bijective images), in the following way. We consider subcategories of **Top** that are productive, are closed only under closed subspaces (like in Theorem A), and only under surjective images (like in a weakened variant of Theorem 1, replacing "bimorphic" with "epimorphic"). Such a theorem is available, but only for T_3 spaces.

Theorem F. (Kannan-Soundararajan [KS], Theorem) *Let $\mathcal{C}$ be a subcategory of $\mathbf{T_3}$. Then the following are equivalent.*

1) $\mathcal{C}$ is productive, closed-hereditary, and is closed under surjective images.

2) Either

A) $\mathrm{Ob}\,\mathcal{C} = \{X \mid X \text{ is } T_3 \text{ and } |X| \leq 1\}$, or

B) there exists a class $\mathcal{F}$ of ultrafilters p on some sets S_p, such that $\mathrm{Ob}\,\mathcal{C}$ consists of all T_3 spaces X satisfying the following property. For $p \in \mathcal{F}$ and $f : S_p \to X$ any function there exists a continuous extension of f, namely $\overline{f} : S_p \cup \{p\} \to X$, where $S_p \cup \{p\}$ has the subspace topology inherited from $\beta[(S_p)_d]$, and where $(S_p)_d$ is the discrete space on S_p.

Here none of the properties in 1) can be omitted, without invalidating the implication 1) $\Longrightarrow$ 2) of the theorem. This is shown by the following examples in $\mathbf{T_3}$: finite spaces, connected spaces, zero-dimensional compact T_2 spaces (the 1-st and 3-rd examples are taken from [P]).

Remark. Let us exclude the trivial case $\mathrm{Ob}\,\mathcal{C} = \{X \mid X \text{ is } T_3 \text{ and } |X| \leq 1\}$. Then the underlying set functor $U : \mathcal{C} \to \mathbf{Set}$ has a left adjoint F that has a natural transformation to the functor $X \to \beta(X_d)$, with all components embeddings ([KS], essentially Step 4, p. 143, applied to T_3 spaces). Then we can recover a (maximal) class $\mathcal{F}$, by considering the spaces FX, for all $X \in \mathrm{Ob}\,\mathbf{Set}$: the points of all these spaces FX will give the class of ultrafilters mentioned in Theorem F. (This is the construction of [KS], Step 6, p. 144, except that there fixed ultrafilters are not considered, but that does not change matters). Thus we have for this (maximal) class $\mathcal{F}$ that for $X \in \mathrm{Ob}\,\mathbf{Set}$, and for any $\mathbf{Set}$-morphism $f : Y \to X$ there exists a domain-codomain extension of f to a $\mathbf{T_3}$-morphism $Ff : FY \to FX$, i.e., $(Ff)(FY) \subset FX$. In short: "$\mathcal{F}$ is closed under images". (This is a special case of the statement of Theorem F, 2), B) applied to the space FX rather than X in Theorem F 2) B)). Observe that [KS] Theorem did not contain this property of $\mathcal{F}$ explicitly. In fact this property is necessary (and sufficient) for F to be a left adjoint of U even when restricted to the minimal class $\{FX \mid X \in \mathrm{Ob}\,\mathbf{Set}\}$ (Kleisli adjunction, [AHS], 20.39, 20.B).

On the other hand, for given $\mathcal{F}$, the class $\mathcal{C}$ constructed in Theorem F is a maximal subclass of $\mathbf{T_3}$ for which F and η are left adjoint to U and unit of adjunction.

Recall that all subcategories considered are isomorphism closed. Observe also that F and η are defined only up to isomorphisms, forming respective commutative diagrams. We can eliminate these ambiguities by using some specific construction of $\beta(X_d)$, e.g., with ultrafilters, and considering

$$(*) \qquad \eta_X X \subset UFX \text{ and } FX \subset \beta(X_d)$$

(thus the usual embeddings are realized by embeddings of subsets/subspaces).

Let us denote, for given (maximal) $\mathcal{C}$ and (maximal) $\mathcal{F}$, by $\mathcal{C}(\mathcal{F})$ and $\mathcal{F}(\mathcal{C})$ the (maximal) subcategory $\mathcal{C}$ constructed for $\mathcal{F}$ in Theorem F, 2), B) and the (maximal) class $\mathcal{F}$ constructed in the proof of [KS], Theorem, Step 6.

[KS] did not completely clarify the situation. Namely, for a category $\mathcal{C}$ there exists a class $\mathcal{F}$ of ultrafilters making 2) B) of Theorem F true. However the questions, which classes $\mathcal{F}$ of ultrafilters arise this way, and possibly when are the corresponding categories $\mathcal{C}$ equal, are not considered there. As already mentioned, the class $\mathcal{F}$ is "closed under images", so we need to consider this question only for classes of ultrafilters "closed under images".

The class of the spaces X described in Theorem F, 2), B) is the largest class $\mathcal{C}_{\max}(\mathcal{F})$ (in $\mathbf{T_3}$!) for which F and η are left adjoint to U and unit of adjunction. This $\mathcal{C}_{\max}(\mathcal{F})$ determines uniquely F and η as left adjoint and unit of adjunction, by convention (*).

Similarly, the class $\mathcal{F}_{\max}(\mathcal{C})$ gives exactly the Kleisli adjunction (minimal adjunction) associated to the adjunction $F \dashv U$.

Beginning with the Kleisli adjunction, then taking the maximal (in $\mathbf{T_3}$) adjunction with given F and η, and turning once more to the Kleisli adjunction clearly gives back the original Kleisli adjunction (by (*)).

Beginning with the maximal (in $\mathbf{T_3}$) adjunction with given F and η, then taking the Kleisli adjunction, and turning once more to the maximal (in $\mathbf{T_3}$) adjunction with given F and η clearly gives back the original maximal adjunction.

This settles the case of maximal subcategories $\mathcal{C}_{\max}(\mathcal{F})$. It will suffice to prove that under 1) of Theorem F each subcategory $\mathcal{C}$ is maximal.

Observe that the proof of [KS], Theorem, Step 7 in fact proves the following. Let $C \in \mathrm{Ob}\,\mathcal{C}_{\max}(\mathcal{F})$. Then $C \in \mathrm{Ob}\,\mathcal{C}$. (Namely there C is the surjective image of FUC, by ε_C, the counit of the adjunction.) This proves $\mathcal{C}_{\max}(\mathcal{F}) \subset \mathcal{C}$, i.e., that each subcategory $\mathcal{C}$ in 1) of Theorem F is maximal.

That is, we have shown the following addition to Theorem F.

Proposition. *In Theorem F, 2), B), we may additionally suppose that $\mathcal{F}$ is "closed under images"* (definition cf. above). *Under this restriction, Theorem F, 2), B) establishes a bijection between the subcategories satisfying Theorem F, 1), and the classes $\mathcal{F}$ of ultrafilters "closed under images".* ■

If in Theorem F 1) we write instead of closedness under surjective images closedness under bijections and closedness under extremal epi images, then 0-dimensional compact T_2 spaces are epireflective and closed under bijective images in $\mathbf{T_3}$, but are not of the form in 2).

Problem 1. Find a fourth example (if it exists) that is epireflective and closed under extremal epi images, but is not of the form in 2) (the proof in [KS] seems to use, by ε_C, both closedness under bijections and extremal epi images).

Extensions of Theorem F cf. in the paper of Hager [Ha], to the case of a concrete category. say, over **Set**. Then his theorem is specialized to $\mathbf{T_2Prox}$ and $\mathbf{T_2Unif}$. (And also for T_2 cozero spaces, where a T_2 *cozero space* can be easiest defined as the cozero sets of all uniformly continuous real valued functions for some T_2 uniformity on the underlying set, and a cozero morphism is a set morphism, such that the inverse image of a cozero set is also a cozero set. More about this cf. in [Ha].) For details we have to refer to [Ha].

