

Knot lattice homology in L -spaces

PETER OZSVÁTH
 ANDRÁS I. STIPSICZ
 ZOLTÁN SZABÓ

*Department of Mathematics, Princeton University
 Princeton, NJ, 08544*

*Rényi Institute of Mathematics
 Budapest, Hungary and
 Institute for Advanced Study, Princeton, NJ, 08540
 Department of Mathematics, Princeton University
 Princeton, NJ, 08544*

Email: `petero@math.princeton.edu` `stipsicz@math-inst.hu` `szabo@math.princeton.edu`

Abstract We show that the knot lattice homology of a knot in an L -space is equivalent to the knot Floer homology of the same knot (viewed these invariants as filtered chain complexes over the polynomial ring $\mathbb{Z}/2\mathbb{Z}[U]$). Suppose that G is a negative definite plumbing tree which contains a vertex w such that $G - w$ is a union of rational graphs. Using the identification of knot homologies we show that for such graphs the lattice homology $\mathbb{H}\mathbb{F}^-(G)$ is isomorphic to the Heegaard Floer homology $\mathrm{HF}^-(Y_G)$ of the corresponding rational homology sphere Y_G .

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1 Introduction

Suppose that G is a negative definite plumbing tree determining the rational homology sphere Y_G . *Heegaard Floer homology* (as it is introduced in [15, 16]) associates to Y_G a finitely generated $\mathbb{F}[U]$ -module $\mathrm{HF}^-(Y_G)$ (where \mathbb{F} denotes the field $\mathbb{Z}/2\mathbb{Z}$ of two element), which splits according to spin^c structures of Y_G and also admits an absolute Maslov grading [13]. This $\mathbb{F}[U]$ -module is defined as the homology of a chain complex $\mathrm{CF}^-(Y)$ associated to a Heegaard diagram of the 3-manifold Y_G . The definition of the chain complex, in turn, involves Floer theoretic constructions for symplectic manifolds and Lagrangians in them associated to the Heegaard diagram of Y_G . In particular, the boundary map of the chain complex counts certain pseudo-holomorphic disks in high-dimensional symplectic manifolds.

In [7] Némethi defined an invariant of negative definite plumbing graphs which is computed as the homology of a chain complex associated to the graph in a purely combinatorial manner. As it was shown in [7], the resulting *lattice homology* $\mathbb{H}\mathbb{F}^-(G)$ captures interesting information about singularities with resolution graph G . Heegaard Floer homology and lattice homology share a number of common properties. The algebraic structures of the two theories are similar, both satisfy certain exact triangles ([2, 9, 16]), and for certain plumbings it is easy to verify that the two homologies are, in fact, isomorphic $\mathbb{F}[U]$ -modules [7].

As a further common property, both theories admit refinements for knots in 3-manifolds [11, 17]. This refinement on the Heegaard Floer side is defined for a pair (Y, K) of a knot $K \subset Y$ in the 3-manifold Y , and admits the shape of a filtration A on the chain complex $\text{CF}^-(Y)$ computing the Heegaard Floer homology $\text{HF}^-(Y)$. (At least this is the case for 3-manifolds with $b_1(Y) = 0$.) In lattice homology, the knots are of a special type. They are specified by a tree Γ_{v_0} with a distinguished vertex v_0 such that $G = \Gamma_{v_0} - v_0$ is a negative definite plumbing tree. Once again, the invariant is a pair $(\mathbb{C}\mathbb{F}^-(G), A)$, where $\mathbb{C}\mathbb{F}^-(G)$ is the chain complex associated to the background plumbing graph G computing the lattice homology, while A is a filtration on this $\mathbb{F}[U]$ -module. (For more about these refinements see [11, 17], the discussion below and also the beginning paragraphs of Sections 2 and 3.)

It is natural to expect that the two homology theories are isomorphic for all negative definite plumbing trees of spheres. Indeed, such isomorphisms have been already verified for a number of families of graphs, cf. [7, 11, 12]. Such an isomorphism provides a convenient description of the important 3-manifold invariants $\text{HF}^-(Y_G)$ — at least for these special 3-manifolds.

In the present paper we show that the filtered chain complexes in the two theories for specific graphs Γ_{v_0} are filtered chain homotopic, and resting on this result we extend the family of plumbing trees/forests for which the two theories produce isomorphic homology groups. The types of graphs for which these ideas apply will be specified below.

In order to state our results, we need to consider a few definitions, notions and constructions. Suppose that Γ_{v_0} is a tree with a distinguished vertex v_0 , and all further vertices of Γ_{v_0} admit some integral framing. As before, let G denote the plumbing graph $\Gamma_{v_0} - v_0$. As a plumbing graph, G actually gives rise to a surgery presentation of the plumbed 3-manifold Y_G : replace each vertex by an unknot, link two unknots if and only if the corresponding vertices are connected by an edge in G , and decorate the unknots with the integers

attached to the corresponding vertices. After performing the surgeries on the unknots corresponding to the vertices of G (with the given framing), we get the 3-manifold Y_G and the vertex v_0 defines a knot $K = K_{v_0}$ in Y_G . The plumbing graph G also determines a simply connected 4-manifold X_G we get by plumbing disk bundles over spheres together according to G , and $K \subset \partial X_G = Y_G$.

Suppose that the plumbing tree G is negative definite. Denote the homology class in the plumbed 4-manifold corresponding to the vertex v_i of the graph by E_i . According to [1] there is a nonzero element $Z = \sum_i n_i \cdot E_i \in H_2(X_G; \mathbb{Z})$ with the property that n_i are all nonnegative integers, for every i we have $Z \cdot E_i \leq 0$, and for any other $Z' = \sum_i n'_i E_i$ with the same properties $n_i \leq n'_i$ holds for every i . (The dot product is computed using the intersection matrix M_G on $H_2(X_G; \mathbb{Z})$.)

Definition 1.1 *The plumbing graph G is rational if for the class $Z = \sum_i n_i E_i$ discussed above, the equality*

$$\left(\sum_i n_i E_i\right)^2 = 2 \sum_i n_i + \sum_i n_i E_i^2 - 2$$

holds. (This condition is equivalent to requiring that the geometric genus $p(Z) = \frac{1}{2}(Z^2 + K \cdot Z) + 1$ of the class Z vanishes.)

For a simple algorithm deciding whether a graph is rational, see Remark 3.2. (For further, more analytic characterizations of rationality, see for example [5].) Following [6] we say that the negative definite plumbing tree G is *almost rational* if there is a vertex w in G with the property that by decreasing the framing on w sufficiently we get a rational graph. The main result of the paper is the following theorem:

Theorem 1.2 *Suppose that Γ_{v_0} is a tree/forest with a distinguished vertex v_0 , $G = \Gamma_{v_0} - v_0$ is negative definite, and each of its components is rational. Then the filtered chain complex $(\mathbb{CF}^-(G), A)$ of Γ_{v_0} in lattice homology is filtered chain homotopy equivalent to the filtered complex $(\mathbb{CF}^-(Y_G), A)$ of the pair (Y_G, K_{v_0}) in Heegaard Floer homology.*

As a simple corollary we get the following:

Corollary 1.3 *The knot lattice homology of a knot in S^3 is equal to the knot Floer homology of the same knot. \square*

Remark 1.4 *The knots produced by the above construction are quite special. It can be shown that if v_0 is a leaf and $G = \Gamma_{v_0} - v_0$ represents S^3 , then G can be sequentially blown down, and therefore the knot represented by the vertex v_0 is an iterated torus knot. If v_0 is not a leaf, then Γ_{v_0} can be presented as the connected sum of other graphs with the distinguished vertex being a leaf, and hence the knot corresponding to v_0 in the general case is the connected sum of certain iterated torus knots.*

