Partitioning orthogonal polygons into $\leq$ 8-vertex pieces, with application to an art gallery theorem

Ervin Győri[a,b,1], Tamás Róbert Mezei[b,*]

[a] *MTA Alfréd Rényi Institute of Mathematics, Reáltanoda u. 13–15, 1053 Budapest, Hungary*
[b] *Central European University, Department of Mathematics and its Applications, Nádor u. 9, 1051 Budapest, Hungary*

Abstract

We prove that every simply connected orthogonal polygon of n vertices can be partitioned into $\lfloor \frac{3n+4}{16} \rfloor$ (simply connected) orthogonal polygons of at most 8 vertices. It yields a new and shorter proof of the theorem of A. Aggarwal that $\lfloor \frac{3n+4}{16} \rfloor$ mobile guards are sufficient to control the interior of an n-vertex orthogonal polygon. Moreover, we strengthen this result by requiring combinatorial guards (visibility is only needed at the endpoints of patrols) and prohibiting intersecting patrols. This yields positive answers to two questions of O'Rourke [7, Section 3.4]. Our result is also a further example of the "metatheorem" that (orthogonal) art gallery theorems are based on partition theorems.

Keywords: art gallery, polyomino partition, mobile guard

1. Introduction

First let us define the basic object studied in this paper.

Definition 1. *A simply connected orthogonal polygon or simple polyomino P is the closed region of the plane bounded by a closed (not self-intersecting) polygon, whose angles are all $\pi/2$ (convex) or $3\pi/2$ (reflex). We denote the number of P's vertices by $n(P)$.*

This definition implies that $n(P)$ is even. We want to emphasize the combinatorial structure of P rather than its geometry, so in this paper we refer to such objects as simple polyominoes, or polyominoes for short. When we talk about not necessarily simply connected polyominoes, we explicitly state it.

Kahn, Klawe and Kleitman [6] in 1980 proved that $\lfloor n/4 \rfloor$ guards are sometimes necessary and always sufficient to cover the interior of a simple polyomino of n vertices. Later the first author of this paper provided a simple and short proof of

*Corresponding author
Email addresses: ervin@renyi.hu (Ervin Győri), tamasrobert.mezei@gmail.com (Tamás Róbert Mezei)
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Theorem 2 ([3], [7, Thm. 2.5]). *Every simple polyomino of n vertices can be partitioned into $\lfloor n/4 \rfloor$ polyominoes of at most 6 vertices.*

Theorem 2 is a deeper result than that of Kahn, Klawe and Kleitman, and gave the first hint of the existence of a "metatheorem": (orthogonal) art gallery theorems have underlying partition theorems. The general case was proved by Hoffmann [5].

Theorem 3 ([5]). *Every (not necessarily simply connected!) orthogonal polygon with n vertices can be covered by $\lfloor n/4 \rfloor$ guards.*

Hoffmann's method (partitioning into smaller pieces that can be covered by one guard) is another example of the metatheorem.

In this paper, we present further evidence that the metatheorem holds, namely we prove the following partition theorem:

Theorem 4. *Any simple polyomino of n vertices can be partitioned into at most $\lfloor \frac{3n+4}{16} \rfloor$ simple polyominoes of at most 8 vertices.*

A mobile guard is one who can patrol a line segment, and it covers a point x of the gallery if there is a point y on its patrol such that the line segment $[x, y]$ is contained in the gallery. The mobile guard art gallery theorem for simple polyominoes follows immediately from Theorem 4, as a polyomino of at most 8 vertices can be covered by a mobile guard:

Theorem 5 ([1], proof also in [7, Thm. 3.3]). $\lfloor \frac{3n+4}{16} \rfloor$ *mobile guards are sufficient for covering an n vertex simple polyomino.*

Theorem 4 is a stronger result than Theorem 5 and it is interesting on its own. It fits into the series of results in [3], [5], [7, Thm. 2.5], [4] showing that rectilinear art gallery theorems are based on theorems on partitions of polyominoes into smaller ("one guardable") polyominoes.

Moreover, Theorem 4 directly implies the following corollary which strengthens the previous theorem and answers two questions raised by O'Rourke [7, Section 3.4].

Corollary 6. $\lfloor \frac{3n+4}{16} \rfloor$ *mobile guards are sufficient for covering an n vertex simple polyomino such that the patrols of two guards do not pass through one another and visibility is only required at the endpoints of the patrols.*

The proof of Theorem 4 is similar to the proofs of Theorem 2 in that it finds a suitable cut and then uses induction on the parts created by the cut. However, here a cut along a line segment connecting two reflex vertices is not automatically good. In case we have no such cuts, we also rely heavily on a tree structure of

the polyomino (Section 3). However, while O'Rourke's proof only uses straight cuts, in our case this is not sufficient: Figure 1 shows a polyomino of 14 vertices which cannot be cut into 2 polyominoes of at most 8 vertices using cuts along straight lines. Therefore we must consider L-shaped cuts too.

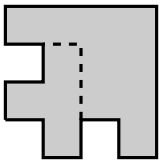


Figure 1: An L-shaped cut creating a partition into 8-vertex polyominoes

Interested readers can find a thorough introduction to the subject of art gallery problems in [7].

2. Definitions and preliminaries

Let P, P_1, P_2 be simple polyominoes of n, n_1, n_2 vertices, respectively. If $P = P_1 \cup P_2$, $\text{int}(P_1) \cap \text{int}(P_2) = \emptyset$, $0 < n_1, n_2$ and $n_1 + n_2 \leq n + 2$ are satisfied, we say that P_1, P_2 form an admissible partition of P, which we denote by $P = P_1 \overset{\star}{\cup} P_2$. Also, we call $L = P_1 \cap P_2$ a cut in this case. We may describe this relationship concisely by $L(P_1, P_2)$. If, say, we have a number of cuts L_1, L_2, etc., then we usually write $L_i(P_1^i, P_2^i)$. Generally, if a subpolyomino is denoted by P_x^y, then y refers to a cut and $x \in \{1, 2\}$ is the label of the piece in the partition created by said cut. Furthermore, if

$$\left\lfloor \frac{3n_1 + 4}{16} \right\rfloor + \left\lfloor \frac{3n_2 + 4}{16} \right\rfloor \leq \left\lfloor \frac{3n + 4}{16} \right\rfloor . \tag{1}$$

is also satisfied, we say that P_1, P_2 form an induction-good partition of P, and we call L a **good cut**.