Richter [R80/81], [R82], [R85a], [R85b], [R89], [R91a], [R91b], [R92a], [R92b], [R99] contain much related material.

We will prove analogues of these theorems for **Top**, **Prox**, $\mathbf{T_2}$**Unif**, **Unif**.

§3. Theorems

The first three theorems will deal with epireflective categories closed under epimorphic or bimorphic images.

First we give a simple proof of an analogue of Theorem A for **Top**, with less hypotheses.

Theorem 1. *Let $\mathcal{C}$ be a subcategory of* **Top**. *Then the following are equivalent:*

1) $\mathcal{C}$ an epireflective subcategory of **Top**, *closed under bimorphic images.*

2) Either $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Top} \mid |X| \le 1\}$, *or* $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Top} \mid X$ *is indiscrete*$\}$, *or* $\mathcal{C} = \mathbf{Top}$.

If we write 1) as being closed under products, extremal subobjects and bimorphic images, then none of these properties can be omitted, without invalidating the implication 1) $\Longrightarrow$ 2) *of the theorem.*

Next we give the analogue of Theorem A for $\mathbf{T_2}$**Unif**.

Theorem 2. *Let $\mathcal{C}$ be a subcategory of* $\mathbf{T_2Unif}$. *Then the following are equivalent:*

1) $\mathcal{C}$ is an epireflective subcategory of $\mathbf{T_2Unif}$, *closed under epi images.*

2) Either $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{T_2Unif} \mid |X| \le 1\}$, *or* $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{T_2Unif} \mid X$ *is compact T_2*$\}$, *or there exists an infinite cardinal λ_0, such that* $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{T_2Unif} \mid X$ *has a covering character at most λ_0*$\}$, *or* $\mathcal{C} = \mathbf{T_2Unif}$.

If we write 1) as being closed under products, extremal subobjects, bimorphic images and extremal epi images, then none of these properties can be omitted, without invalidating the implication 1) $\Longrightarrow$ 2) *of the theorem.*

Next we turn to a common analogue of Theorems 1 and 2, for **Unif**.

Theorem 3. *Let $\mathcal{C}$ be a subcategory of* **Unif**. *Then the following are equivalent:*

1) $\mathcal{C}$ is an epireflective subcategory of **Unif**, *closed under bimorphic images.*

2) Either $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Unif} \mid |X| \le 1\}$, *or* $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Unif} \mid X$ *is indiscrete*$\}$, *or there exists an infinite cardinal λ_0 such that* $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Unif} \mid X$ *has a covering character at most λ_0*$\}$, *or* $\mathcal{C} = \mathbf{Unif}$.

If we write 1) as being closed under products, extremal subobjects and bimorphic images, then none of these properties can be omitted, without invalidating the implication 1) $\Longrightarrow$ 2) *of the theorem.*

Problem 2. There arises the question about the uniform version of Theorem F of Kannan-Soundararajan. That is, we suppose closedness under products, closed subspaces and surjective images. This would be a common generalization of Theorems 2 and 3. As mentioned after Theorem F, Theorems 2 and 3 have a common generalization in Hager [Ha]. However, the description in [Ha] does not seem to imply in an evident way our concrete descriptions in our Theorems 2 and 3. (It uses for the description some class of epimorphisms of the category, of which there are illegitimely many; and it does not seem to be evident how to identify these classes and concretize the description in our concrete cases.) Also, our proofs are independent of [Ha].

However this seems to be a question much more complicated than Theorem F of Kannan-Soundararajan. Let us restrict our attention to the case of $\mathbf{T_2Unif}$. Of course, we have as examples all T_2 uniform spaces, whose underlying topological spaces form a class (of Tychonoff spaces!) as described in Theorem F. Moreover, if we take a cardinal $\lambda_0 \geq \aleph_0$, we have an example all those uniform spaces, whose underlying topological spaces form a class as described in Theorem F, and whose covering characters are at most λ_0. (For $\lambda_0 = \aleph_0$ we have examples for proximities.) The problem is again that we cannot identify the class of epimorphisms whose existence is stated in [Ha] and cannot concretize the description in [Ha]. So a concrete, usable description still is missing.

The next three theorems will deal with epireflective subcategories, or subcategories closed under products and closed subspaces, which are algebraic or varietal.

Theorem 4. *Let* $\mathbf{T} = \mathbf{Top}$ *or* $\mathbf{T} = \mathbf{Prox}$ *or* $\mathbf{T} = \mathbf{Unif}$. *Let* $\mathcal{C}$ *be a subcategory of* $\mathbf{T}$. *Then the following are equivalent:*

1) $\mathcal{C}$ *is an epireflective subcategory of* $\mathbf{T}$, *that is algebraic.*

2) Either $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{T} \mid |X| \leq 1\}$, *or* $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{T} \mid X \text{ is indiscrete}\}$.

If we write 1) as being closed under products, extremal subobjects, and being algebraic, then none of these properties can be omitted, without invalidating the implication 1) $\Longrightarrow$ 2) *of the theorem.*

Although the cases $\mathbf{T} = \mathbf{Prox}$ and $\mathbf{T} = \mathbf{Unif}$ of Theorem 4 are covered by the next theorem (namely the minimal non-trivial epireflective subcategory of compact T_2 proximity or uniform spaces is that of 0-dimensional compact T_2 proximity or uniform spaces, and its hereditary hull, taken in **Prox** or **Unif**, contains also non-compact proximity or uniform spaces), its proof in Theorem 4 is much simpler than the proof of Theorem 5.

Theorem 5. *Let* $\mathcal{C}$ *be a subcategory of* **Unif**. *Then the following are equivalent:*

1) $\mathcal{C}$ *is closed under products and closed subspaces and is algebraic.*

2) Either $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Unif} \mid X \text{ is indiscrete}\}$, *or* $\mathcal{C}$ *is an epireflective subcategory of* $\{X \in \operatorname{Ob}\mathbf{Unif} \mid X \text{ is compact } T_2\}$.

None of the properties of 1) of this theorem can be omitted, without invalidating the implication 1) $\Longrightarrow$ 2) *of the theorem.*

Next we turn to an analogue of Theorem E for **Unif**. This theorem implies the description of epireflective and varietal subcategories both in $\mathbf{T_2Unif}$ and **Unif**.

Theorem 6. *Let* $\mathcal{C}$ *be a subcategory of* **Unif**. *Then the following are equivalent:*

1) $\mathcal{C}$ *is a subcategory of* **Unif**, *closed under products and closed subspaces, that is varietal.*

2) Either $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Unif} \mid |X| \leq 1\}$, *or* $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Unif} \mid X \text{ is indiscrete}\}$, *or* $\operatorname{Ob}\mathcal{C} = \{X \in \operatorname{Ob}\mathbf{Unif} \mid X \text{ is compact } T_2\}$.

None of the properties in 1) can be omitted, without invalidating the implication 1) $\Longrightarrow$ 2) *of the theorem.*

Problem 3. What remains open, is the following question, that would include Theorems D, E and 4 (for **Top**), and would be an analogue of Theorem 5. Namely, can one describe all subcategories $\mathcal{C}$ of **Top**, closed under products and closed subspaces, and being algebraic? Are there more such subcategories than described in Theorems D and E? (Observe that the uniform case is settled by Theorem 5. Thus the situation is just the converse of the situation mentioned in Problem 2, where the uniform case seems to be much more complicated.)

§4. PROOFS

Proof of Theorem 1. We only need to prove 1) $\Longrightarrow$ 2).