As a further corollary we verify the following result:

Theorem 1.5 *Suppose that G is a negative definite plumbing tree with the property that it admits a vertex w such that all the components of $G - w$ are rational graphs. Then the lattice homology $\mathbb{H}\mathbb{F}^-(G)$ of G is isomorphic (as a Maslov-graded $\mathbb{F}[U]$ -module) to the Heegaard Floer homology $\mathbb{H}\mathbb{F}^-(Y_G)$ of the 3-manifold Y_G defined by G .*

Notice that every almost rational graph (in the sense of [7]), and so in particular every graph with one bad vertex is considered by the above theorem. For such graphs the stated isomorphism was already proved by Némethi [7] — indeed, the result for almost rational graphs will be used in the proof of our theorem. Theorem 1.5, however, applies to many more graphs, among which we can find type- k graphs for arbitrary k . (Recall [12] that a plumbing graph is said to be of type- k if we can find k vertices $\{v_{i_1}, \dots, v_{i_k}\}$ on which we can change the framings $\{m_{i_1}, \dots, m_{i_k}\}$ in such a way that the resulting graph is rational.) For similar results see also [7, 10, 11].

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2 Heegaard Floer homology

We start our discussion by briefly recalling the relevant notions and constructions of Heegaard Floer homology. For a thorough discussion the reader is advised to consult [15, 16, 17, 21].

Suppose that $K \subset Y$ is a given knot, and for simplicity assume that Y is a rational homology sphere, i.e. $b_1(Y) = 0$. Consider a doubly pointed Heegaard diagram $\mathfrak{D}_K = (\Sigma, \alpha, \beta, w, z)$ compatible with (Y, K) . Let \mathbb{T}_α (and \mathbb{T}_β ,

respectively) denote product tori $\times_{i=1}^g \alpha_i$ (and $\times_{i=1}^g \beta_i$, resp.) in the g -fold symmetric product $\text{Sym}^g(\Sigma)$ of the genus- g surface Σ . The diagram $\mathfrak{D} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ (and a spin^c structure \mathbf{s}) gives rise to a chain complex $\text{CF}^-(\mathfrak{D}, \mathbf{s})$ over $\mathbb{F}[U]$, which is naturally a subcomplex of $\text{CF}^\infty(\mathfrak{D}, \mathbf{s}) = \text{CF}^-(\mathfrak{D}, \mathbf{s}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U^{-1}, U]$. For both cases the generators of the modules are given by the intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ in $\text{Sym}^g(\Sigma)$, and the boundary map counts certain pseudo-holomorphic disks in this symmetric product (after fixing appropriate almost complex structure on it). Since Y is a rational homology sphere (and so, in particular, all spin^c structures are torsion), the chain complexes come with absolute Maslov gradings [13], taking values in \mathbb{Q} . The boundary map of the complex decreases this grading by one (and therefore the Maslov grading descends to the homologies), while multiplication by U decreases the Maslov grading by two. Indeed, since Y is a rational homology sphere, for a fixed spin^c structure \mathbf{s} the $\mathbb{F}[U]$ -module $\text{HF}^-(Y, \mathbf{s}) = H_*(\text{CF}^-(\mathfrak{D}, \mathbf{s}), \partial^-)$ has the algebraic structure of a sum of $\mathbb{F}[U]$ with a finitely generated U -torsion module. The Maslov grading of the generator of the free part is called the *d-invariant* $d(Y, \mathbf{s})$ of the underlying spin^c 3-manifold (Y, \mathbf{s}) . Our present grading conventions on HF^- are different from those from [13]: according to our present conventions, the generator of $\text{HF}^-(S^3) \cong \mathbb{F}[U]$ has Maslov grading equal to 0, rather than -2 . However, the d invariants are the same.

We use the further basepoint z of \mathfrak{D}_K to equip the module $\text{CF}^-(\mathfrak{D}, \mathbf{s})$ by the Alexander grading A , inducing the *Alexander filtration* on $\text{CF}^-(\mathfrak{D}, \mathbf{s})$. At first glance the theory only provides a relative Alexander grading on $\text{CF}^-(\mathfrak{D}, \mathbf{s})$ for a fixed spin^c structure \mathbf{s} . For a rational homology sphere, however, these gradings can be coordinated for the various spin^c structures, and by requiring a global symmetry we get an absolute lift, which induces the filtration we considered above. The Alexander grading in general is not assumed to be integer valued. Nevertheless, for every spin^c structure \mathbf{s} there is a rational number $i_{\mathbf{s}} \in [0, 1)$ with the property that for any generator \mathbf{x} with spin^c structure \mathbf{s} the Alexander grading $A(\mathbf{x})$ is congruent to $i_{\mathbf{s}}$ mod 1. Multiplication by U decreases the Alexander grading by 1.

Notice that the filtered chain complex $(\text{CF}^-(\mathfrak{D}, \mathbf{s}), A)$ naturally gives rise to the doubly filtered chain complex $(\text{CF}^\infty(\mathfrak{D}, \mathbf{s}), j, A)$ where $(\text{CF}^\infty(\mathfrak{D}, \mathbf{s}), j, A)$ is naturally an $\mathbb{F}[U^{-1}, U]$ -module and the filtration j measures the negative of the exponent of U . In particular

$$\{x \in \text{CF}^\infty(\mathfrak{D}, \mathbf{s}) \mid j(x) \leq 0\}$$

is naturally isomorphic to $\text{CF}^-(\mathfrak{D}, \mathbf{s})$.

The double filtration allows us to consider the following subcomplexes:

$$\begin{aligned} B(\mathbf{s}) &= \{x \in \mathrm{CF}^\infty(\mathfrak{D}, \mathbf{s}) \mid j(x) \leq 0\}, \\ C_i(\mathbf{s}) &= \{x \in \mathrm{CF}^\infty(\mathfrak{D}, \mathbf{s}) \mid A(x) \leq i\}, \\ A_i(\mathbf{s}) &= C_i(\mathbf{s}) \cap B(\mathbf{s}) = \{x \in \mathrm{CF}^\infty(\mathfrak{D}, \mathbf{s}) \mid j(x) \leq 0, A(x) \leq i\}. \end{aligned}$$

There are natural embeddings $\psi = \psi_i(\mathbf{s}): A_i(\mathbf{s}) \rightarrow A_{i+1}(\mathbf{s})$, $v = v_i(\mathbf{s}): A_i(\mathbf{s}) \rightarrow B(\mathbf{s})$ and $h = h_i(\mathbf{s}): A_i(\mathbf{s}) \rightarrow C_i(\mathbf{s})$. The map $\phi = \phi_{i+1}(\mathbf{s}): A_{i+1}(\mathbf{s}) \rightarrow A_i(\mathbf{s})$ is defined by composing the multiplication-by- U map with the natural embedding. (The rational number i above is congruent to $i_{\mathbf{s}} \in [0, 1)$.)

The doubly filtered chain complex, and in particular the subcomplexes defined (and the maps considered) above can be conveniently used to describe a chain complex computing Heegaard Floer homology for any surgery along K [20, 22]. (In the following we will concentrate exclusively on integral surgeries.)