Lemma 7. *An admissible partition $P = P_1 \overset{\star}{\cup} P_2$ is also and induction-good partition if*

(a) $n_1 + n_2 = n + 2$ *and* $n_1 \equiv 2, 8,$ *or* $14 \pmod{16}$,

or

(b) $n_1 + n_2 = n$ *and* $n_1 \equiv 0, 2, 6, 8, 12,$ *or* $14 \pmod{16}$,

or

(c) $n \not\equiv 14 \mod 16$ *and either*

$$n_1 + n_2 = n \text{ and } n_1 \equiv 10 \pmod{16} \quad \textbf{or} \quad n_1 + n_2 = n + 2 \text{ and } n_1 \equiv 12 \pmod{16}.$$

Proof. Using the fact that the floor function satisfies the triangle inequality, the proof reduces to an easy case-by-case analysis, which we leave to the reader. □

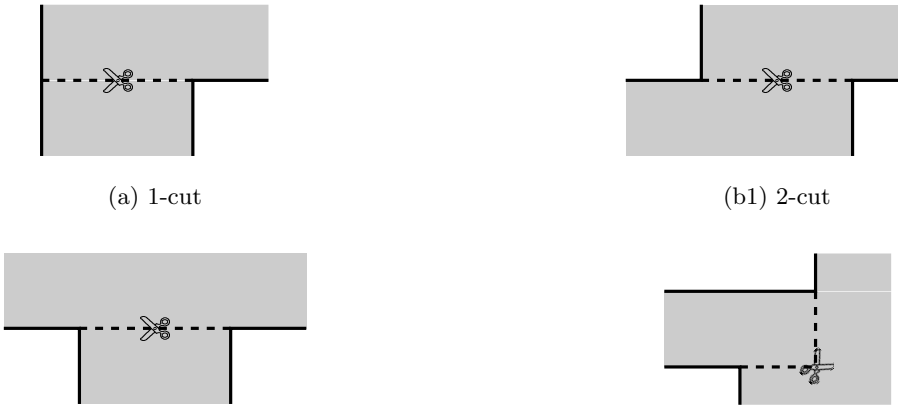
Any cut L in this paper falls into one of the following 3 categories (see Figure 2):

(a) 1-cuts: L is a line segment, and exactly one of its endpoints is a (reflex) vertex of P.

(b) 2-cuts: L is a line segment, and both of its endpoints are (reflex) vertices of P.

(c) L-cuts: L consists of two connected line segments, and both endpoints of L are (reflex) vertices of P.

Note that for 1-cuts and L-cuts the size of the parts satisfy $n_1 + n_2 = n + 2$, while for 2-cuts we have $n_1 + n_2 = n$.



(a) 1-cut (b1) 2-cut

(b2) 2-cut (c) L-cut

Figure 2: Examples for all types of cuts. Light gray areas are subsets of $\mathrm{int}(P)$.

In the proof of Theorem 4 we are searching for an induction-good partition of P. As a good cut defines an induction-good partition, it is sufficient to find a good cut. We could hope that a good cut of a subpolyomino of P is extendable to a good cut of P, but unfortunately a good cut of a subpolyomino may only be an admissible cut with respect to P (if it is a cut of P at all). Lemma 7, however, allows us to look for cut-systems containing a good cut. Fortunately, it is sufficient to consider non-crossing, nested cut-systems of at most 3 cuts, defined as follows.

Definition 8 (Good cut-system)**.** *The cuts $L_1(P_1^1, P_2^1)$, $L_2(P_1^2, P_2^2)$ and $L_3(P_1^3, P_2^3)$ (possibly $L_2 = L_3$) constitute a good cut-system if $P_1^1 \subset P_1^2 \subseteq P_1^3$, and the set*

$$\big\{n(P_1^i) \mid i \in \{1,2,3\}\big\} \cup \big\{n(P_1^i) + 2 \mid i \in \{1,2,3\} \text{ and } L_i \text{ is a 2-cut}\big\}$$

contains three consecutive even elements modulo 16 (ie., the union of their residue classes contains a subset of the form $\{a, a+2, a+4\} + 16\mathbb{Z}$). If this is the case we also define their kernel as $\ker\{L_1, L_2, L_3\} = (P_1^1 - L_1) \cup (P_2^3 - L_3)$, which will be used in Lemma 16.

Lemma 7(a) and 7(b) immediately yield that any good cut-system contains a good cut.

Remark 9. *It is easy to see that if a set of cuts satisfies this definition, then they obviously satisfy it in the reverse order too (the order of the generated parts are also switched). Actually, these are exactly the two*

orders in which they do so. Thus the kernel is well-defined, and when speaking about a good cut-system it is often enough to specify the set of participating cuts.

3. Tree structure

Any reflex vertex of a polyomino defines a (1- or 2-) cut along a horizontal line segment whose interior is contained in $\mathrm{int}(P)$ and whose endpoints are the reflex vertex and another point on the boundary of the polyomino. Next we define a graph structure derived from P, which is a standard tool in the literature, for example it is called the R-graph of P in [4].

Definition 10 (Horizontal cut tree)**.** *The horizontal cut tree T is obtained as follows. First, partition P into a set of rectangles by cutting along all of the horizontal cuts of P. Let $V(T)$, the vertex set of T be the set of resulting (internally disjoint) rectangles. Two rectangles of T are connected by an edge in $E(T)$ iff their boundaries intersect.*

The graph T is indeed a tree as its connectedness is trivial and since any cut creates two internally disjoint polyominoes, T is also cycle-free. We can think of T as a sort of dual of the planar graph determined by P and its horizontal cuts. The nodes of T represent rectangles of P and edges of T represent horizontal 1- and 2-cuts of P. For this reason we may refer to nodes of T as rectangles. This nomenclature also helps in distinguishing between vertices of P (points) and nodes of T. Moreover, for an edge $e \in E(T)$, we may denote the cut represented by e by simply e, as the context should make it clear whether we are working in the graph T or the polyomino P itself.

Note that the vertical sides of rectangles are also edges of the polyomino. We will also briefly use vertical cut trees, which can be defined analogously.

Definition 11. *Let T be the horizontal cut tree of P. Define $t : E(T) \to \mathbb{N}$ as follows: given any edge $\{R_1, R_2\} \in E(T)$, let*

$$t(\{R_1, R_2\}) = n(R_1 \cup R_2) - 8.$$

Observe that

$$t(e) = \begin{cases} 0, & \text{if } e \text{ represents a 2-cut;} \\ -2, & \text{if } e \text{ represents a 1-cut.} \end{cases}$$

The following claim is used throughout the paper to count the number of vertices of subpolyominoes.

Claim 12. *Let T be the horizontal cut tree of P. Then*

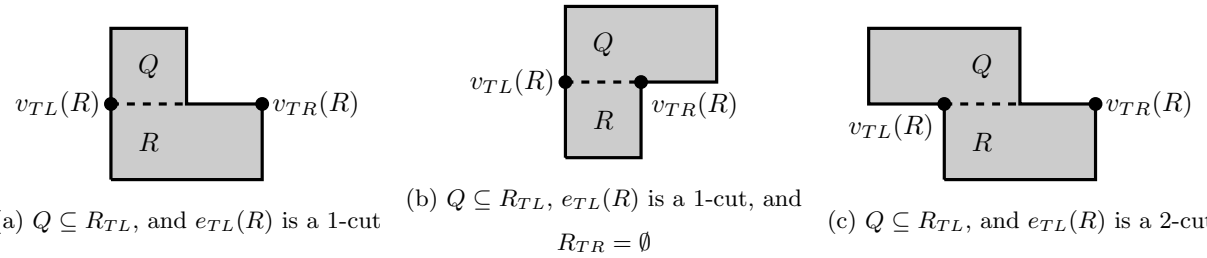
$$n(P) = 4|V(T)| + \sum_{e \in E(T)} t(e).$$

Proof. The proof is straightforward. □

Remark 13. *The equality still holds even if some of the rectangles of T are cut into several rows (and the corresponding edges, for which the function t takes -4, are added to T).*

3.1. Extending cuts and cut-systems

The following two technical lemmas considerably simplify our analysis in Section 4, where many cases distinguished by the relative positions of reflex vertices of P on the boundary of a rectangle need to be handled. For a rectangle R let us denote its top left, top right, bottom left, and bottom right vertices with $v_{TL}(R)$, $v_{TR}(R)$, $v_{BL}(R)$, and $v_{BR}(R)$, respectively.