The empty product, i.e., the one-point space belongs to $\mathrm{Ob}\,\mathcal{C}$, as well as its (closed) subspace the empty set. Hence, $\{X \in \mathrm{Ob}\,\mathbf{Top} \mid |X| \le 1\} \subset \mathrm{Ob}\,\mathcal{C}$. If here we have equality, we are done.

Therefore we may suppose that some space X belongs to $\mathrm{Ob}\,\mathcal{C}$, where $|X| \ge 2$. Then any of its two-point subspaces belongs to $\mathrm{Ob}\,\mathcal{C}$ as well, hence we may suppose $|X| = 2$. Then its bijective image the two-point indiscrete subspace belongs to $\mathrm{Ob}\,\mathcal{C}$ as well, hence we may suppose that X is the two-point indiscrete space I_2. Then any subspace of any power of I_2 belongs to $\mathrm{Ob}\,\mathcal{C}$, hence $\{X \in \mathrm{Ob}\,\mathbf{Top} \mid X$ is indiscrete$\} \subset \mathrm{Ob}\,\mathcal{C}$. If here we have equality, we are done.

Therefore we may suppose that some space X belongs to $\mathrm{Ob}\,\mathcal{C}$, where X is non-indiscrete. Then X has a non-indiscrete two-point subspace, that of course belongs to $\mathrm{Ob}\,\mathcal{C}$, hence we may suppose $|X| = 2$. Then the Sierpiński space is a bijective image of X, therefore it belongs to $\mathrm{Ob}\,\mathcal{C}$ as well. Since any T_0 topological space is a subspace of a power of the Sierpiński space, hence $\{X \in \mathrm{Ob}\,\mathbf{Top} \mid X$ is $T_0\} \subset \mathrm{Ob}\,\mathcal{C}$. Finally, any topological space is a subspace of a product of a T_0 space and an indiscrete space. Hence $\mathcal{C} = \mathbf{Top}$.

There remains to give three examples. All but the 1-st, 2-nd, or 3-rd properties are satisfied by the subclasses of **Top** consisting of finite spaces, of connected spaces (both being closed even under all surjective images), or of T_0 spaces, respectively. $\blacksquare$

We begin the proof of Theorem 2 with a simple lemma, that is known. For 1) of Lemma 1 (for realcompact spaces), cf. [GJ], Theorem 8.9, and for 2) of Lemma 1 (also for realcompact spaces), cf. [GJ], Theorem 8.13. A categorical generalization of both 1) and 2), namely that epireflective subcategories are strongly closed under limits, with an explanation that 1) and 2) are particular cases of this general statement, cf. in [H68], §9.3. We state our Lemma for $\mathbf{T_2Unif}$.

Lemma 1. ([GJ], [H68], cited just before this Lemma) *Let $\mathcal{E}$ be an epireflective subcategory of* $\mathbf{T_2Unif}$.

1) Let $X \in \mathrm{Ob}\,\mathbf{T_2Unif}$, and let X_α, for $\alpha \in A$, be subspaces of X, such that for each $\alpha \in A$ we have $X_\alpha \in \mathrm{Ob}\,\mathcal{E}$. Then $\cap_{\alpha \in A} X_\alpha \in \mathrm{Ob}\,\mathcal{E}$.

2) Let $X \in \mathrm{Ob}\,\mathcal{E}$, $Y \in \mathrm{Ob}\,\mathbf{T_2Unif}$, $Z \subset Y$, $Z \in \mathrm{Ob}\,\mathcal{E}$ and let $f : X \to Y$ be uniformly continuous. Then $f^{-1}(Z) \in \mathrm{Ob}\,\mathcal{E}$. $\blacksquare$

Next we give a certain uniform analogue of well-known theorems for topological spaces, cf. [E], Exercises 4.2.D and 4.4.J and Theorem 4.4.15, about representing topological, or metric spaces as images of certain spaces under certain types of mappings. In particular, these statements characterize the class of first countable T_0 spaces, or metric spaces, as the open, or perfect images of subspaces of Baire spaces $D_\lambda^{\aleph_0}$ — where D_λ is a discrete topological space of cardinality λ, and where λ equals the weight of the space to be represented — respectively. Some more specialized theorems of this type cf., e.g., in [M]. (About inverse limits of uniform spaces, to be used in the proof of Lemma 2, cf. [I], §IV, subchapter "Inverse limits".)

Lemma 2. *Let M be a complete metric space with covering character at most λ_0, where λ_0 is an infinite cardinal. Then there is a dense, uniformly continuous map from a closed subspace of a countable product (taken in* $\mathbf{T_2Unif}$*) of discrete uniform spaces, of cardinalities less than λ_0, to M.*

Proof. Let (M, ϱ) be our complete metric space, with $\operatorname{cov\,char} X \leq \lambda_0$. By replacing the original metric ϱ by $(1-\varepsilon)\varrho/(1+\varrho)$, if necessary, we may assume that $\operatorname{diam} M < 1$. For each integer $n \geq 0$ we will define sets $M_n \subset M$ as follows. M_n is a maximal subset of M, containing M_{n-1} (for $n = 0$ we let $M_{-1} = \emptyset$), such that any two different points of M_n have a distance at least $1/2^n$. Clearly $|M_0| = 1$, and for all n we have $|M_n| < \lambda_0$. (This is true also for $\operatorname{cov\,char} M = \aleph_0$, i.e., when M is precompact.)

For $n \geq 0$ we define maps $f_n : M_{m+1} \to M_n$, such that f_n is identity on M_n, and else, for $m_{n+1} \in M_{n+1} \setminus M_n$, we have

$$\varrho(m_{n+1}, f_n(m_{n+1})) < 1/2^n. \tag{*}$$

The existence of $f_n(m_{n+1})$ follows from the maximality property of M_n. Of course, inequality (*) holds for all $m_{n+1} \in M_{n+1}$. The same maximality property, for each n, implies that $\cup_{n=0}^{\infty} M_n$ is dense in M.

We define a partial order $\leq$ on $\cup_{n=0}^{\infty} M_n$, as the transitive (and reflexive) hull of the relation

$$\{(f_n(m_{n+1}), m_{n+1}) \mid n \geq 0,\ m_{n+1} \in M_{n+1}\}.$$

This gives a tree structure on $\cup_{n=0}^{\infty} M_n$, and any two points of $\cup_{n=0}^{\infty} M_n$ have a greatest lower bound. The 0-th, 1-st, 2-nd, ... levels of the tree are $M_0, M_1 \setminus M_0, M_2 \setminus M_1, \ldots$.

Then $M_0 \xleftarrow{f_0} M_1 \xleftarrow{f_1} \ldots$ forms an inverse system of complete, and in fact, uniformly discrete uniform spaces. Its inverse limit $\varprojlim(M_n, f_n)$ is a complete uniform space, and $\cup_{n=0}^{\infty} M_n$ has a natural embedding i to $\varprojlim(M_n, f_n)$: to $m_n \in M_n$ we let correspond the thread (branch)

$$\begin{cases} i(m_n) := \langle f_0 f_1 \ldots f_{n-1}(m_n), f_1 \ldots f_{n-1}(m_n), \ldots, \\ f_{n-2} f_{n-1}(m_n), f_{n-1}(m_n), m_n, m_n, m_n, \ldots \rangle. \end{cases}$$

We define a metric d on $i(\cup_{n=0}^{\infty} M_n)$ as follows. For $m_{n_1} \in M_{n_1} \setminus M_{n_1 - 1}$ and $m_{n_2} \in M_{n_2} \setminus M_{n_2 - 1}$, we let $d(im_{n_1}, im_{n_2}) := 1/2^n$, where the greatest lower bound of m_{n_1} and m_{n_2} is on the n-th level, where $n \geq 0$.