Before stating the theorem connecting the filtered chain complexes and Heegaard Floer homologies of surgeries, we introduce the following notation. Suppose that \mathbf{s} is a given spin^c structure on Y . Let $\mathbf{s}_{[K]}$ denote the spin^c structure we get by twisting \mathbf{s} with the Poincare dual of $[K]$. In particular, $c_1(\mathbf{s}_{[K]}) = c_1(\mathbf{s}) + 2PD[K]$. (If K is null-homologous, then $\mathbf{s} = \mathbf{s}_{[K]}$.) More generally, for any integer $n \in \mathbb{Z}$ the spin^c structure $\mathbf{s}_{n[K]}$ is given by applying the previous construction $|n|$ -times, using $[K]$ for positive n and $-[K]$ for negative n .

For the following theorem, it is useful to use a completed version of Heegaard Floer homology, $\mathbf{HF}^-(Y)$. This can be defined as the homology of $\mathrm{CF}^-(Y) \otimes_{\mathbb{F}[U]} \mathbb{F}[[U]]$, where $\mathbb{F}[[U]]$ denotes the ring of formal power series in U . There are corresponding completions $\mathbf{B}(\mathbf{s})$, $\mathbf{A}_i(\mathbf{s})$, and $\mathbf{C}_i(\mathbf{s})$ of $B(\mathbf{s})$, $A_i(\mathbf{s})$, and $C_i(\mathbf{s})$ respectively. The maps ϕ, ψ, v and h naturally extend to these completions.

Theorem 2.1 ([20]) *Let $Y_p(K)$ denote the result of some integer surgery along K and fix a spin^c structure \mathbf{t} on $Y_p(K)$. Suppose furthermore that $b_1(Y_p(K)) = 0$. Then there are rational numbers i and α , a spin^c structure $\mathbf{s} \in \mathrm{Spin}^c(Y)$ (depending on p and \mathbf{t}) and chain maps $N_n: \mathbf{C}_{i+\alpha \cdot n}(\mathbf{s}_{n[K]}) \rightarrow \mathbf{B}(\mathbf{s}_{(n+1)[K]})$ such that the chain complexes*

$$\begin{aligned} \mathbb{A}_{\mathbf{t}} &= \prod_{n=-\infty}^{\infty} \mathbf{A}_{i+\alpha \cdot n}(\mathbf{s}_{n[K]}) \\ \mathbb{B}_{\mathbf{t}} &= \prod_{n=-\infty}^{\infty} \mathbf{B}(\mathbf{s}_{n[K]}) \end{aligned}$$

and the map $f: \mathbb{A}_{\mathbf{t}} \rightarrow \mathbb{B}_{\mathbf{t}}$ defined on the component $\mathbf{A}_{i+\alpha \cdot n}(\mathbf{s}_{n[K]})$ as $v_{i+\alpha n} + N_n \circ h_{i+\alpha n}$ provide a mapping cone $MC(\mathbb{A}_{\mathbf{t}}, \mathbb{B}_{\mathbf{t}}, f)$ with the property that its homology is isomorphic to $\mathbf{HF}^-(Y_p(K), \mathbf{t})$. \square

In the above theorem one needs to use the completed version of the theory once $p > 0$ is allowed. (In fact, we will use this result for sufficiently large positive integer p .) On the other hand, as our next lemma shows, for torsion spin^c structures the two theories (before and after the above completion) determine each other. Indeed, let C be a free chain complex of free chain complex over $\mathbb{F}[U]$ with finitely generated homology. A *Maslov grading* on C is a grading with values in \mathbb{Q} with the property that the differential drops the grading by 1 and multiplication by U drops it by 2. The following is a standard algebraic fact:

Lemma 2.2 *Let C be a free, Maslov-graded chain complex over $\mathbb{F}[U]$, with finitely generated homology. Let $C[[U]]$ denote its completion, thought of as a chain complex over $\mathbb{F}[[U]]$. Then, the homology of C and the homology of $C[[U]]$ contain the same information; in particular, if C and C' are two finitely-generated, Maslov-graded chain complexes and $H_*(C[[U]]) \cong H_*(C'[[U]])$, then $H_*(C) \cong H_*(C')$.*

Proof By standard homological algebra, $H_*(C)$ is a direct sum of finitely many cyclic modules, each of which inherits a (rational) Maslov grading. It follows that $H_*(C)$ has the following form:

$$H_*(C) \cong \mathbb{F}[U]^{n_0} \oplus \bigoplus_{i=1}^k (\mathbb{F}[U]/U^{n_i}).$$

But in this case,

$$H_*(C[[U]]) \cong \mathbb{F}[[U]]^{n_0} \oplus \bigoplus_{i=1}^k (\mathbb{F}[U]/U^{n_i}),$$

concluding the argument. \square

Recall that the rational homology sphere Y is called an *L-space* if $\mathbf{HF}^-(Y, \mathbf{s})$ is isomorphic to $\mathbb{F}[U]$ for every spin^c structure $\mathbf{s} \in \text{Spin}^c(Y)$. With a slight extension of the results of [19] (where surgeries along certain knots in S^3 were examined) we get the following:

Theorem 2.3 *Suppose that Y is an L-space, and $K \subset Y$ is a knot such that there is a sufficiently large positive integer p with the property that $Y_p(K)$ (p -surgery on K) is also an L-space. Then,*

- (1) $H_*(B(\mathbf{s})) = H_*(C_i(\mathbf{s})) = H_*(A_i(\mathbf{s})) = \mathbb{F}[U]$ for every spin^c structure \mathbf{s} on Y .
- (2) The map $\psi_i(\mathbf{s})_*$ induced by the embedding of $A_i(\mathbf{s})$ into $A_{i+1}(\mathbf{s})$ is multiplication by $U^{d_i(\mathbf{s})}$ with $d_i(\mathbf{s}) \in \{0, 1\}$.
- (3) The maps $v_i(\mathbf{s})_*$ and $h_i(\mathbf{s})_*$ (induced by the maps $v_i(\mathbf{s})$ and $h_i(\mathbf{s})$ on homology) are multiplications by $U^{a_i(\mathbf{s})}$ and $U^{b_i(\mathbf{s})}$ for some nonnegative integers $a_i(\mathbf{s}), b_i(\mathbf{s})$, respectively.
- (4) If $d_i(\mathbf{s}) = 0$ then $a_i(\mathbf{s}) = a_{i+1}(\mathbf{s})$ and $b_i(\mathbf{s}) = b_{i+1}(\mathbf{s}) - 1$, while if $d_i(\mathbf{s}) = 1$ then $a_i(\mathbf{s}) = a_{i+1}(\mathbf{s}) + 1$ and $b_i(\mathbf{s}) = b_{i+1}(\mathbf{s})$.
- (5) Finally, for all i, j large enough $a_i(\mathbf{s}) = 0$ and $b_{-j}(\mathbf{s}) = 0$.

Proof Since $H_*(B(\mathbf{s}))$ is isomorphic to $\text{HF}^-(Y, \mathbf{s})$ and $C_i(\mathbf{s})$ is chain homotopic to $B(\mathbf{s}_K)$ by the map N encountered in Theorem 2.1, our assumption on Y determines the first two of the three homologies of the first claim. For the computation of the third homology we invoke the fact that for p large enough the surgery provides an L -space, hence $H_*(A_i(\mathbf{s}))$ cannot contain any U -torsion submodules. Indeed, by Lemma 2.2 a U -torsion submodule in $H_*(A_i(\mathbf{s}))$ implies the existence of a U -torsion submodule in $H_*(\mathbf{A}_i(\mathbf{s}))$, but such submodules are in $\ker v_*$ and $\ker h_*$, eventually giving rise to nontrivial U -torsion in the homology of the mapping cone $MC(\mathbb{A}_t, \mathbb{B}_t, f)$. This, however, contradicts the assumption that $Y_p(K)$ is an L -space (implying, by Lemma 2.2, that $\text{HF}^-(Y_p(K), \mathbf{t}) = \mathbb{F}[[U]]$ for all spin^c structure).