(a) $Q \subseteq R_{TL}$, and $e_{TL}(R)$ is a 1-cut

(b) $Q \subseteq R_{TL}$, $e_{TL}(R)$ is a 1-cut, and $R_{TR} = \emptyset$

(c) $Q \subseteq R_{TL}$, and $e_{TL}(R)$ is a 2-cut

(d) $R_{TL} = R_{TR} = \emptyset$, R is either a corridor or a pocket (see Section 4)

(e) $R_{TL} = \emptyset$, but $Q \subseteq R_{TR}$

Figure 3: $R \cup Q$ in all essentially different relative positions of $R, Q \in V(T)$, where $\{R, Q\} \in E(T)$ and $v_{TL}(R) \in Q$

Definition 14. *Let $R, Q \in V(T)$ be arbitrary. We say that Q is adjacent to R at $v_{TL}(R)$, if $v_{TL}(R) \in Q$ and $v_{TL}(R)$ is not a vertex of the polyomino $R \cup Q$, or $v_{TR}(R) \notin Q$. Such situations are depicted on Figures 3a, 3b, and 3c. However, in the case of Figure 3d and 3e we have $v_{TR}(R) \in Q \not\subseteq R_{TL}$ $(= \emptyset)$.*

If Q is adjacent to R at $v_{TL}(R)$, let $e_{TL}(R) = \{R, Q\}$; by cutting P along the dual of $e_{TL}(R)$, ie., $R \cap Q$, we get two polyominoes, and we denote the part containing Q by R_{TL}. If there is no such Q, let $e_{TL}(R) = \emptyset$ and $R_{TL} = \emptyset$. These relations can be defined analogously for top right (R_{TR}, $e_{TR}(R)$), bottom left (R_{BL}, $e_{BL}(R)$), and bottom right (R_{BR}, $e_{BR}(R)$).

Lemma 15. *Let R be an arbitrary rectangle such that $R_{BL} \neq \emptyset$. Let U be the remaining portion of the polyomino, ie., $P = R_{BL} \overset{\star}{\cup} U$ is a polyomino-partition. Take an admissible polyomino-partition $U = U_1 \overset{\star}{\cup} U_2$ such that $v_{BL}(R) \in U_1$. We can extend this to an admissible polyomino-partition of P where the two parts are $U_1 \cup R_{BL}$ and U_2.*

Proof. Let $Q_1 = R \cap U_1$ and let $Q_2 \in V(T)$ be the rectangle which is a subset of R_{BL} and adjacent to R.

Observe that U_1 and R_{BL} only intersect on R's bottom side, therefore their intersection is a line segment L and so $U_1 \cup R_{BL}$ is a polyomino. Trivially, $P = (U_1 \cup R_{BL}) \overset{\star}{\cup} U_2$ is partition into polyominoes, so only admissibility remains to be checked.

Let the horizontal cut tree of U_1 and R_{BL} be T_{U_1} and $T_{R_{BL}}$, respectively. The horizontal cut tree of $U_1 \cup R_{BL}$ is $T_{U_1} \cup T_{R_{BL}} + \{Q_1, Q_2\}$, except if $t(\{Q_1, Q_2\}) = -4$. Either way, by referring to Remark 13 we can use Claim 12 to write that

$$\begin{aligned} n(U_1 \cup R_{BL}) + n(U_2) - n(P) &= n(U_1) + n(R_{BL}) + t(\{Q_1, Q_2\}) + n(U_2) - n(P) = \\ &= \Big(n(U_1) + n(U_2) - n(U)\Big) + \Big(n(U) + n(R_{BL}) - n(P)\Big) + t(\{Q_1, Q_2\}) = \\ &= (n(U_1) + n(U_2) - n(U)) - t(\{R, Q_2\}) + t(\{Q_1, Q_2\}). \end{aligned} \tag{2}$$

Now it is enough to prove that $t(\{Q_1, Q_2\}) \leq t(\{R, Q_2\})$. If $t(\{R, Q_2\}) = 0$ this is trivial. The remaining case is when $t(\{R, Q_2\}) = -2$. This means that $v_{BL}(R)$ is not a vertex of $R \cup Q_2$, therefore it is not a vertex of $Q_1 \cup Q_2$ either, implying that $n(Q_1 \cup Q_2) < 8$. □

Lemma 16. *Let $R \in V(T)$ be such that $R_{BL} \neq \emptyset$. Let U be the other half of the polyomino, ie., $P = R_{BL} \overset{\star}{\cup} U$. If U has a good cut-system $\mathcal{L}$ such that $v_{BL}(R) \in \ker \mathcal{L}$ then P also has a good cut-system.*

Proof. Let us enumerate the elements of $\mathcal{L}$ as L_i where $i \in I$. Take $L_i(U_1^i, U_2^i)$ such that $v_{BL}(R) \in U_1^i$. Using Lemma 15 extend L_i to a cut $L_i'(P_1^i, P_2^i)$ of P such that $U_2^i = P_2^i$.

Equation (2) and the statement following it implies that $n(P_1^i) + n(P_2^i) = n(P) + 2 \implies n(U_1^i) + n(U_2^i) = n(U) + 2$. In other words, if L_i is a 2-cut then so is L_i'. Therefore

$$\begin{aligned} \big\{n(U_2^i) \mid i \in I\big\} \cup \big\{n(U_2^i) + 2 \mid i \in I \text{ and } L_i \text{ is a 2-cut}\big\} \subseteq \\ \subseteq \big\{n(P_2^i) \mid i \in I\big\} \cup \big\{n(P_2^i) + 2 \mid i \in I \text{ and } L_i' \text{ is a 2-cut}\big\} \end{aligned}$$

and by referring to Remark 9 we get that $\{L_i' \mid i \in I\}$ is a good cut-system of P. □

4. Proof of Theorem 4

We will prove Theorem 4 by induction on the number of vertices. For $n \leq 8$ the theorem is trivial.

For $n > 8$, let P be the polyomino which we want to partition into smaller polyominoes. It is enough to prove that P has a good cut. The rest of this proof is an extensive case study. Let T be the horizontal cut tree of P. We need two more definitions.

- A **pocket** in T is a degree-1 rectangle R, whose only incident edge in T is a 2-cut of P and this cut covers the entire top or bottom side of R.

- A **corridor** in T is a rectangle R of degree ≥ 2 in T, which has an incident edge in T which is a 2-cut of P and this cut covers the entire top or bottom side of R.

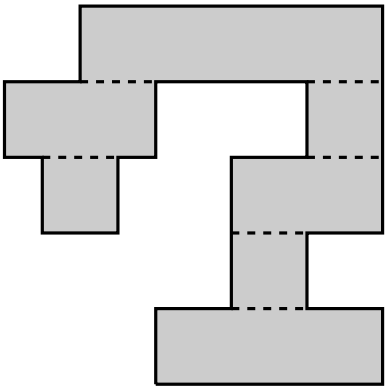
We distinguish 4 cases.