This can be extended to a metric d on $\varprojlim(M_n, f_n)$ as follows. The distance of two different threads (branches) is $1/2^n$, if the threads are identical exactly on the 0-th, 1-st, 2-nd, ..., n-th levels. (This metric is non-Archimedean, i.e., we have $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, thus, in particular, $d(im_1, im_3) \leq \max\{d(im_1, im_2), d(im_2,$

$im_3)\}$.) Then $i(\cup_{n=0}^{\infty} M_n)$ is dense in $\varprojlim(M_n, f_n)$. Observe that $\varprojlim(M_n, f_n)$ is a closed subspace of the product $\prod_{n=0}^{\infty} M_n$.

Let us map $i(\cup_{m=0}^{\infty} M_n)$ to $\cup_{m=0}^{\infty} M_n$ by the left inverse j of the embedding i, when i is considered here as a map from $\cup_{m=0}^{\infty} M_n$ to $i(\cup_{m=0}^{\infty} M_n)$. We assert that j is a Lipschitz map with Lipschitz constant 4. In fact, let $m_{n_1} \in M_{n_1} \setminus M_{n_1-1}$ and $m_{n_2} \in M_{n_2} \setminus M_{n_2-1}$ have greatest lower bound on level n; thus $d(im_{n_1}, im_{n_2}) = 1/2^n$. Then

$$\begin{cases} \varrho(m_{n_1}, m_n) \leq \varrho\left(m_{n_1}, f_{n_1-1}(m_{n_1})\right) + \varrho\left(f_{n_1-1}(m_{n_1}), f_{n_1-2} f_{n_1-1}(m_{n_1})\right) + \\ \cdots + \varrho\left(f_{n+1} f_{n+2} \dots f_{n_1-2} f_{n_1-1}(m_{n_1}), f_n f_{n+1} f_{n+2} \dots f_{n_1-2} f_{n_1-1}(m_{n_1})\right) \\ < 1/2^{n_1-1} + 1/2^{n_1-2} + \dots + 1/2^n < 2/2^n. \end{cases}$$

Similarly, $\varrho(m_{n_2}, m_n) < 2/2^n$, hence

$$\varrho(m_{n_1}, m_{n_2}) < 4/2^n = 4d(im_{n_1}, im_{n_2}),$$

as claimed above.

Now recall that $i(\cup_{n=0}^{\infty} M_n)$ is dense in the complete metric space $\varprojlim(M_n, f_n)$, and $\cup_{n=0}^{\infty} M_n$ is dense in the complete metric space M. Then j has an extension $\varphi : \varprojlim(M_n, f_n) \to M$, that is Lipschitz with constant 4, hence is a uniformly continuous and dense map. ■

Proof of Theorem 2. We only need to prove 1) $\Longrightarrow$ 2).

1. Like in the proof of Theorem 1, second paragraph, we see that $\{X \in \operatorname{Ob} \mathbf{Unif} \mid |X| \leq 1\} \subset \operatorname{Ob} \mathcal{C}$. If here we have equality, we are done.

Now suppose that here we do not have equality, i.e., $\mathcal{C}$ contains a T_2 uniform space X with at least two points. Then X has a closed subspace consisting of two points, i.e., the discrete two-point space D_2, that therefore belongs to $\operatorname{Ob} \mathcal{C}$. Then all closed subspaces of all finite powers of D_2 belong to $\operatorname{Ob} \mathcal{C}$, hence $\{X \in \operatorname{Ob} \mathbf{T_2Unif} \mid X \text{ is a finite discrete space}\} \subset \operatorname{Ob} \mathcal{C}$.

Also $D_2^{\aleph_0} \in \operatorname{Ob} \mathcal{C}$ (power meant in $\mathbf{T_2Unif}$), i.e., the Cantor set with its unique compatible uniformity belongs to $\operatorname{Ob} \mathcal{C}$. Then also its uniformly continuous image $[0,1]$ belongs to $\operatorname{Ob} \mathcal{C}$, and all its powers $[0,1]^{\alpha}$ belong to $\operatorname{Ob} \mathcal{C}$, as well as all their closed subspaces. That is, $\{X \in \operatorname{Ob} \mathbf{T_2Unif} \mid X \text{ is compact } T_2\} \subset \operatorname{Ob} \mathcal{C}$. If here we have equality, we are done. (Up to this point, the proof is essentially the same, as in [W], p. 51, [HS71] and [P].)

Now suppose that here we do not have equality, i.e., $\operatorname{Ob} \mathcal{C}$ contains a T_2 uniform space X that is not compact. Then its uniformly continuous image pX, its precompact reflection, is homeomorphic to X, hence also is non-compact, and belongs to $\operatorname{Ob} \mathcal{C}$. Thus we may assume that $X \in \operatorname{Ob} \mathcal{C}$ is precompact, non-compact. Then γX, its completion, is a proper superset of X. Further, γX is compact T_2.

Following [P], choose $a \in X$ and $b \in \gamma X \setminus X$. Then $\{a, b\}^{\aleph_0} \subset (\gamma X)^{\aleph_0}$, and $\{a, b\}^{\aleph_0}$ is the Cantor set with its unique compatible uniformity, C, say. By $X \in \operatorname{Ob} \mathcal{C}$ we have $X^{\aleph_0} \in \operatorname{Ob} \mathcal{C}$, hence also any subspace of $(\gamma X)^{\aleph_0}$, containing $X^{\aleph_0}$ (that is dense in $(\gamma X)^{\aleph_0}$), belongs to $\operatorname{Ob} \mathcal{C}$. In particular, $(\gamma X)^{\aleph_0} \setminus \{\langle b, b, \dots \rangle\} \in \operatorname{Ob} \mathcal{C}$. This last subspace has as closed subspace $\{a, b\}^{\aleph_0} \setminus \{\langle b, b, \dots \rangle\}$. Hence, using for C be the usual ternary representation of the Cantor set, we have $C \setminus \{0\} \in \operatorname{Ob} \mathcal{C}$. Then, for the usually constructed surjection $f : C \to [0,1]$, we have $f(C \setminus \{0\}) =$

$(0,1] \in \mathrm{Ob}\,\mathcal{C}$.

2. The class of cardinalities λ (finite of infinite), for which the discrete space D_λ of cardinality λ belongs to $\mathrm{Ob}\,\mathcal{C}$, forms an initial segment of all cardinalities, i.e., it is of the form $\{\lambda \mid \lambda < \lambda_0\}$ or $\{\lambda \mid \lambda \text{ is a cardinal}\}$. In the first case, by the last sentence of the second paragraph of **1**, we have $\lambda_0 \geq \aleph_0$.

3. We begin with the case when this initial segment is $\{\lambda \mid \lambda < \lambda_0\}$. No T_2 uniform space in $\mathrm{Ob}\,\mathcal{C}$ can have a covering character greater than λ_0, since such a space contains a closed subspace D_{λ_0}, and then we would have $D_{\lambda_0} \in \mathrm{Ob}\,\mathcal{C}$.

Thus it remains to show that also conversely, a T_2 uniform space with covering character at most λ_0 belongs to $\mathrm{Ob}\,\mathcal{C}$. We will prove this in three steps:
1) for complete metric spaces, with the induced uniformities,
2) for any metric spaces, with the induced uniformities,
3) for any T_2 uniform spaces.

3.1. Let M be a complete metric space with $\mathrm{cov\,char}\, M \leq \lambda_0$. By Lemma 2 there is a dense, uniformly continuous map from a closed subspace of a countable product (taken in $\mathbf{T_2Unif}$) $\prod_{n=1}^\infty D_{\lambda_n}$ to M — where D_{λ_n} is a discrete uniform space of cardinality λ_n $(< \lambda_0)$.