The fact that $H_*(A_i(\mathbf{s}))$ is equal to $\mathbb{F}[U]$ then follows from the fact that the embedding v induces an isomorphism on homologies with $\mathbb{F}[U^{-1}, U]$ -coefficients, and $H_*(\text{CF}^\infty(Y, \mathbf{s})) = \mathbb{F}[U^{-1}, U]$: it shows that $H_*(A_i(\mathbf{s}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U^{-1}, U]) = \mathbb{F}[U^{-1}, U]$, implying $H_*(A_i(\mathbf{s})) = \mathbb{F}[U]$. This observation concludes the proof of (1).

For (2) consider now the two maps $\psi = \psi_i(\mathbf{s}): A_i(\mathbf{s}) \rightarrow A_{i+1}(\mathbf{s})$ and $\phi = \phi_{i+1}(\mathbf{s}): A_{i+1}(\mathbf{s}) \rightarrow A_i(\mathbf{s})$, where the latter is the natural embedding after multiplication by U . Obviously the composition is simply multiplication by U , hence it induces the same map on homologies. Therefore the induced maps ψ_* and ϕ_* are either multiplications by 1 or by U , in such a way that their product is multiplication by U . This concludes the proof of the claim about $d_i(\mathbf{s})$.

Considering the maps v_* and h_* over $\mathbb{F}[U^{-1}, U]$, we get isomorphisms, therefore these induced maps are nonzero with $\mathbb{F}[U]$ -coefficients. Since both maps are between two copies of $\mathbb{F}[U]$, the third claim follows.

By taking the commutative triangle involving $H_*(A_i(\mathbf{s}))$, $H_*(A_{i+1}(\mathbf{s}))$ and $H_*(B(\mathbf{s}))$, the claimed change of $a_i(\mathbf{s})$ immediately follows. Similarly, the commutative square involving $H_*(A_i(\mathbf{s}))$, $H_*(A_{i+1}(\mathbf{s}))$, $H_*(C_i(\mathbf{s}))$ and $H_*(C_{i+1}(\mathbf{s}))$, together with the multiplication by U map from $C_{i+1}(\mathbf{s})$ to $C_i(\mathbf{s})$ (inducing an isomorphism on homology) verifies the claim about $b_i(\mathbf{s})$.

Finally, if the monotone sequences $\{a_i(\mathbf{s})\}$, $\{b_{-j}(\mathbf{s})\}$ stabilize on any other value, Theorem 2.1 would produce homologies which are not finitely generated $\mathbb{F}[[U]]$ -modules, contradicting basic properties of Heegaard Floer homology groups. This observation proves (5) and concludes the proof of the theorem. \square

Using information about the results of surgeries on K , and by applying Theorem 2.1 we get additional information about the exponents $a_i(\mathbf{s})$, $b_i(\mathbf{s})$ and $d_i(\mathbf{s})$.

Lemma 2.4 *The quantity $\min\{a_i(\mathbf{s}), b_i(\mathbf{s})\}$ can be determined from the Heegaard Floer homology of $Y_n(K)$ for $n \in \mathbb{Z}$ large enough in absolute value and from a suitably chosen spin^c structure on it.*

Proof Indeed, for $|n|$ large enough and an appropriate choice of the spin^c structure \mathbf{t} on $Y_n(K)$, the mapping cone computing $\mathbf{HF}^-(Y_n(K), \mathbf{t})$ involves the maps $v_i: \mathbf{A}_i(\mathbf{s}) \rightarrow \mathbf{B}(\mathbf{s})$ and $h_i: \mathbf{A}_i(\mathbf{s}) \rightarrow \mathbf{C}_i(\mathbf{s})$, together with some other similar maps, for which one of the exponents $a_{i+\alpha \cdot n}$ or $b_{i+\alpha \cdot n}$ is equal to 0. Therefore those parts can be contracted when computing the homology, and we are left with the complex $g: \mathbb{F}[[U]] \rightarrow \mathbb{F}[[U]] \oplus \mathbb{F}[[U]]$, where g maps the generator of the domain into $(U^{a_i(\mathbf{s})}, U^{b_i(\mathbf{s})})$. The homology of this complex is isomorphic to $\mathbb{F}[[U]] \oplus \mathbb{F}[U]/(U^m)$ with $m = \min\{a_i(\mathbf{s}), b_i(\mathbf{s})\}$, hence the $\mathbb{F}[[U]]$ -module structure of the resulting homology recovers $\min\{a_i(\mathbf{s}), b_i(\mathbf{s})\}$. \square

There is a further property of the mapping cone considered in Theorem 2.1, which we state presently and will use in our subsequent argument. Notice that in the notation of Theorem 2.1, there is a natural map of the chain complex $\mathbf{B}(\mathbf{s}_{n[K]})$ (computing the homology group $\mathbf{HF}^-(Y, \mathbf{s}_{n[K]})$) into the mapping cone $MC(\mathbb{A}_{\mathbf{t}}, \mathbb{B}_{\mathbf{t}}, f)$ by embedding $\mathbf{B}(\mathbf{s}_{n[K]})$ into the subcomplex $\mathbb{B}_{\mathbf{t}}$ of the mapping cone. The resulting composition therefore gives a map $\varphi: \mathbf{B}(\mathbf{s}_{n[K]}) \rightarrow MC(\mathbb{A}_{\mathbf{t}}, \mathbb{B}_{\mathbf{t}}, f)$, inducing a map $\varphi_*: \mathbf{HF}^-(Y, \mathbf{s}_{n[K]}) \rightarrow \mathbf{HF}^-(Y_p(K), \mathbf{t})$. For the next statement recall that a 4-dimensional spin^c cobordism (W, \mathbf{u}) between the spin^c 3-manifolds (Y_1, \mathbf{s}_1) and (Y_2, \mathbf{s}_2) induces a map $F_{W, \mathbf{u}}: \mathbf{HF}^-(Y_1, \mathbf{s}_1) \rightarrow \mathbf{HF}^-(Y_2, \mathbf{s}_2)$ and a corresponding map $\mathbf{F}_{W, \mathbf{u}}: \mathbf{HF}^-(Y_1, \mathbf{s}_1) \rightarrow \mathbf{HF}^-(Y_2, \mathbf{s}_2)$ on the completed theories. This map is defined on the chain complex level by

appropriately counting certain holomorphic *triangles* in the symmetric product $\text{Sym}^g(\Sigma)$, see [18].