Case 1. T is a path, Figure 4(a);

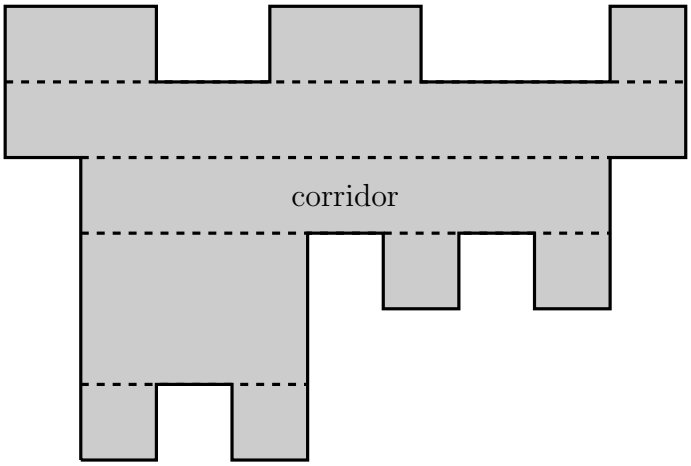
Case 2. T has a corridor, Figure 4(b);

Case 3. T does not have a corridor, but it has a pocket, Figure 4(c);

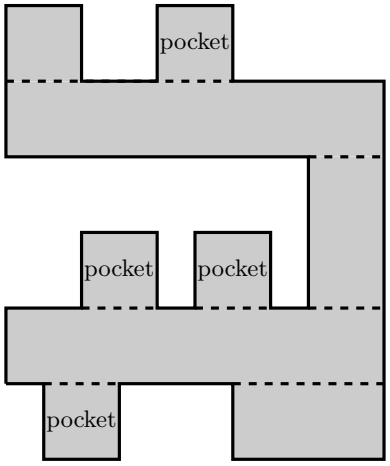
Case 4. None of the previous cases apply, Figure 4(d).



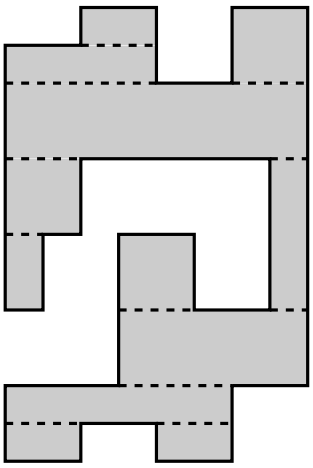
(a) T is a path.



(b) T has a corridor.



(c) T does not have a corridor, but it has a pocket.



(d) T does not have a corridor or a pocket, and it is not a path.

Figure 4: The 4 cases of the proof.

Case 1. T is a path

Claim 17. *If an edge incident to a degree-2 vertex R of T is a 2-cut of P then the incident edges of R form a good cut-system.*

Proof. Let the two incident edges of R be e_1 and e_2. Let their generated partitions be $e_1(P_1^1, P_2^1)$ and $e_2(P_1^2, P_2^2)$, such that $R \subseteq P_2^1 \cap P_1^2$. Then $P_1^2 = P_1^1 \cup R$, so

$$n(P_1^2) = n(P_1^1) + n(R) + t(e_1) = n(P_1^1) + 4.$$

Definition 8 is satisfied by $\{e_1, e_2\}$, as $\{n(P_1^1), n(P_1^2)\} \cup \{n(P_1^1) + 2\}$ is a set of three consecutive even elements. □

Claim 18. *If there are two rectangles R_1 and R_2 which are adjacent degree-2 vertices of T then the union of the set of incident edges of R_1 and R_2 form a good cut-system.*

Proof. Let the two components of $T - R_1 - R_2$ be T_1 and T_2, so that $e_1, e_2, f \in E(T)$ joins T_1 and R_1, R_1 and R_2, R_2 and T_2, respectively. Obviously, $\cup V(T_1) \subset (\cup V(T_1)) \cup R_1 \subset (\cup V(T_1)) \cup R_1 \cup R_2$. If one of $\{e_1, e_2, f\}$ is a 2-cut, we are done by the previous claim. Otherwise

$$n((\cup V(T_1)) \cup R_1) = n(\cup V(T_1)) + n(R_1) + t(e_1) = n(\cup V(T_1)) + 2,$$
$$n((\cup V(T_1)) \cup R_1 \cup R_2) = n(\cup V(T_1)) + n(R_1) + n(R_2) + t(e_1) + t(f) = n(\cup V(T_1)) + 4,$$

and so $\{n(\cup V(T_1)), n((\cup V(T_1)) \cup R_1), n((\cup V(T_1)) \cup R_1 \cup R_2)\}$ are three consecutive even elements. This concludes the proof that $\{e_1, e_2, f\}$ is a good cut-system of P. □

Suppose T is a path. If T is a path of length ≤ 3 such that all of its edges are 1-cuts, then $n(P) \leq 8$. Also, if T is path of length 2 and its only edge represents a 2-cut, then $n(P) = 8$. Otherwise either Claim 17 or Claim 18 can be applied to provide a good cut-system of P.

Case 2. T has a corridor

Let $e = \{R', R\} \in E(T)$ be a horizontal 2-cut such that R' is a wider rectangle than R and $\deg(R) \geq 2$. Let the generated partition be $e(P_1^e, P_2^e)$ such that $R' \subseteq P_1^e$. We can handle all possible cases as follows.

(a) If $n(P_1^e) \not\equiv 4, 10 \bmod 16$ or $n(P_2^e) \not\equiv 4, 10 \bmod 16$, then e is a good cut by Lemma 7(b).

(b) If $\deg(R) = 2$, we find a good cut using Claim 17.

(c) If $R_{BL} = \emptyset$, then $L(P_1^L, P_2^L)$ such that $R' \subseteq P_1^L$ in Figure 5 is a good cut, since $n(P_1^L) = n(P_1^e) + 4 - 0 \equiv 8, 14 \bmod 16$.

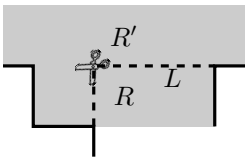


Figure 5: L is a good cut

(d) If $R_{BL} \neq \emptyset$ and $\deg(R) \geq 3$, then let us consider the following five cuts of P (Figure 6): $L_1(R_{BL}, R \cup P_1^e)$, $L_2(R_{BL} \cup Q_1, Q_2 \cup Q_3 \cup P_1^e)$, $L_3(R_{BL} \cup Q_1 \cup Q_2, Q_3 \cup P_1^e)$, $L_4(Q_3, R_{BL} \cup Q_1 \cup Q_2 \cup P_1^e)$, and $L_5(Q_3 \cup Q_2, R_{BL} \cup Q_1 \cup P_1^e)$.

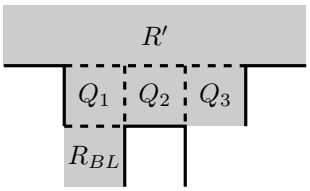


Figure 6: $\deg(R) \geq 3$ and $R_{BL} \neq \emptyset$

The first piece of these partitions have the following number of vertices (respectively).

(1) $n(R_{BL})$

(2) $n(R_{BL} \cup Q_1) = n(R_{BL}) + n(Q_1) + (t(e_{BL}(R)) - 2) = n(R_{BL}) + t(e_{BL}(R)) + 2$

(3) $n(R_{BL} \cup Q_1 \cup Q_2) = n(R_{BL}) + n(Q_1 \cup Q_2) + t(e_{BL}(R)) = n(R_{BL}) + t(e_{BL}(R)) + 4$

(4) $n(Q_3)$

(5) $n(Q_3 \cup Q_2) = n(Q_3) + n(Q_2) - 2 = n(Q_3) + 2$

- If $t(e_{BL}(R)) = 0$, then $\{L_1, L_2, L_3\}$ is a good cut-system, so one of them is a good cut.
- If $t(e_{BL}(R)) = -2$, and none of the 5 cuts above are good cuts, then using Lemma 7(b) on L_2 and L_3 gives $n(R_{BL}) \equiv 4, 10 \bmod 16$. The same argument can be used on L_4 and L_5 to conclude that $n(Q_3) \equiv 4, 10 \bmod 16$. However, previously we derived that

$$n(P_2^e) \equiv 4, 10 \bmod 16$$

$$n(R_{BL} \cup Q_1 \cup Q_2 \cup Q_3) = n(R_{BL} \cup Q_1 \cup Q_2) + n(Q_3) - 2 = n(R_{BL}) + n(Q_3) \equiv 4, 10 \bmod 16$$

This is only possible if $n(R_{BL}) \equiv n(Q_3) \equiv 10 \bmod 16$. Let $e_{BL}(R) = \{R, S\}$.