Therefore we have for each n that $D_{\lambda_n} \in \mathrm{Ob}\,\mathcal{C}$, hence $\prod_{n=1}^\infty D_{\lambda_n} \in \mathrm{Ob}\,\mathcal{C}$, hence all closed subspaces of $\prod_{n=1}^\infty D_{\lambda_n}$ belong to $\mathrm{Ob}\,\mathcal{C}$, as well as all dense images of these closed subspaces belong to $\mathrm{Ob}\,\mathcal{C}$. Therefore $M \in \mathrm{Ob}\,\mathcal{C}$ for any complete metric space M with $\mathrm{cov\,char}\, M \leq \lambda_0$, with the induced uniformity.

3.2. Let (M, ϱ) be a metric space with $\mathrm{cov\,char}\, M \leq \lambda_0$. As in the proof of Lemma 2 we may assume $\mathrm{diam}\, M < 1$. Then its completion $\gamma(M, \varrho) =: (\gamma M, \tilde{\varrho})$ has the same covering character, hence, by **3.1**, belongs to $\mathrm{Ob}\,\mathcal{C}$. Let $m_0 \in \gamma M$ be arbitrary, but fixed. Then $(\gamma M) \setminus \{m_0\} = f^{-1}((0,1])$, where $f : \gamma M \to [0,1]$ is defined as $f(m) := \tilde{\varrho}(m_0, m)$, for each $m \in \gamma M$. Since $\gamma M \in \mathrm{Ob}\,\mathcal{C}$ and $(0,1] \in \mathrm{Ob}\,\mathcal{C}$, therefore, by 2) of Lemma 1, $(\gamma M) \setminus \{m_0\} = f^{-1}((0,1]) \in \mathrm{Ob}\,\mathcal{C}$. Then 1) of Lemma 1 implies that any subspace of γM belongs to $\mathrm{Ob}\,\mathcal{C}$. In particular, $M \in \mathrm{Ob}\,\mathcal{C}$, for any metric space M with $\mathrm{cov\,char}\, M \leq \lambda_0$, with the induced uniformity.

3.3. Let X be a T_2 uniform space with $\mathrm{cov\,char}\, X \leq \lambda_0$. Then X is a subspace of a product of metric spaces M_α, for $\alpha \in A$. We may suppose that the restriction of each projection $\pi_\alpha : \prod_{\alpha \in A} M_\alpha \to M_\alpha$ to X is surjective. Then, for each $\alpha \in A$, we have $\mathrm{cov\,char}\, M_\alpha \leq \lambda_0$, hence, by **3.2**, $M_\alpha \in \mathrm{Ob}\,\mathcal{C}$. Now let us embed each M_α to $N_\alpha := M_\alpha \times [0,1]$, via $m_\alpha \mapsto (m_\alpha, 0)$, for $m_\alpha \in M_\alpha$. Then $\mathrm{cov\,char}\, N_\alpha = \mathrm{cov\,char}\, M_\alpha \leq \lambda_0$, for each $\alpha \in A$.

Let $n_\alpha \in N_\alpha$ be arbitrary. Then $\mathrm{cov\,char}\,(N_\alpha \setminus \{n_\alpha\}) = \mathrm{cov\,char}\, N_\alpha \leq \lambda_0$, hence, by **3.2**,

$$N_\alpha \setminus \{n_\alpha\} \in \mathrm{Ob}\,\mathcal{C}, \text{ and therefore } \prod_{\alpha \in A} (N_\alpha \setminus \{n_\alpha\}) \in \mathrm{Ob}\,\mathcal{C}.$$

Since n_α is not an isolated point of N_α, therefore $\prod_{\alpha \in A}(N_\alpha \setminus \{n_\alpha\})$ is dense in $\prod_{\alpha \in A} N_\alpha$, hence any subspace of $\prod_{\alpha \in A} N_\alpha$, containing $\prod_{\alpha \in A}(N_\alpha \setminus \{n_\alpha\})$ (as a dense subspace), belongs to $\mathrm{Ob}\,\mathcal{C}$. In particular,

$$\Big(\prod_{\alpha \in A} N_\alpha\Big) \setminus \{\langle n_\alpha \rangle\} \in \mathrm{Ob}\,\mathcal{C}, \text{ for arbitrary } \langle n_\alpha \rangle \in \prod_{\alpha \in A} N_\alpha.$$

By 1) of Lemma 1, then any subspace of $\prod_{\alpha \in A} N_\alpha$ belongs to $\mathrm{Ob}\,\mathcal{C}$. In particular, any subspace of $\prod_{\alpha \in A} M_\alpha$, e.g., X, belongs to $\mathrm{Ob}\,\mathcal{C}$, for any T_2 uniform space X with $\mathrm{cov\,char}\, X \leq \lambda_0$.

Together with the first paragraph of **3** this gives that $\mathrm{Ob}\,\mathcal{C} = \{X \in \mathrm{Ob}\,\mathbf{T_2Unif} \mid X$ has a covering character at most $\lambda_0\}$.

4. There remains the case, from the case distinction in **2**, when all uniformly discrete spaces D_λ (of cardinality λ) belong to $\mathrm{Ob}\,\mathcal{C}$. Then by the above proof, for any cardinal λ_0, all T_2 uniform spaces with covering character at most λ_0 belong to $\mathrm{Ob}\,\mathcal{C}$. That is, all T_2 uniform spaces belong to $\mathrm{Ob}\,\mathcal{C}$, hence $\mathcal{C} = \mathbf{T_2Unif}$.

5. There remains to give four examples. These are (except the second one) the same as in [P]. All but the 1-st, 2-nd, 3-rd or 4-th properties are satisfied by the subclasses of $\mathbf{T_2Unif}$ consisting of finite spaces, of spaces with connected topology, of spaces where the closure of any at most countably infinite set is compact, or of 0-dimensional compact T_2 uniform spaces, respectively. ■

Proof of Theorem 3. We will follow the proofs of Theorems 1 and 2. We only need to prove 1) $\Longrightarrow$ 2).

1. Like in the proof of Theorem 1, second paragraph, we see that $\{X \in \mathrm{Ob}\,\mathbf{Unif} \mid |X| \leq 1\} \subset \mathrm{Ob}\,\mathcal{C}$. If here we have equality, we are done.

Now suppose that here we do not have equality, i.e., $\mathrm{Ob}\,\mathcal{C}$ contains a uniform space X with at least two points. Then X has a subspace consisting of two points, thus some two-point space belongs to $\mathrm{Ob}\,\mathcal{C}$. Then its bijective image, the two-point indiscrete space I_2 also belongs to $\mathrm{Ob}\,\mathcal{C}$. Then any subspace of any power of I_2 belongs to $\mathrm{Ob}\,\mathcal{C}$, hence $\{X \in \mathrm{Ob}\,\mathbf{Unif} \mid X$ is indiscrete$\} \subset \mathrm{Ob}\,\mathcal{C}$. If here we have equality, we are done.

Now suppose that here we do not have equality, i.e., $\mathcal{C}$ contains a non-indiscrete uniform space X. Then X has a subspace consisting of two points, that is a discrete two-point space D_2, and that has to belong to $\mathrm{Ob}\,\mathcal{C}$. Then all subspaces of all finite powers of D_2 belong to $\mathrm{Ob}\,\mathcal{C}$, i.e., $\{X \in \mathrm{Ob}\,\mathbf{Unif} \mid X$ is a finite discrete space$\} \subset \mathrm{Ob}\,\mathcal{C}$.