Proposition 2.5 ([20, 22]) *The map φ_* described above is equal to the map $\mathbf{F}_{W_p(K), \mathbf{u}}$ induced on the Heegaard Floer homology groups by the spin^c cobordism $(W_p(K), \mathbf{u})$ where*

- *the 4-dimensional cobordism $W_p(K)$ is defined by the surgery (viewed as a 4-dimensional 2-handle attachment), and*
- *the spin^c structure \mathbf{u} is given by the property that it restricts as $\mathbf{s}_{n[K]}$ and \mathbf{t} on the boundary components Y and $Y_p(K)$ of $W_p(K)$, and its first Chern class takes the value $i + \alpha \cdot n$ on the generator of $H_2(W_p(K); \mathbb{Z})$. \square*

As a map induced by a 4-dimensional spin^c cobordism, φ_* has a well-defined degree shift. (Recall that since Y and $Y_p(K)$ are assumed to be rational homology spheres, all spin^c structures on them are torsion, and therefore the Heegaard Floer homology groups admit absolute Maslov gradings.) By [13] this quantity is equal to

$$\frac{1}{4}(c_1^2(\mathbf{u}) - 3\sigma(W_p(K)) - 2\chi(W_p(K))). \quad (2.1)$$

This expression then provides us a way to control the difference $a_i(\mathbf{s}) - b_i(\mathbf{s})$. Indeed, suppose that g generates the homology $H_*(\mathbf{A}_i(\mathbf{s}))$, and it has Maslov grading $\mu(g)$. Then the boundary map in the mapping cone (pointing to $\mathbf{B}(\mathbf{s})$) maps it to an element of Maslov grading $\mu(g) - 1$. This element (in $H_*(\mathbf{B}(\mathbf{s}))$) is equal $U^{a_i(\mathbf{s})}$ -times the generator of $H_*(\mathbf{B}(\mathbf{s}))$. The generator of $H_*(\mathbf{B}(\mathbf{s}))$, however, in the mapping cone has Maslov grading

$$d(Y, \mathbf{s}) + \frac{1}{4}(c_1^2(\mathbf{u}) - 3\sigma(W_p(K)) - 2\chi(W_p(K))),$$

since in $\mathbf{HF}^-(Y, \mathbf{s})$ the Maslov grading of the generator is by definition equal to $d(Y, \mathbf{s})$ and the map $\mathbf{F}_{W_p(K), \mathbf{u}}$ shifts degree by the quantity given in Equation (2.1). In conclusion, for the Maslov grading $\mu(g)$ of the generator $g \in H_*(\mathbf{A}_i(\mathbf{s}))$ we get that

$$\mu(g) = -2a_i(\mathbf{s}) + d(Y, \mathbf{s}) + \frac{1}{4}(c_1^2(\mathbf{u}) - 3\sigma(W_p(K)) - 2\chi(W_p(K)) - 1). \quad (2.2)$$

The same argument, now applied to the map $\mathbf{A}_i(\mathbf{s}) \rightarrow \mathbf{B}(\mathbf{s}_{[K]})$ gives

$$\mu(g) = -2b_i(\mathbf{s}) + d(Y, \mathbf{s}_{[K]}) + \frac{1}{4}(c_1^2(\mathbf{u}') - 3\sigma(W_p(K)) - 2\chi(W_p(K)) - 1). \quad (2.3)$$

Subtracting Equation (2.2) from Equation (2.3) we arrive to the following conclusion:

Theorem 2.6 *The difference $a_i(\mathbf{s}) - b_i(\mathbf{s})$ is equal to*

$$\frac{1}{2}(d(Y, \mathbf{s}) - d(Y, \mathbf{s}_{[K]})) + \frac{1}{8}(c_1^2(\mathbf{u}) - c_1^2(\mathbf{u}'))$$

where the spin^c structures \mathbf{u}, \mathbf{u}' on the cobordism $W_p(K)$ are determined by the spin^c structures \mathbf{s} and $\mathbf{s}_{[K]}$ on Y , by \mathbf{t} on $Y_p(K)$ and by the values i and $i + \alpha$ of $c_1(\mathbf{u}), c_1(\mathbf{u}')$ on the generator of $H_2(W_p(K); \mathbb{Z})$. \square

2.1 The connected sum formula

By taking the connected sum of the pairs (Y_1, K_1) and (Y_2, K_2) in the points of K_1 and K_2 , we get the connected sum $(Y_1 \# Y_2, K_1 \# K_2)$. As it was shown in [22, Theorem 5.1], the chain complex $\text{CF}^-(Y_1 \# Y_2, \mathbf{s}_1 \# \mathbf{s}_2)$ is the tensor product of the chain complexes $\text{CF}^-(Y_1, \mathbf{s}_1)$ and $\text{CF}^-(Y_2, \mathbf{s}_2)$ (over $\mathbb{F}[U]$), and the Alexander grading $A_\#$ on the connected sum is simply the sum of the Alexander gradings of the individual Alexander gradings A_1, A_2 :

$$A_\#(x \otimes y) = A_1(x) + A_2(y).$$

In a similar manner, the doubly filtered chain complexes of (Y_i, K_i) (for $i = 1, 2$) determine the doubly filtered chain complexes of the connected sum $(Y_1 \# Y_2, K_1 \# K_2)$.

3 Lattice homology

With some modifications, the results proved in the previous section can be verified in the lattice homology setting as well. We go through the statements and arguments below.

We start by a short recollection of lattice homology and knot lattice homology. (For more details, see [7, 11]). Lattice homology was introduced by Némethi [7], and it associates an algebraic invariant to a negative definite plumbing graph. For simplicity suppose that G is a negative definite plumbing tree/forest on the vertex set $V = \text{Vert}(G)$, giving rise to the plumbing 4-manifold X_G and its boundary 3-manifold Y_G . Then the $\mathbb{F}[U^{-1}, U]$ -module $\mathbb{CF}^\infty(G)$, freely generated by the pairs $[K, E]$ with $K \in H^2(X_G; \mathbb{Z})$ characteristic and $E \subset V$ admits a natural j -filtration (by the negative of the exponent of U) and a boundary map ∂ such that the subcomplex $\mathbb{CF}^-(G) = \{x \in \mathbb{CF}^\infty(G) \mid j(x) \leq 0\}$ computes the *lattice homology* $\mathbb{HF}^-(G)$ of G . (For the definition of the boundary map see [7, 11].) The chain complex splits according to the spin^c structures of G (or, equivalently, of Y_G), and the homology $\mathbb{HF}^-(G, \mathbf{s})$

of $\mathbb{CF}^-(G, \mathbf{s})$ for a spin^c structure $\mathbf{s} \in \text{Spin}^c(Y_G)$ further splits according to the δ -grading of $\mathbb{CF}^-(G, \mathbf{s})$ given for a generator $[K, E]$ by the cardinality $|E|$ of E :

$$\mathbb{HF}^-(G, \mathbf{s}) = \bigoplus_{i=0}^{|V|} \mathbb{HF}_i^-(G, \mathbf{s}).$$

The chain complex comes with an absolute Maslov grading which descends to a Maslov grading on the homologies, having the property that multiplication by U is of degree -2 . The formula for this grading gr for a generator $[K, E]$ reads as follows:

$$\text{gr}([K, E]) = 2g[K, E] + |E| + \frac{1}{4}(K^2 + |V|), \quad (3.1)$$

where $g[K, E] = \min\{\sum_{u \in I} K(u) + (\sum_{u \in I} u)^2 \mid I \subset E\}$. (Recall that since Y_G is a rational homology sphere, the square K^2 of $K \in H^2(X_G; \mathbb{Q})$ is well-defined and $K^2 \in \mathbb{Q}$.)

The algebraic structure of $\mathbb{HF}^-(G, \mathbf{s})$ is similar to that of the Heegaard Floer group of (Y_G, \mathbf{s}) : it is a finitely generated $\mathbb{F}[U]$ -module, which is a direct sum of a free part isomorphic to $\mathbb{F}[U]$, and a U -torsion part. The Maslov grading of the generator of the free part is the d -invariant $d^L(G, \mathbf{s})$ in lattice homology.