  - If $\deg(S) = 2$, then let $E(T) \ni e' \neq e_{BL}(R)$ be the other edge of S. Let the partition generated by it be $e'(P_1^{e'}, P_2^{e'})$ such that $R' \subseteq P_1^{e'}$. We have

$$n(R_{BL}) = n(P_2^{e'}) + n(S) + t(e')$$

$$n(P_2^{e'}) = n(R_{BL}) - 4 - t(e') \equiv 6 - t(e') \bmod 16$$

Either e' is a 1-cut, in which case $n(P_2^{e'}) \equiv 8 \bmod 16$, or e' is a 2-cut, giving $n(P_2^{e'}) \equiv 6 \bmod 16$. In any case, Lemma 7 says that e' is a good cut.

– If $\deg(S) = 3$, then we can partition P as in Figure 7. Since $n(Q_5 \cup Q_6) = 4 + n(Q_6) - 2$, by Lemma 7(a) the only case when neither $L_6(Q_5 \cup Q_6, Q_4 \cup R \cup P_1^e)$ nor $L_7(Q_6, Q_4 \cup Q_5 \cup R \cup P_1^e)$ is a good cut of P is when $n(Q_6) \equiv 4, 10 \bmod 16$. Also,

$$10 \equiv n(Q_4 \cup Q_5 \cup Q_6) = n(Q_4) + n(Q_5) + n(Q_6) - 2 - 2 \equiv n(Q_4) + n(Q_6) \bmod 16.$$

∗ If $n(Q_6) \equiv 10 \bmod 16$, then $n(Q_4) \equiv 0 \bmod 16$, hence

$$n(Q_4 \cup Q_5 \cup Q_{11} \cup Q_{12}) = n(Q_4 \cup Q_5) + 4 - 4 = n(Q_4) + n(Q_5) - 2 \equiv 2 \bmod 16,$$

showing that $L_8(Q_4 \cup Q_5 \cup Q_{11} \cup Q_{12}, Q_6 \cup Q_{13} \cup Q_2 \cup Q_3 \cup P_1^e)$ is a good cut.

∗ If $n(Q_6) \equiv 4 \bmod 16$,

$$\begin{aligned} n(Q_6 \cup Q_{13} \cup Q_2 \cup Q_3) &= n(Q_6) + n(Q_{13} \cup Q_2) + n(Q_3) - 2 - 2 \equiv \\ &\equiv n(Q_6) + 10 \equiv 14 \mod 16, \end{aligned}$$

therefore $L_9(Q_6 \cup Q_{13} \cup Q_2 \cup Q_3, Q_4 \cup Q_5 \cup Q_{11} \cup Q_{12} \cup P_1^e)$ is a good cut.

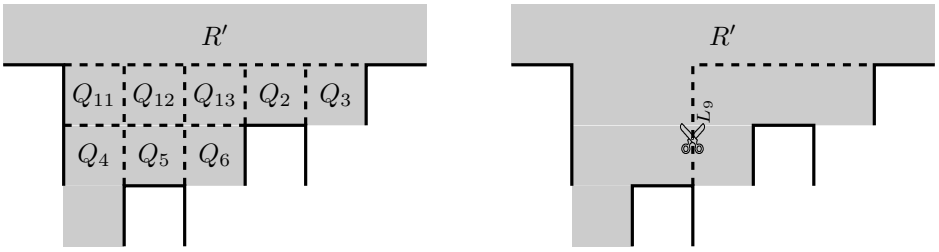


Figure 7: $\deg(Q) \geq 3$ and $Q_{BL} \neq \emptyset$

In all of the above subcases we found a good cut.

Case 3. There are no corridors in T, but there is a pocket

Let S be a (horizontal) pocket. Also, let R be the neighbor of S in T. If $\deg(R) = 2$, then Claim 17 provides a good cut-system of P. However, if $\deg(R) \geq 3$, we have two cases.

Case 3.1. If R is adjacent to at least two pockets

Let U be the union of R and its adjacent pockets, and let T_U be its **vertical** cut tree. It contains at least 4 reflex vertices, therefore $V(T_U) \geq 3$.

- If $V(T_U) = 3$, then $|E(T_U)| = 2$. Thus $t(e) = 0$ for any $e \in E(T_U)$, and Claim 17 gives a good cut-system $\mathcal{L}$ of U such that all 4 vertices of R are contained in $\ker \mathcal{L}$.

- If $V(T_U) \geq 4$, then Claim 18 gives a good cut-system $\mathcal{L}$ of U such that all 4 vertices of R are contained in $\ker \mathcal{L}$.

Since there are no corridors in P, we have

$$P = \Big(\big((U \cup R_{BL}) \cup R_{TL}\big) \cup R_{BR}\Big) \cup R_{TR}.$$

By applying Lemma 16 repeatedly, the good cut-system $\mathcal{L}$ can be extended to a good cut-system of P.

Case 3.2. If S is the only pocket adjacent to R

We may assume without loss of generality that S intersect the top side of R. Again, define U as the union of R and its adjacent pockets.

- If $R_{TL} \neq \emptyset$, let $V = U \overset{\star}{\cup} R_{TL}$. The cut-system $\{L_1, L_2, L_3\}$ in Figure 8 is a good cut-system of V, and all 4 vertices of R are contained in $\ker\{L_1, L_2, L_3\}$. By applying Lemma 16 repeatedly, we get a good cut-system of P.

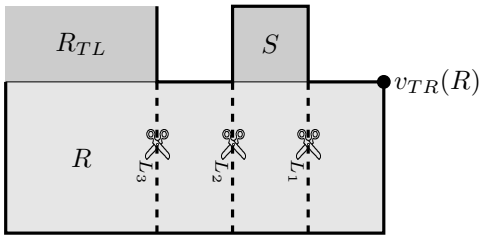


Figure 8: $\{L_1, L_2, L_3\}$ is a good cut-system of $V = R \cup R_{TL} \cup S$

- If $R_{TR} \neq \emptyset$, the case can be solved analogously to the previous case.
- Otherwise $R_{BL} \neq \emptyset$ and $R_{BR} \neq \emptyset$. Let $L_1(U_1^1, U_2^1)$ and $L_2(U_1^2, U_2^2)$ be the vertical cuts (from right to left) defined by the two reflex vertices of U, such that $v_{BR}(R) \in U_1^1 \subset U_1^2$. Let $V = R_{BL} \overset{\star}{\cup} U$. As before, L_1 and L_2 can be extended to cuts of V, say $L_1'(U_1^1, V_2^1)$, $L_2'(U_1^2, V_2^2)$. We claim that together with $e_{BL}(R)(U, V_2^3)$, they form a good cut-system $\mathcal{L}$ of V. This is obvious, as $\{n(U_1^1), n(U_1^2), n(U)\} = \{4, 6, 8\}$. Since $v_{BR}(R) \in \ker \mathcal{L}$, P also has a good cut-system by Lemma 16.