2. The class of cardinalities λ (finite or infinite), for which the discrete space D_λ of cardinality λ belongs to $\mathrm{Ob}\,\mathcal{C}$, forms an initial segment of all cardinalities, i.e., it is of the form $\{\lambda \mid \lambda < \lambda_0\}$, or $\{\lambda \mid \lambda$ is a cardinal$\}$. In the first case, by the last sentence of **1**, we have $\lambda_0 \geq \aleph_0$.

3. We begin with the case when this initial segment is $\{\lambda \mid \lambda < \lambda_0\}$. No uniform space in $\mathrm{Ob}\,\mathcal{C}$ can have a covering character greater than λ_0, since such a space contains a (closed) subspace D_{λ_0}, and then we would have $D_{\lambda_0} \in \mathrm{Ob}\,\mathcal{C}$.

Thus it remains to show that, also conversely, a uniform space with covering character at most λ_0 belongs to $\mathrm{Ob}\,\mathcal{C}$.

Let $\lambda < \lambda_0$, and let I be an indiscrete space. Then by **1** and **2** we have that $I, D_\lambda \in \mathrm{Ob}\,\mathcal{C}$, hence each subspace of $D_\lambda \times I$ belongs to $\mathrm{Ob}\,\mathcal{C}$. That is, each uniform space, with underlying set X, say, that has a covering base consisting of a single partition P of cardinality λ, belongs to $\mathrm{Ob}\,\mathcal{C}$. Let us denote this space X by X_P.

Now let Y be a uniform space with underlying set X, having a covering base $\mathcal{P}$ consisting of all partitions P of X, of cardinalities $|P| < \lambda_0$. Then Y can be embedded to $\prod_{P \in \mathcal{P}} X_P$ ($\in \mathrm{Ob}\,\mathcal{C}$) via the diagonal map. That is, the subspace of this product space, which is the diagonal, is isomorphic to Y. Therefore also $Y \in \mathrm{Ob}\,\mathcal{C}$.

Of course, Y has another covering base, consisting of all covers of X of cardinalities less than λ_0. This implies that any uniform structure on the underlying set X, having a covering base consisting of covers of cardinalities less than λ_0, is a bijective image of Y, hence belongs to $\mathrm{Ob}\,\mathcal{C}$ as well. That is, any uniform space, with covering character at most λ_0, belongs to $\mathrm{Ob}\,\mathcal{C}$.

Together with the first paragraph of **3** this gives that $\mathrm{Ob}\,\mathcal{C} = \{X \in \mathrm{Ob}\,\mathbf{Unif} \mid X$ has a covering character at most $\lambda_0\}$.

4. There remains the case, from the case distinction in **2**, when all uniformly discrete spaces D_λ (of cardinality λ) belong to $\mathrm{Ob}\,\mathcal{C}$. Then by the above proof, for any cardinal λ_0, all uniform spaces with covering character at most λ_0 belong to $\mathrm{Ob}\,\mathcal{C}$. That is, all uniform spaces belong to $\mathrm{Ob}\,\mathcal{C}$, hence $\mathcal{C} = \mathbf{Unif}$.

5. There remains to give three examples. The first one is the same as in [P]. All but the 1-st, 2-nd or 3-rd properties are satisfied by the subclasses of **Unif** consisting of finite spaces, of spaces with connected topology, or by T_2 uniform spaces, respectively. ■

Before the proof of Theorems 4, 5, 6 we give a lemma. Lemma 3, 1) is surely known (algebraic subcategories of **Set**), but could not locate it, therefore we give its simple proof.

Lemma 3. *Let* $\mathbf{T}$ *be* **Top**, **Prox** *or* **Unif**. *Let* $\mathcal{C} \subset \mathbf{T}$ *be algebraic.*

1) If $\mathrm{Ob}\,\mathcal{C} \subset \{X \in \mathrm{Ob}\,\mathbf{T} \mid X$ *is indiscrete*$\}$, *then* $\mathrm{Ob}\,\mathcal{C} = \{X \in \mathrm{Ob}\,\mathbf{T} \mid |X| = 1\}$, *or* $\mathrm{Ob}\,\mathcal{C} = \{X \in \mathrm{Ob}\,\mathbf{T} \mid |X| \le 1\}$, *or* $\mathrm{Ob}\,\mathcal{C} = \{X \in \mathrm{Ob}\,\mathbf{T} \mid X$ *is indiscrete*$\}$.

2) If $\mathbf{T}$ *is* **Prox** *or* **Unif**, *and* $\mathcal{C}$ *is closed under products and closed subspaces, then either* $\mathrm{Ob}\,\mathcal{C} = \{X \in \mathrm{Ob}\,\mathbf{T} \mid X$ *is indiscrete*$\}$, *or* $\mathrm{Ob}\,\mathcal{C} \subset \{X \in \mathrm{Ob}\,\mathbf{T} \mid X$ *is* $T_2\}$.

Proof. **1.** We begin with the proof of 1).

The category $\mathcal{C}$, as a category in its own right, has products, which are preserved by the underlying set functor. Therefore the empty product, the one-point algebra (space) belongs to $\mathrm{Ob}\,\mathcal{C}$. Then either $\emptyset \notin \mathrm{Ob}\,\mathcal{C}$ or $\emptyset \in \mathrm{Ob}\,\mathcal{C}$. Therefore

$$\{X \in \mathrm{Ob}\,\mathbf{T} \mid |X| = 1\} \subset \mathrm{Ob}\,\mathcal{C}\ (\not\ni \emptyset), \text{ or } \{X \in \mathrm{Ob}\,\mathbf{T} \mid |X| \le 1\} \subset \mathrm{Ob}\,\mathcal{C}.$$

If in one of these inclusions we have equality, we are done.

Therefore we may suppose that some space X belongs to $\mathrm{Ob}\,\mathcal{C}$, where $|X| \ge 2$.

Then X is indiscrete, and all powers X^α of X, taken in $\mathcal{C}$, belong to $\mathrm{Ob}\,\mathcal{C}$. Now, the underlying set functor U of $\mathcal{C}$ preserves products, hence these products are the indiscrete structures on $(UX)^\alpha$ (i.e., the powers taken in $\mathbf{T}$), hence indiscrete spaces of arbitrarily large cardinality belong to $\mathrm{Ob}\,\mathcal{C}$. Let us consider $X^\alpha \in \mathrm{Ob}\,\mathcal{C}$. Let us consider any subset Y of X^α. Let $x_1, x_2 \in X$, with $x_1 \ne x_2$ (recall $|X| \ge 2$). Let us consider the $\mathbf{T}$-morphisms (hence $\mathcal{C}$-morphisms) $f, g : X^\alpha \to X$, defined by $f(z) = x_1$ for all $z \in X^\alpha$, and $g(z) = x_1$ for all $z \in Y$ and $g(z) = x_2$ for all $z \in X^\alpha \setminus Y$. Recall that the equalizer of f, g is preserved by the underlying set functor of $\mathcal{C}$, hence $Y \in \mathrm{Ob}\,\mathcal{C}$ (up to isomorphy, but $\mathrm{Ob}\,\mathcal{C}$ is isomorphism closed). Therefore all indiscrete spaces of cardinality at most $|X^\alpha|$ belong to $\mathrm{Ob}\,\mathcal{C}$. Hence $\{C \in \mathrm{Ob}\,\mathbf{T} \mid C$ is indiscrete$\} \subset \mathrm{Ob}\,\mathcal{C}$. Since the converse inclusion holds by hypothesis, we have here in fact equality. (Cf. also the proof of [R91a], Proposition 1.1.)