The significance of rationality and almost rationality of a graph G in the present context comes from the following theorem of Némethi:

Theorem 3.1 (Némethi, [7]) *Suppose that G is a negative definite plumbing tree.*

- *If G is a rational graph then $\mathbb{HF}^-(G, \mathbf{s}) = \mathbb{HF}_0^-(G, \mathbf{s}) = \mathbb{F}[U]$ for every spin^c structure \mathbf{s} .*
- *If G is almost rational, then $\mathbb{HF}^-(G, \mathbf{s}) = \mathbb{HF}_0^-(G, \mathbf{s})$, that is, the homology is supported in the lowest δ -grading.*
- *For an almost rational graph G and spin^c structure \mathbf{s} the lattice homology $\mathbb{HF}^-(G, \mathbf{s})$ is isomorphic (as a Maslov graded $\mathbb{F}[U]$ -module) to $\mathbb{HF}^-(Y_G, \mathbf{s})$. \square*

Remark 3.2 *In fact, according to [7] the property that $\mathbb{HF}^-(G, \mathbf{s}) = \mathbb{F}[U]$ for every spin^c structure does characterize rational graphs. We will not use this direction of Némethi's result in our subsequent discussions.*

There is a fairly simple combinatorial algorithm due to Laufer [3], which decides whether a negative definite tree is rational or not. The algorithm proceeds as follows. Consider $Z_1 = \sum E_i$ and determine all values of $Z_1 \cdot E_i$. If there is a product which is at least 2, then the algorithm stops and the graph is not

rational. If all the products satisfy $Z_1 \cdot E_i \leq 0$ then the algorithm stops again, and the graph is rational. If there is an index i such that $Z_1 \cdot E_i = 1$, then define $Z_2 = Z_1 + E_i$ and repeat the previous step with Z_2 in the role of Z_1 . Iterating the above procedure we get a sequence of vectors Z_1, Z_2, \dots . According to [3] this procedure stops after finitely many steps, and hence determines whether the graph is rational or not.

Suppose now that Γ_{v_0} is a tree with a distinguished vertex v_0 , and with framings on all the other vertices. Assume that the plumbing tree/forest $G = \Gamma_{v_0-v_0}$ is a negative definite graph. As it is discussed in [11], in this situation v_0 induces a filtration, the *Alexander filtration*, on $\mathbb{CF}^-(G, \mathbf{s})$ and on $\mathbb{CF}^\infty(G, \mathbf{s})$, turning the latter into a doubly filtered chain complex (exactly as we saw it in the Heegaard Floer setup).

As in the previous section, the doubly filtered chain complex allows us to define various subcomplexes. Fix a spin^c structure $\mathbf{s} \in \text{Spin}^c(Y_G)$. In order to keep track which theory we are in, we will add a superscript L referring to *lattice* homology. Consider therefore the subcomplexes

$$B^L(\mathbf{s}) = \{x \in \mathbb{CF}^\infty(G, \mathbf{s}) \mid j(x) \leq 0\},$$

$$C_i^L(\mathbf{s}) = \{x \in \mathbb{CF}^\infty(G, \mathbf{s}) \mid A(x) \leq i\},$$

$$A_i^L(\mathbf{s}) = C_i^L(\mathbf{s}) \cap B^L(\mathbf{s}) = \{x \in \mathbb{CF}^\infty(G, \mathbf{s}) \mid j(x) \leq 0, A(x) \leq i\}.$$

(Once again $i \equiv i_{\mathbf{s}} \pmod{1}$, where $i_{\mathbf{s}} \in [0, 1) \cap \mathbb{Q}$ is a rational number attached to the spin^c 3-manifold (Y_G, \mathbf{s}) .) The important maps are again the natural embeddings $\psi_i^L(\mathbf{s}): A_i^L(\mathbf{s}) \rightarrow A_{i+1}^L(\mathbf{s})$, $v_i^L(\mathbf{s}): A_i^L(\mathbf{s}) \rightarrow B^L(\mathbf{s})$ and $h_i^L(\mathbf{s}): A_i^L(\mathbf{s}) \rightarrow C_i^L(\mathbf{s})$. As before, $\phi_i^L(\mathbf{s}): A_{i+1}^L(\mathbf{s}) \rightarrow A_i^L(\mathbf{s})$ is defined as the composition of multiplication by U and the natural embedding. Following similar notations in the Heegaard Floer context, the spin^c structure given by twisting \mathbf{s} with $n[v_0]$ will be denoted by \mathbf{s}_{nv_0} .

Theorem 3.3 ([11]) *Let $G_{-k}(v_0)$ denote the graph we get from Γ_{v_0} by attaching framing $-k$ to v_0 with $k \in \mathbb{N}$ in such a manner that the resulting plumbing graph is negative definite. Fix a spin^c structure \mathbf{t} on $G_{-k}(v_0)$. Then there are rational numbers i and α , a spin^c structure $\mathbf{s} \in \text{Spin}^c(G)$ (depending on k and \mathbf{t}) and chain maps $N_n^L: C_{i+\alpha \cdot n}^L(\mathbf{s}_{nv_0}) \rightarrow B^L(\mathbf{s}_{(n+1)v_0})$ such that the chain complexes*

$$\mathbb{A}_{\mathbf{t}}^L = \bigoplus_{n=-\infty}^{\infty} A_{i+\alpha \cdot n}^L(\mathbf{s}_{nv_0})$$

$$\mathbb{B}_{\mathbf{t}}^L = \bigoplus_{n=-\infty}^{\infty} B^L(\mathbf{s}_{nv_0})$$

and the map $f^L: \mathbb{A}_{\mathbf{t}}^L \rightarrow \mathbb{B}_{\mathbf{t}}^L$ defined on the component $A_{i+\alpha \cdot n}^L(\mathbf{s}_{nv_0})$ as $v_{i+\alpha \cdot n}^L + N_n^L \circ h_{i+\alpha \cdot n}^L$ provide a mapping cone $MC(\mathbb{A}_{\mathbf{t}}^L, \mathbb{B}_{\mathbf{t}}^L, f^L)$ with the property that its homology is isomorphic to $\mathbb{H}\mathbb{F}^-(G_{-k}(v_0), \mathbf{t})$. Moreover, the lowest δ -grading $\mathbb{H}\mathbb{F}_0^-(G_{-k}(v_0), \mathbf{t})$ is supported in the homology of the subcomplex $\mathbb{B}_{\mathbf{t}}^L$. \square

Remark 3.4 Since we assume both G and $G_{-k}(v_0)$ to be negative definite, in the above theorem we did not need to consider the completed version of the theory.

The property in the Heegaard Floer discussion of requiring L -space surgery along the knot $K \subset Y$ is now substituted with the following observation.

Lemma 3.5 Suppose that Γ_{v_0} has the property that the plumbing graph $G = \Gamma_{v_0} - v_0$ is rational and connected (i.e. v_0 is a leaf). If we equip v_0 with framing $-k$ with $k \in \mathbb{N}$ large enough, then the resulting plumbing tree $G_{-k}(v_0)$ is almost rational.

Proof Since v_0 is a leaf, it is connected to a single vertex w . It is not hard to see that by decreasing the framing on w sufficiently, we get a rational graph, concluding the proof. (Cf. the algorithm in Remark 3.2 about verifying that a graph is rational.) \square

The following lemma will be useful in the comparison with Heegaard Floer homology:

Lemma 3.6 Suppose that Γ_{v_0} has the property that the plumbing graph $G = \Gamma_{v_0} - v_0$ is rational and connected (i.e. v_0 is a leaf). Then, if we equip v_0 with a framing p for $p \in \mathbb{N}$ large enough, the three-manifold $Y_{G_p(v_0)}$ is an L -space.