Case 4. T is not a path and it does not contain either corridors or pockets

By the assumptions of this case, any two adjacent rectangles are adjacent at one of their vertices, so the maximum degree in T is 3 or 4. We distinguish between several sub-cases.

Case 4.1. There exists a rectangle of degree ≥ 3 such that its top or bottom side is entirely contained by one of its neighboring rectangle;

Case 4.2. Every rectangle of degree ≥ 3 is such that its top and bottom side are not entirely contained by any of their neighboring rectangle;

Case 4.2.1. There exist at least two rectangles of degree ≥ 3;

Case 4.2.2. There is exactly one rectangle of degree ≥ 3.

Case 4.1. There exists a rectangle of degree ≥ 3 such that its top or bottom side is entirely contained by one of its neighboring rectangle

Let R be a rectangle and R' its neighbor, such that the top or bottom side of R is $\subset \partial R'$. Moreover, choose R such that if we partition P by cutting $e = \{R, R'\}$, the part containing R is minimal (in the set theoretic sense).

Without loss of generality, the top side of R is contained entirely by a neighboring rectangle R' and $R_{TL} = \emptyset$.

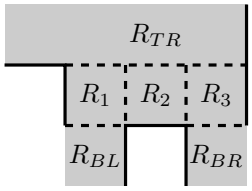


Figure 9

This is pictured in Figure 9, where $R = R_1 \cup R_2 \cup R_3$. We can cut off R_{BL}, $R_{BL} \cup R_1$, and $R_{BL} \cup R_1 \cup R_2$, whose number of vertices are respectively

(1) $n(R_{BL})$

(2) $n(R_{BL} \cup R_1) = n(R_{BL}) + n(R_1) + (t(e_{BL}(R))) - 2) = n(R_{BL}) + t(e_{BL}(R)) + 2$

(3) $n(R_{BL} \cup R_1 \cup R_2) = n(R_{BL}) + n(R_1 \cup R_2) + t(e_{BL}(R))) = n(R_{BL}) + t(e_{BL}(R)) + 4$

If $t(e_{BL}(R)) = 0$, then one of the 3 cuts is a good cut by Lemma 7(a).

Otherwise $t(e_{BL}(R)) = -2$, thus either one of the 3 cuts is a good cut or $n(R_{BL}) \equiv 4, 10 \bmod 16$. Let S be the rectangle for which $e_{BL}(R) = \{R, S\}$. Since $e_{BL}(R)$ is a 1-cut containing the top side of S, we cannot have $\deg(S) = 3$, as it contradicts the choice of R. We distinguish between two cases.

***Case 4.1.1.* $\deg(S) = 1$**

Let $U = R' \cup R \cup R_{BL} \cup R_{BR}$, which is depicted on Figure 10a. It is easy to see that $L_1(Q_1, U_2^1)$, $L_2(Q_1 \cup Q_2, U_2^2)$, and $L_3(Q_1 \cup Q_2 \cup Q_3, U_2^3)$ in Figure 10b is a good cut-system of U.

(a) (b)

Figure 10

As all 4 vertices of S are contained in $\ker\{L_1, L_2, L_3\}$, we can extend this good cut-system to P by reattaching S_{TL}, S_{BL}, S_{TR} (if non-empty) via Lemma 16. Therefore P has a good cut.

Case 4.1.2. $\mathbf{deg(S) = 2}$

Let f be the edge of S which is different from $e_{BL}(R) = e_{TL}(S)$. Let the partition generated by it be $f(P_1^f, P_2^f)$, where $S \subseteq P_2^f$. We have $n(P_1^f) = n(R_{BL}) - n(S) - t(f)$.

- If $t(f) = -2$, then $n(P_1^f) \equiv 2, 8 \bmod 16$, so f is a good cut by Lemma 7(a).
- If $t(f) = 0$, then $n(P_1^f) \equiv 0, 6 \bmod 16$, so f is a good cut by Lemma 7(b).

Case 4.2. Every rectangle of degree ≥ 3 is such that its top and bottom side are not entirely contained by any of their neighboring rectangle

Let R be a rectangle of degree ≥ 3 and $e = \{R, S\}$ be one of its edges. Let the partition generated by e be $e(P_1^e, P_2^e)$, where $R \subset P_1^e$ and $S \subseteq P_2^e$. If e is a 1-cut then by the assumptions of this case $\deg(S) \leq 2$.

- If $\deg(S) = 1$ and $t(e) = -2$, then $n(P_2^e) + t(e) = 2$.
- If $\deg(S) = 1$ and $t(e) = 0$, then $n(P_2^e) + t(e) = 4$.
- If $\deg(S) = 2$ and one of the edges of S is a 0-cut, then P has a good cut by Claim 17.
- If $\deg(S) = 2$ and both edges of S, e and (say) f are 1-cuts: Let the partition generated by f be $P = P_1^f \overset{\star}{\cup} P_2^f$, such that $S \in P_1^f$. Then $n(P_2^e) = n(P_2^f) + n(S) + t(f) = n(P_2^f) + 2$. Either one of e and f is a good cut, or by Lemma 7(a) we have $n(P_2^f) \equiv 4, 10 \mod 16$. In other words, $n(P_2^e) + t(e) \equiv 4, 10 \mod 16$. Similarly, $n(P_1^f) = n(P_1^e) + 4 - 2 = n(P_1^e) + 2$, so $n(P_1^e) \equiv 4, 10 \mod 16$.
- If $\deg(S) \geq 3$, then $t(e) = 0$. Either e is a good cut, or by Lemma 7(b) we have $n(P_2^e) + t(e) \equiv 4, 10 \mod 16$. Lemma 7(b) also implies $n(P_1^e) \equiv 4, 10 \mod 16$.

From now on, we assume that none of the edges of the neighbors of a degree ≥ 3 rectangle represent a good cut, so in particular we have

$$n(P_2^e) + t(e) \equiv 2, 4, \text{ or } 10 \mod 16.$$

In addition to the simple analysis we have just conducted, we deduce an easy claim to be used in the following subcases.

Claim 19. *Let $R \in V(T)$ be of degree ≥ 3 and suppose both $R_{BR} \neq \emptyset$ and $R_{TR} \neq \emptyset$. Then P has two admissible cuts L_1 and L_2 such that they form a good cut-system or*

- ***(i)*** *one of the parts generated by L_1 has size $\Big(n(R_{BR}) + t(e_{BR}(R))\Big) + \Big(n(R_{TR}) + t(e_{TR}(R))\Big) + 2$,*
  and
- ***(ii)*** *one of the parts generated by L_2 has size $\Big(n(R_{BR}) + t(e_{BR}(R))\Big) + \Big(n(R_{TR}) + t(e_{TR}(R))\Big) + 4$.*