2. We turn to the proof of 2).

If $\mathrm{Ob}\,\mathcal{C} \subset \{X \in \mathrm{Ob}\,\mathbf{T} \mid X \text{ is } T_2\}$, we are done. Therefore let $\mathrm{Ob}\,\mathcal{C} \not\subset \{X \in \mathrm{Ob}\,\mathbf{T} \mid X \text{ is } T_2\}$, and let us choose $C \in \mathcal{C}$ that is not T_2, i.e., that is not T_0. Then some point of C is not closed, and its closure, X, say, is a closed indiscrete subset of C, with $|X| > 1$. Then, by closed hereditariness and productivity of $\mathcal{C}$, we have $\emptyset, X \in \mathrm{Ob}\,\mathcal{C}$, and also any power X^α belongs to $\mathrm{Ob}\,\mathcal{C}$. Then repeating the considerations in 1) we obtain $\{X \in \mathrm{Ob}\,\mathbf{T} \mid X \text{ is indiscrete}\} \subset \mathrm{Ob}\,\mathcal{C}$. If here we have equality, we are done.

Therefore we may assume that some non-indiscrete space C belongs to $\mathrm{Ob}\,\mathcal{C}$. Then the indiscrete space I on UC also belongs to $\mathrm{Ob}\,\mathcal{C}$, and we have a bijection $b : C \to I$ that is not an isomorphism in $\mathbf{T}$, hence it is not an isomorphism in $\mathcal{C}$ either. However, for an algebraic category $\mathcal{C}$, the underlying set functor U reflects isomorphisms, and we have a contradiction. ■

Proof of Theorem 4. We only need to prove 1) $\Longrightarrow$ 2).

1. The left adjoint of the underlying set functor $U : \mathcal{C} \to \mathbf{Set}$ will be denoted by F. Objects of $\mathbf{T}$ will be called *spaces*, and if we investigate an object of $\mathcal{C}$, it will be called an *algebra*.

Like in the proof of Theorem 1, second paragraph, we see that

$$\{X \in \mathrm{Ob}\,\mathbf{T} \mid |X| \le 1\} \subset \mathrm{Ob}\,\mathcal{C}. \tag{*}$$

If here we have equality, we are done. Therefore let $\mathrm{Ob}\,\mathcal{C}$ contain an object C with $|UC| \ge 2$.

2. We distinguish two cases:

1) $\mathrm{Ob}\,\mathcal{C} \subset \{\text{indiscrete spaces in } \mathrm{Ob}\,\mathbf{T}\}$, or

2) there exists $C \in \mathrm{Ob}\,\mathcal{C}$, such that C (as an object of $\mathbf{T}$) is not indiscrete.

3. In the first case, by Lemma 3, 1) and (*), we have $\mathrm{Ob}\,\mathcal{C} = \{X \in \mathrm{Ob}\,\mathbf{T} \mid |X| \le 1\}$ or $\mathrm{Ob}\,\mathcal{C} = \{X \in \mathrm{Ob}\,\mathbf{T} \mid X \text{ is indiscrete}\}$.

4. In the second case $\mathrm{Ob}\,\mathcal{C}$ contains a non-indiscrete object C, hence as its subspace, also contains a non-indiscrete object with two points. Hence we may suppose that $|UC| = 2$. Then, for $\mathbf{T} = \mathbf{Prox}$ and $\mathbf{T} = \mathbf{Unif}$ we have that C is the two-point discrete space D_2. For $\mathbf{T} = \mathbf{Top}$ we have that C is the two-point discrete space, or the Sierpiński space. However, observe that the square of the Sierpiński space contains a two-point discrete subspace, so we may assume that C is the two-point discrete space D_2 for $\mathbf{T} = \mathbf{Top}$ as well.

Let $UC = UD_2 = \{c_1, c_2\}$. Let S be a set with $|S| \ge 2$, and let us consider the free algebra $FS \in \mathrm{Ob}\,\mathcal{C}$. We have the unit of adjunction $\eta_S : S \to UFS$. Then for any $\mathbf{Set}$-morphism $f : S \to \{c_1, c_2\} = UC$, there exists a $\mathcal{C}$-morphism $\varphi : FS \to C$ such that $f = (U\varphi) \circ \eta_S$. This readily implies that η_S is an injection. Moreover it also implies that $\eta_S S$ ($\subset UFS$), considered as a subspace of FS, that (by hereditariness of $\mathcal{C}$) satisfies $\eta_S S \in \mathrm{Ob}\,\mathcal{C}$, also satisfies the following. It has a topology/proximity finer than (thus equal to) the one projectively generated by all $\mathbf{Set}$-morphisms to $\{c_1, c_2\} = UC$, i.e., the discrete topology/proximity on $U(\eta_S S)$. Or it has a uniformity finer than the finest precompact uniformity on $U(\eta_S S)$ (i.e., the one having as a covering base all finite partitions of $U(\eta_S S)$), respectively.

Now let $\mathbf{T} = \mathbf{Top}$ or $\mathbf{T} = \mathbf{Prox}$. Then, by epireflectivity, this discrete subspace $\eta_S S$ belongs to $\mathrm{Ob}\,\mathcal{C}$ (and the discrete spaces of cardinality at most 1 belong to $\mathrm{Ob}\,\mathcal{C}$ by (*)). Hence $\mathrm{Ob}\,\mathcal{C} \supset \{\text{discrete spaces in } \mathbf{T}\}$. For $\mathbf{T} = \mathbf{Unif}$ the same reasoning gives only that the uniformity $\eta_S S$ on $U(\eta_S S)$, finer than the finest precompact uniformity on $U(\eta_S S)$, belongs to $\mathrm{Ob}\,\mathcal{C}$.

The two-point discrete space D_2 in $\mathbf{T}$ belongs to $\mathrm{Ob}\,\mathcal{C}$. Hence, by epireflectivity, also $D_2^{\aleph_0} \in \mathrm{Ob}\,\mathcal{C}$, where the power is taken in $\mathbf{T}$. However, $D_2^{\aleph_0}$ is the Cantor set, or the Cantor set with its unique compatible proximity, or the Cantor set with its unique compatible uniformity, respectively. Let

$$S := U(D_2^{\aleph_0}) = (UD_2)^{\aleph_0}.$$

For simplicity, we assume that the embedding $\eta_S : S \to UFS$ is pointwise identical. Then for $\mathbf{T} = \mathbf{Top}$ and $\mathbf{T} = \mathbf{Prox}$ the space $\eta_S S$ (as a subspace of FS) is discrete, hence is strictly finer than $D_2^{\aleph_0}$. For $\mathbf{T} = \mathbf{Unif}$ the space $\eta_S S$ is finer than the finest precompact uniformity on $U(\eta_S S)$. In all three cases the space $\eta_S S$ is strictly finer than $D_2^{\aleph_0}$. Thus the identical bijection $b : \eta_S S \to D_2^{\aleph_0}$ is not an isomorphism in $\mathbf{T}$, hence not an isomorphism in $\mathcal{C}$ either. However, for an algebraic category $\mathcal{C}$, the underlying set functor U reflects isomorphisms, and we have a contradiction. Hence case 2) in **2** from our case distinction cannot exist.

5. There remains to give three examples. All but the 1-st, 2-nd or 3-rd properties are satisfied by examples 1) and 2) of [HS71] and by 3) the category of T_2 topological, proximity or uniform spaces (observe that their underlying set functors do not reflect isomorphisms therefore they are not algebraic). In 1) we mean discrete topological, proximity or uniform spaces, which form even a varietal category [HS71]. In 2) we mean powers of a compact T_2 topological space, or of the same space with its unique compatible proximity or uniformity, consisting of more than one point, that is strongly rigid — i.e., whose only continuous self-maps are the identity and the constant maps (such spaces exist, cf. [HS71]) — together with the empty space, which category is even varietal, cf. [R92a], p. 368. ■

A large part of the next proof is taken from [R82] and [R85a].