Proof Let G' be the graph obtained by decreasing the framing on w by one, and replacing the edge connecting w and v_0 by a string of edges with $p - 1$ new vertices, all labelled with -2 . By simple Kirby calculus, $Y_{G_p(v_0)} \cong Y_{G'}$. Applying Laufer's algorithm (Remark 3.2), it is easy to see that G' is a rational graph. The lemma now follows from Theorem 3.1. \square

Recall that by Theorem 3.1 the property of being almost rational implies that the lattice homology of any graph $G_{-k}(v_0)$ with sufficiently large $k \in \mathbb{N}$ is supported in $\mathbb{H}\mathbb{F}_0^-(G_{-k}(v_0))$. Assume now that Γ_{v_0} has the special property that v_0 is a leaf and $G = \Gamma_{v_0} - v_0$ is a rational graph. The structure theorem

in this setting has a similar shape as it was described in Theorem 2.3 in the Heegaard Floer context (although the proof is slightly different).

Theorem 3.7 *Suppose that Γ_{v_0} is a given tree with one distinguished vertex v_0 such that the plumbing graph $G = \Gamma_{v_0} - v_0$ is a connected, rational graph. Then*

- (1) $H_*(B^L(\mathbf{s})) = H_*(C_i^L(\mathbf{s})) = H_*(A_i^L(\mathbf{s})) = \mathbb{F}[U]$ for every spin^c structure \mathbf{s} on G .
- (2) The map $\psi_i^L(\mathbf{s})_*$ induced by the embedding of $A_i^L(\mathbf{s})$ into $A_{i+1}^L(\mathbf{s})$ on the homologies is multiplication by $U^{d_i^L(\mathbf{s})}$ with $d_i^L(\mathbf{s}) \in \{0, 1\}$.
- (3) The maps $v_i^L(\mathbf{s})_*$ and $h_i^L(\mathbf{s})_*$ are multiplications by $U^{a_i^L(\mathbf{s})}$ and $U^{b_i^L(\mathbf{s})}$, respectively, where $a_i^L(\mathbf{s}), b_i^L(\mathbf{s})$ are nonnegative integers.
- (4) If $d_i^L(\mathbf{s}) = 0$ then $a_i^L(\mathbf{s}) = a_{i+1}^L(\mathbf{s})$ and $b_i^L(\mathbf{s}) = b_{i+1}^L(\mathbf{s}) - 1$, while if $d_i^L(\mathbf{s}) = 1$ then $a_i^L(\mathbf{s}) = a_{i+1}^L(\mathbf{s}) + 1$ and $b_i^L(\mathbf{s}) = b_{i+1}^L(\mathbf{s})$.
- (5) Finally, for all i, j large enough $a_i^L(\mathbf{s}) = 0$ and $b_{-j}^L(\mathbf{s}) = 0$.

Proof By definition $B^L(\mathbf{s})$ is chain homotopic to $\mathbb{CF}^-(G, \mathbf{s})$, hence by our assumption on G it follows from Theorem 3.1 that $H_*(B^L(\mathbf{s})) = \mathbb{F}[U]$. The map N encountered in Theorem 3.3 (multiplied with the appropriate U -power) provides a chain isomorphism between $C_i^L(\mathbf{s})$ and $B^L(\mathbf{s}_{v_0})$, hence the assumption on G also shows that $H_*(C_i^L(\mathbf{s})) = \mathbb{F}[U]$.

The argument for $A_i^L(\mathbf{s})$ is slightly more complicated. Any U -torsion of $H_*(A_i^L(\mathbf{s}))$ must be in the kernel of v_* and h_* , since their images are in free $\mathbb{F}[U]$ -modules. From these maps the homology $\mathbb{H}\mathbb{F}^-(G_{-k}(v_0))$ can be computed for any k . By Theorem 3.1 and Lemma 3.5 these homologies are concentrated on level 0, while the kernel of v_*^L and h_*^L determine nontrivial subgroups of $\mathbb{H}\mathbb{F}_{>0}^-$. From this observation we conclude that $A_i^L(\mathbf{s})$ is a free $\mathbb{F}[U]$ -module. The rank over $\mathbb{F}[U]$ can be determined using the Universal Coefficient Formula, and the fact that v_*^L provides isomorphisms between $A_i^L(\mathbf{s}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U^{-1}, U]$ and $B^L(\mathbf{s}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U^{-1}, U]$ (and $\mathbb{H}\mathbb{F}^\infty(G, \mathbf{s}) = \mathbb{F}[U^{-1}, U]$ for every negative definite tree G). Therefore $H_*(A_i(\mathbf{s}))$ is a free $\mathbb{F}[U]$ -module with $H_*(A_i(\mathbf{s}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U^{-1}, U]) = \mathbb{F}[U^{-1}, U]$, implying that $H_*(A_i(\mathbf{s})) = \mathbb{F}[U]$, concluding the proof of (1).

The nontriviality of the induced maps with $\mathbb{F}[U^{-1}, U]$ -coefficients show that the U -equivariant maps v_*^L and h_*^L are both nontrivial, and hence are multiplications by some U -power $U^{a_i^L(\mathbf{s})}$ and $U^{b_i^L(\mathbf{s})}$. The rest of the proof of the theorem now proceeds exactly as the proof of Theorem 2.3. \square

Since knot lattice homology and lattice homology are connected by the same mapping cone description which connects knot Floer homology and Heegaard Floer homology, the proof of Lemma 2.4 applies verbatim in the present context, giving:

Lemma 3.8 *The quantity $\min\{a_i^L(\mathbf{s}), b_i^L(\mathbf{s})\}$ can be determined from the lattice homology of $G_{-k}(v_0)$ for $k \in \mathbb{N}$ large enough and from a suitably chosen spin^c structure \mathbf{t} on it. \square*

Similar considerations as in the Heegaard Floer context allow us to get information about the difference $a_i(\mathbf{s}) - b_i(\mathbf{s})$. As before, in these arguments we will rely on Maslov gradings. Since no maps induced by cobordisms have been introduced for lattice homologies (but such maps play an important role in the proof of Theorem 2.6), we give a slightly modified argument. Below we spell out the details of the lattice homological counterpart of Theorem 2.6.

Recall that the lattice chain complex $\mathbb{CF}^-(Y, \mathbf{s})$ admits a Maslov grading. In the mapping cone construction of Theorem 3.3 the chain complexes $A_i(\mathbf{s}_{nv_0})$ and $B(\mathbf{s}_{nv_0})$ get Maslov gradings from the mapping cone. For these gradings the maps v, h (as boundary maps) are of degree -1 . On the other hand, $B(\mathbf{s}_{nv_0})$ is isomorphic as a chain complex to $\mathbb{CF}^-(G, \mathbf{s}_{nv_0})$, where the isomorphism is given in [11, Proposition 5.5]. This isomorphism sends a generator $[L, H]$ in $\mathbb{CF}^-(G_{-k}(v_0))$ (with the property that $L \in H^2(X_G; \mathbb{Z})$ characteristic and $H \subset \text{Vert}(G_{-k}(v_0)) = \text{Vert}(G) \cup \{v_0\}$ satisfying $v_0 \notin H$) to $[L|_G, H]$. Suppose now that g is a generator of $H_*(A_i(\mathbf{s}_{nv_0})) = \mathbb{F}[U]$, and it is of Maslov grading $\mu(g)$.

Since G is a rational graph, we get that the generator of $\mathbb{H}\mathbb{F}^-(G, \mathbf{s})$ can be represented by sum of pairs of the form $[K, \emptyset]$, since lattice homology in this case is supported in δ -grading 0. According to Equation (3.1), the grading of such an element is $\text{gr}([K, \emptyset]) = \frac{1}{4}(K^2 + |V_G|)$. On the other hand, the generator of $\mathbb{H}\mathbb{F}^-(G, \mathbf{s})$ is, by definition, of Maslov grading $d^L(G, \mathbf{s})$.