Proof. Let $U = R \cup R_{BL} \cup R_{BR}$. Let $L_1(U_1^1, U_2^1)$ and $L_2(U_1^2, U_2^2)$ be the vertical cuts of U defined by the two reflex vertices of U that are on the boundary of R, such that $v_{BR}(R) \in U_1^1 \subset U_1^2$. By Lemma 15, L_1 and L_2 can be extended to cuts of $V = R \cup R_{BL} \cup R_{BR} \cup R_{TR}$, say $L_1'(V_1^1, U_2^1)$, $L_2'(V_1^2, U_2^2)$. If one of L_1' or L_2' is a 2-cut, then similarly to Claim 17 one can verify they form a good cut-system of V which we can extend to P. Otherwise

$$\begin{aligned}
n(V_1^1) &= n(R \cap U_1^1) + n(R_{BR}) + n(R_{TR}) + (t(e_{BR}(R)) - 2) + t(e_{TR}(R)) = \\
&= \Big(n(R_{BR}) + t(e_{BR}(R))\Big) + \Big(n(R_{TR}) + t(e_{TR}(R))\Big) + 2, \\
n(V_1^2) &= n(R \cap U_1^2) + n(R_{BR}) + n(R_{TR}) + t(e_{BR}(R)) + t(e_{TR}(R)) = \\
&= \Big(n(R_{BR}) + t(e_{BR}(R))\Big) + \Big(n(R_{TR}) + t(e_{TR}(R))\Big) + 4.
\end{aligned}$$

Lastly, we extend L_1' and L_2' to P by reattaching R_{TL} using Lemma 15. This step does not affect the parts V_1^1 and V_1^2, so we are done. □

Case 4.2.1. There exist at least two rectangles of degree ≥ 3

In the subgraph T' of T which is the union of all paths of T which connect two degree ≥ 3 rectangles, let R be a leaf and $e = \{R, S\}$ its edge in the subgraph. As defined in the beginning of Case 4.2, the set of incident edges of R (in T) is $\{e_i \mid 1 \leq i \leq \deg(R)\}$, and without loss of generality we may suppose that $e = e_{\deg(R)}$. The analysis also implies that for all $1 \leq i \leq \deg(R) - 1$ we have $n(P_2^{e_i}) + t(e_i) = 2, 4$.

By the assumptions of this case $\deg(S) \geq 2$, therefore $n(P_1^e) \equiv 4$ or $10 \mod 16$. If $\deg(S) \geq 3$ let $Q = S$. Otherwise $\deg(S) = 2$ and let Q be the second neighbor of R in T'. The degree of Q cannot be 1 by its choice. If $\deg(Q) = 2$, then we find a good cut using Claim 18. In any case, we may suppose from now on that $\deg(Q) \geq 3$.

Let $\{f_i \mid 1 \le i \le \deg(Q)\}$ be the set of incident edges of Q such that they generate the partitions $P = P_1^{f_i} \overset{\star}{\cup} P_2^{f_i}$ where $R \subset P_1^{f_i}$ and $Q \subset P_2^{f_1}$. We have

$$n(P_1^e) = n(R) + \sum_{i=1}^{\deg(R)-1} \Big(n(P_2^{e_i}) + t(e_i)\Big) \in 4 + \{2,4\} + \{2,4\} + \{0,2,4\} = \{8,10,12,14,16\},$$

so the only possibility is $n(P_1^e) = 10$.

- If $\deg(S) \ge 3$, e is a 2-cut (by the assumption of Case 4.2), so by Lemma 7(c) either e is a good cut or $n(P) \equiv 14 \bmod 16$. Since $Q = S$ and $e = f_1$, we have

$$n(P_1^{f_1}) + t(f_1) = n(P_1^e) + t(e) = 10$$
$$n(P_2^{f_1}) = n(P) - n(P_1^{f_1}) - t(f_1) \equiv 14 - 10 \equiv 4 \bmod 16.$$

- If $\deg(S) = 2$, either e is a 1-cut or we find a good cut using Claim 17. Also, $f_1 = \{S, Q\}$ is a 1-cut too (otherwise apply Claim 17), so

$$n(P_1^{f_1}) + t(f_1) = n(P_1^e) + n(S) + t(e) + t(f_1) = 10.$$

  By Lemma 7(c), either f_1 is a good cut (as $n(P_1^{f_1}) = 12$) or $n(P) \equiv 14 \bmod 16$. Thus

$$n(P_2^{f_1}) = n(P) - n(P_1^{f_1}) - t(f_1) \equiv 14 - 12 + 2 \equiv 4 \bmod 16$$

We have

$$n(P) = n(Q) + \Big(n(P_1^{f_1}) + t(f_1)\Big) + \sum_{i=2}^{\deg(Q)} \Big(n(P_2^{f_i}) + t(f_i)\Big) \in$$
$$\in 14 + \{2,4,10\} + \{2,4,10\} + \{0,2,4,10\} \mod 16.$$

The only way we can get 14 mod 16 on the right hand side is when $\deg(Q) = 4$ and out of

$$n(Q_{BL}), n(Q_{BR}), n(Q_{TL}), n(Q_{TR}) \mod 16$$

one is 2, another is 4, and two are 10 mod 16.

The last step in this case is to apply Claim 19 to Q. If it does not give a good cut-system then it gives an admissible cut where one of the parts has size congruent to $2 + 10 + 2 = 14$ or $2 + 4 + 2 = 8$ modulo 16, therefore we find a good cut anyway.

Case 4.2.2. There is exactly one rectangle of degree ≥ 3

Let R be the rectangle of degree ≥ 3 in T, and let $\{e_i \mid 1 \le i \le \deg(R)\}$ the edges of R, which generate the partitions $P = P_1^{e_i} \overset{\star}{\cup} P_2^{e_i}$ where $R \subset P_1^{e_i}$. Then $P_2^{e_i}$ is path for all i. If either Claim 17 or Claim 18 can be

applied, P has a good cut. The remaining possibilities can be categorized into 3 types:

$$\begin{array}{lllll}
\textbf{Type 1:} & t(e_i) = -2, & n(P_2^{e_i}) = 4, & \text{and} & n(P_2^{e_i}) + t(e_i) = 2; \\
\textbf{Type 2:} & t(e_i) = -2, & n(P_2^{e_i}) = 4 + 4 - 2 = 6, & \text{and} & n(P_2^{e_i}) + t(e_i) = 4; \\
\textbf{Type 3:} & t(e_i) = 0, & n(P_2^{e_i}) = 4, & \text{and} & n(P_2^{e_i}) + t(e_i) = 4.
\end{array}$$

Without loss of generality $R_{BR} \neq \emptyset$ and $R_{TR} \neq \emptyset$. We will now use Claim 19. If it gives a good cut-system, we are done. Otherwise

- If exactly one of $e_{BR}(R)$ and $e_{TR}(R)$ is of **type 1**, apply Claim 19(i): it gives an admissible cut which cuts off a subpolyomino of size $2 + 4 + 2 = 8$, so P has a good cut.
- If both $e_{BR}(R)$ and $e_{TR}(R)$ are of **type 1**, apply Claim 19(ii): it gives an admissible cut which cuts off a subpolyomino of size $2 + 2 + 4 = 8$, so P has a good cut.
- If none of $e_{BR}(R)$ and $e_{TR}(R)$ are of **type 1** and $n(P) \not\equiv 14 \pmod{16}$, apply Claim 19(i): it gives an admissible cut which cuts off a subpolyomino of size $4+4+4 = 12$, which is a good cut by Lemma 7(c).

Now we only need to deal with the case where $n(P) = 14$ and neither $e_{BR}(R)$ nor $e_{TR}(R)$ is of **type 1**.

If R still has two edges of **type 1**, again Claim 19(i) gives a good cut of P. If R has at most one edge of **type 1**, we have

$$14 = n(P) = n(R) + \sum_{i=1}^{\deg(R)} \left(n(P_2^{e_i}) + t(e_i) \right) \in 4 + \{2, 4\} + \{4\} + \{4\} + \{0, 4\} = \{14, 16, 18, 20\},$$

and the only way we can get 14 on the right hand is when $\deg(R) = 3$ and both $e_{BR}(R)$ and $e_{TR}(R)$ are of **type 2 or 3** while the third incident edge of R is of **type 1**. We may assume without loss of generality that the cut represented by $e_{TR}(R)$ is longer than the cut represented by $e_{BR}(R)$.