Proof of Theorem 5. **1.** The implication 2) $\Rightarrow$ 1) is evident in the first case. For the second case observe that the category of compact T_2 uniform spaces is canonically concretely isomorphic to the category of compact T_2 topological spaces (via induced topology/unique compatible uniformity). Thus our implication reduces to the analogous implication for the category of compact T_2 topological spaces, that follows from [R82], Corollary 1.5.

2. We repeat **1**, **2**, **3** from the proof of Theorem 4 word for word. However, observe that $\{X \in \mathrm{Ob}\,\mathbf{Unif} \mid |X| \leq 1\}$ is an epireflective subcategory of $\{X \in \mathrm{Ob}\,\mathbf{Unif} \mid X \text{ is compact } T_2\}$.

By the first sentence of **4** from the proof of Theorem 4, $\mathrm{Ob}\,\mathcal{C}$ contains a non-indiscrete object C.

Then by Lemma 3, 2) we have $\mathrm{Ob}\,\mathcal{C} \subset \mathrm{Ob}\,\mathbf{T_2Unif}$.

3. From now on we follow [R82].

Lemma 1.1 of [R82] will become the following. Let every bijection in $\mathcal{C}$ be a uniform isomorphism, let β be a limit ordinal, and let $[0, \beta]$ be the usual (compact) ordinal space, with the unique uniformity compatible with its order topology. Further, let $\mathcal{U}$ be a uniformity on the ordinal space $[0, \beta)$, which is finer than the precompact uniformity inherited from its compactification $[0,\beta]$ (i.e., the coarsest — precompact — uniformity compatible with its order topology). Then

$$[0,\beta] \in \mathrm{Ob}\,\mathcal{C} \Longrightarrow ([0,\beta),\mathcal{U}) \notin \mathrm{Ob}\,\mathcal{C}\,.$$

The proof remains the same.

In the statement of Proposition 1.2 of [R82], **Top** has to be replaced by **Unif**, and of course, D_2 is now a discrete uniform space on two points. The proof remains the same, of course replacing topological products and coproducts by uniform ones.

The statement of Lemma 1.3 of [R82] remains word for word the same, of course replacing **Top** by **Unif**, and also its proof remains the same.

The assertion of Theorem 1.4 of [R82] remains the same, of course replacing **Top** by **Unif**. In the proof the following changes have to be made. Everywhere, rather than the ordinal space $[0,\alpha]$, with its order topology, we consider the unique uniformity compatible with its order topology. Moreover, rather than the ordinal spaces $[0,\beta)$ or $[\alpha+1,\beta)$, with their order topologies (where $\alpha<\beta$), we consider the respective compatible uniformities on them, that are the above mentioned precompact uniformities inherited from their compactifications $[0,\beta]$ or $[\alpha+1,\beta]$ (i.e., their coarsest compatible uniformities). For $\alpha<\beta$, the set $[0,\alpha]$ is not just clopen in $[0,\beta)$, but together with its complement form a uniform cover of $[0,\beta)$. Accordingly, topological coproduct at this place is replaced by uniform coproduct.

Then the statements of Corollaries 1.5 and 1.6 of [R82] remain word for word the same, and also their proofs carry over. (Actually, for [R82], Corollary 1.5, after the first step of its proof, namely that $\mathrm{Ob}\,\mathcal{C} \subset \{X \in \mathrm{Ob}\,\mathbf{T_2Unif} \mid X \text{ is compact } (T_2)\}$, we can use the canonical concrete isomorphism of the categories of compact T_2 topological and compact T_2 uniform spaces mentioned in **1** of this proof, and then just we have to apply the result of [R82], Corollary 1.5, not repeat its proof.)

4. There remains to give three examples (in **Unif**). These are the same as in **5** of the proof of Theorem 4, of course in the third case meaning only the category $\mathbf{T_2Unif}$. Observe that examples 1) and 2) are even varietal, and examples 1) ([HS71]) and 3) are closed even for any subspaces. ■

A large part of the next proof is taken from [HS71] and [R85a].

First proof of Theorem 6. We only need to prove 1) $\Longrightarrow$ 2).

1. Like in the proof of Theorem 1, second paragraph, we see that $\{X \in \mathrm{Ob}\,\mathbf{Unif} \mid |X| \leq 1\} \subset \mathrm{Ob}\,\mathcal{C}$. If here we have equality, we are done.

Therefore we may suppose that some space X belongs to $\mathrm{Ob}\,\mathcal{C}$, where $|X| \geq 2$.

2. Now we make a case distinction. Either

1) $\mathrm{Ob}\,\mathcal{C} \not\subset \mathrm{Ob}\,\mathbf{T_2Unif}$, or

2) $\mathrm{Ob}\,\mathcal{C} \subset \mathrm{Ob}\,\mathbf{T_2Unif}$.

3. We begin with case 1). Then, by Lemma 3, 2), we have $\mathrm{Ob}\,\mathcal{C} = \{X \in \mathrm{Ob}\,\mathbf{Unif} \mid X \text{ is indiscrete}\}$. (Cf. also the proof of [R91a], Proposition 1.1.)

4. We turn to case 2), i.e., when $\mathrm{Ob}\,\mathcal{C} \subset \mathrm{Ob}\,\mathbf{T_2Unif}$. Then we can repeat the proof of Lemmas 1 and 2, Corollaries 1 and 2 and the Theorem from [HS71].

We only have to change the words topological spaces, continuous maps, topological quotient spaces and maps etc. to their uniform counterparts. Thus we obtain that

$$\mathrm{Ob}\,\mathcal{C} = \{X \in \mathrm{Ob}\,\mathbf{Unif} \mid |X| \le 1\}, \text{ or } \mathrm{Ob}\,\mathcal{C} = \{X \in \mathrm{Ob}\,\mathbf{Unif} \mid X \text{ is compact } T_2\}.$$

5. There remains to give three examples (in **Unif**). These are the same as in **5** of the proof of Theorem 4, of course in the third case meaning only the category $\mathbf{T_2Unif}$. Observe that examples 1) ([HS71]) and 3) are closed even for any subspaces. ■

Of course, Theorem 6 also follows from Theorem 5. However, this proof of Theorem 6, although is shorter to write, is in fact more complicated. Namely, in the above first proof of Theorem 6 we used the proof from [HS71], which is simpler than the proof from [R82] of a more general theorem, used in the proof of Theorem 5.

Second proof of Theorem 6. We only deal with 1) $\Longrightarrow$ 2).

Parts **1**, **2**, **3** from the above proof of Theorem 6 are just copied. Thus, in particular, we assume that some uniform space X belongs to $\mathrm{Ob}\,\mathcal{C}$, where $|X| \ge 2$, and $\mathrm{Ob}\,\mathcal{C} \subset \mathrm{Ob}\,\mathbf{T_2Unif}$.

Since a varietal category is algebraic, we have by Theorem 5 that $\mathcal{C}$ is an epireflective subcategory of compact T_2 uniform spaces. By the canonical concrete isomorphism of the categories of compact T_2 uniform spaces and compact T_2 topological spaces (from **1** of the proof of Theorem 5), we have a corresponding epireflective and varietal subcategory $\mathcal{C}'$ of compact T_2 topological spaces. Then [HS71] Theorem or [R82], Corollary 1.6 implies that

$$\begin{cases} \mathrm{Ob}\,\mathcal{C}' = \{\text{compact } T_2 \text{ topological spaces}\}, \\ \text{ hence } \mathrm{Ob}\,\mathcal{C} = \{\text{compact } T_2 \text{ uniform spaces}\}. \end{cases}$$

■

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