- If P is vertically convex, its vertical cut tree is a path, so it has a good cut as deduced in Case 1.
- If P is not vertically convex, but $e_{TR}(R)$ is a **type 2** edge of R, such that the only horizontal cut of R_{TR} is shorter than the cut represented by $e_{TR}(R)$, then $P' = P - R_{BR}$ is vertically convex, and has $(10-4)/2 = 3$ reflex vertices. By Claim 17 or Claim 18, P' has a good cut-system such that its kernel contains $v_{BR}(R)$, since its x coordinate is maximal in P'. Lemma 16 states that P also has a good cut-system.
- Otherwise we find that the top right part of P looks like to one of the cases in Figure 11. It is easy to see that in all three pictures L is an admissible cut which generates two subpolyominoes of 8 vertices.

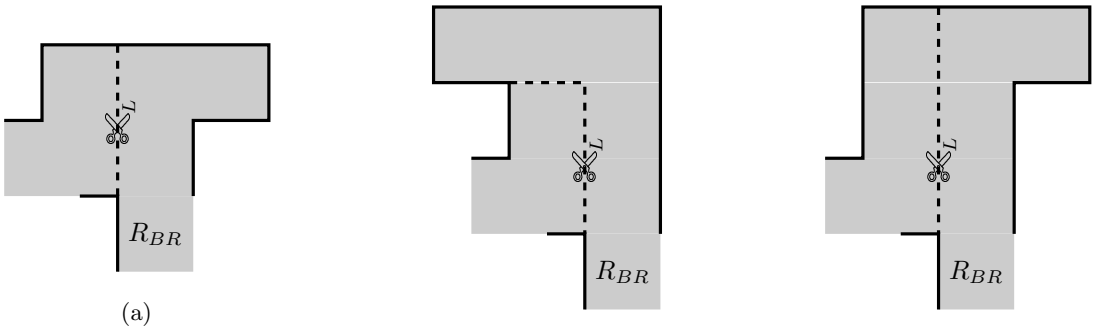


Figure 11

The proof of Theorem 4 is complete. To complement the formal proof, we now demonstrate the algorithm on Figure 12.

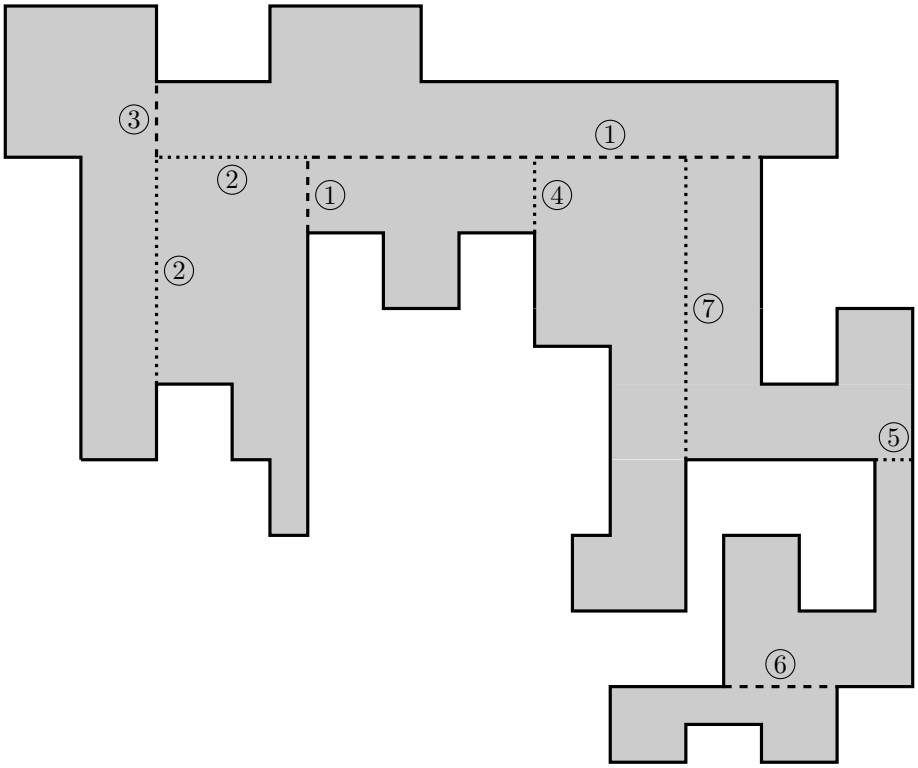


Figure 12: The output of the algorithm on a polyomino $n = 52$ vertices.

First, we resolve a corridor via the L-cut ①, which creates two pieces of 20 and 34($\equiv 2 \bmod 16$) vertices. As a result of this cut, a new corridor emerges in the 20-vertex piece, so we cut the polyomino at ②, cutting off a piece of 8 vertices. The other piece of 14 vertices containing two pockets is further divided by ③ into a piece of 6 and 8 vertices. Another pocket is dealt with by cut ④, which divides the polyomino into an 8-

and a 28-vertex piece. To the larger piece, Case 4.2 applies, and we find cut ⑤, which produces a 16- and a 14-vertex piece. The 16-vertex piece is cut into two 8-vertex pieces by ⑥. Lastly, Figure 11c of Case 4.2.2 applies to the 14-vertex piece, so cut ⑦ divides it into two 8-vertex pieces.

We got lucky with cuts ② and ⑥ in the sense that they both satisfy the inequality in (1) strictly. Hence, only 8 pieces are needed to partition the polyomino on Figure 12 into polyominoes of $\leq$ 8-vertex pieces, instead of the upper bound of $(3 \cdot 52 + 4)/16 = 10$.

5. Conclusion

We have not dealt with algorithmic aspects yet in this paper, but the proof given in the previous section can easily be turned into an $O(n^2)$ algorithm which partitions P into at most $\lfloor \frac{3n+4}{16} \rfloor$ simple polyominoes. The running time can be improved to $O(n)$ by using linear-time triangulation [2] to construct the horizontal cut tree of P (such that the edge list of a vertex is ordered by the x coordinates of the corresponding cuts) and a list of pockets, corridors, and rectangles of Case 4.1, all of which can be maintained in $O(1)$ for the partitions after finding and performing a cut in $O(1)$.

Theorem 4 fills a gap between two already established (sharp) results: in [3] it is proved that polyominoes can be partitioned into at most $\lfloor \frac{n}{4} \rfloor$ polyominoes of at most 6 vertices, and in [4] it is proved that any polyomino in general position (a polyomino without 2-cuts) can be partitioned into $\lfloor \frac{n}{6} \rfloor$ polyominoes of at most 10 vertices. However, we do not know of a sharp theorem about partitioning polyominoes into polyominoes of at most 12 vertices.

Furthermore, for $k \geq 4$, not much is known about partitioning (not necessarily simply connected) orthogonal polygons into polyominoes of at most $2k$ vertices. According to the "metatheorem," the first step in this direction would be proving that an orthogonal polygon of n vertices with h holes can be partitioned into $\lfloor \frac{3n+4h+4}{16} \rfloor$ polyominoes of at most 8 vertices. This would generalize the corresponding art gallery result in [4, Thm. 5.].

Acknowledgment

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