According to [11, Proposition 5.5] the image of $[K, \emptyset]$ in the mapping cone is the class represented by $[L, \emptyset]$, where $L|_G = K$ and L satisfies

$$\frac{1}{2}(L(\Sigma) + \Sigma^2) = i. \quad (3.2)$$

Here $\text{Vert}(G_{-k}(v_0)) = \{v_0, v_1, \dots, v_r\}$ and $\Sigma = v_0 + \sum_{j>0} a_j \cdot v_j$ with the property that $\Sigma \cdot v_j = 0$ for all $j > 0$ and $a_j \in \mathbb{Q}$. The expression of (3.2) uniquely defines L by the formula

$$L(v_0) = 2i - \Sigma^2 - K(\Sigma - v_0).$$

Now the Maslov grading of $[L, \emptyset]$ in the mapping cone is equal to $\frac{1}{4}(L^2 + |V_{G-k}(v_0)|)$, which is obviously equal to

$$\frac{1}{4}(L^2 - K^2) + \frac{1}{4} + d^L(G, \mathbf{s}).$$

Therefore the Maslov grading $\mu(g)$ for the generator of $H_*(A_i(\mathbf{s}))$ is

$$\frac{1}{4}(L^2 - K^2) + \frac{1}{4} + d^L(G, \mathbf{s}) - 2a_i^L(\mathbf{s}) + 1.$$

A similar argument gives that the same Maslov grading $\mu(g)$ is equal to

$$\frac{1}{4}(L'^2 - K'^2) + \frac{1}{4} + d^L(G, \mathbf{s}_{v_0}) - 2b_i^L(\mathbf{s}) + 1,$$

where $[K', \emptyset]$ now represents a generator of $\mathbb{H}\mathbb{F}^-(G, \mathbf{s}_{v_0})$ and L' is its extension. The difference of these two expressions now provides a formula for $a_i^L(\mathbf{s}) - b_i^L(\mathbf{s})$ in terms of d^L -invariants of lattice homology groups of the background G , and squares of certain characteristic cohomology elements:

Theorem 3.9 *With the notations as above, the difference $a_i^L(\mathbf{s}) - b_i^L(\mathbf{s})$ is equal to*

$$\frac{1}{2}(d^L(G, \mathbf{s}) - d^L(G, \mathbf{s}_{v_0})) + \frac{1}{8}((L^2 - K^2) - (L'^2 - K'^2)). \quad \square$$

3.1 The connected sum formula

Suppose that Γ_{v_0} and $\Gamma'_{v'_0}$ are two given graphs with distinguished vertices v_0 and v'_0 , respectively. By adding an edge connecting v_0 and v'_0 to the disjoint union $\Gamma_{v_0} \cup \Gamma'_{v'_0}$ and then contracting this edge we get a new graph $\Delta_{v_0=v'_0}$ with the distinguished vertex $v_0 = v'_0$. The background 3-manifolds Y_G and $Y_{G'}$ corresponding to Γ_{v_0} and $\Gamma'_{v'_0}$ determine the 3-manifold Y corresponding to $D = \Delta_{v_0=v'_0} - (v_0 = v'_0)$ by $Y_D = Y_G \# Y_{G'}$. Moreover, the knot $K_{v_0=v'_0}$ in Y_D corresponding to the distinguished vertex is isotopic to the connected sum $K_{v_0} \# K_{v'_0} \subset Y_G \# Y_{G'} = Y_D$.

It was shown in [11, Section 4] that the lattice homology chain complex $\mathbb{C}\mathbb{F}^-(D, \mathbf{s} \# \mathbf{s}')$ is the tensor product of $\mathbb{C}\mathbb{F}^-(G, \mathbf{s})$ and of $\mathbb{C}\mathbb{F}^-(G', \mathbf{s}')$ over $\mathbb{F}[U]$, and the Alexander gradings A and A' add up to produce the Alexander grading induced by $v_0 = v'_0$ on $\mathbb{C}\mathbb{F}^-(D, \mathbf{s} \# \mathbf{s}')$. Similarly to the Heegaard Floer theory situation, the same simple derivation provides the doubly filtered chain complexes $\mathbb{C}\mathbb{F}^\infty(D, \mathbf{s} \# \mathbf{s}')$.

4 Comparing the two theories

Suppose now that Γ_{v_0} is a tree with a distinguished vertex v_0 such that $G = \Gamma_{v_0} - v_0$ is a negative definite connected rational graph (in particular, v_0 is a leaf). The corresponding 3-manifold Y_G is by Theorem 3.1(1) an L -space with a knot $K = K_{v_0}$ in it. Since by Lemma 3.6 large enough surgery on the knot K is also an L -space, both Theorems 2.3 and 3.7 apply to this situation. In particular, for a fixed spin^c structure \mathbf{s} on Y_G (or G) we get the sequences $\{a_i(\mathbf{s})\}$, $\{a_i^L(\mathbf{s})\}$, $\{b_i(\mathbf{s})\}$, $\{b_i^L(\mathbf{s})\}$, and $\{d_i(\mathbf{s})\}$, $\{d_i^L(\mathbf{s})\}$.

Proposition 4.1 *Suppose that Γ_{v_0} is a tree with a distinguished vertex v_0 such that $G = \Gamma_{v_0} - v_0$ is a negative definite connected rational graph (in particular, v_0 is a leaf). Then, for the sequences $\{a_i(\mathbf{s})\} = \{a_i^L(\mathbf{s})\}$, $\{b_i(\mathbf{s})\} = \{b_i^L(\mathbf{s})\}$ hold, and therefore $d_i(\mathbf{s}) = d_i^L(\mathbf{s})$ for every $i \in i_{\mathbf{s}} + \mathbb{Z}$.*

Proof By Theorems 2.3(4) and 3.7(4) the sequences $\{a_i(\mathbf{s})\}$ and $\{a_i^L(\mathbf{s})\}$ determine the sequences $\{d_i(\mathbf{s})\}$, $\{d_i^L(\mathbf{s})\}$ respectively (and these sequences then determine $\{b_i(\mathbf{s})\}$ and $\{b_i^L(\mathbf{s})\}$), and therefore it is sufficient to verify that $a_i(\mathbf{s}) = a_i^L(\mathbf{s})$ ($i \in i_{\mathbf{s}} + \mathbb{Z}$) to conclude the statement of the proposition.

For the graph $G_{-k}(v_0)$ with $k \in \mathbb{N}$ large enough the lattice homology group $\mathbb{H}\mathbb{F}^-(G_{-k}(v_0), \mathbf{t})$ is isomorphic to $\mathbb{H}\mathbb{F}^-(Y_{G_{-k}(v_0)}, \mathbf{t})$: the graph is almost rational by Lemma 3.5, hence the isomorphism of the two theories follows from Theorem 3.1 of Némethi on almost rational graphs. The combination of Lemmas 2.4 and 3.8, together with the above isomorphism now implies that

$$\min\{a_i(\mathbf{s}), b_i(\mathbf{s})\} = \min\{a_i^L(\mathbf{s}), b_i^L(\mathbf{s})\}. \quad (4.1)$$

In a similar manner, Theorems 2.6 and 3.9 imply that

$$a_i(\mathbf{s}) - b_i(\mathbf{s}) = a_i^L(\mathbf{s}) - b_i^L(\mathbf{s}). \quad (4.2)$$

Indeed, since G is a rational graph, the lattice homology $\mathbb{H}\mathbb{F}^-(G, \mathbf{s})$ and the Heegaard Floer homology $\mathbb{H}\mathbb{F}^-(Y_G, \mathbf{s})$ are isomorphic as Maslov graded $\mathbb{F}[U]$ -modules, and in particular, the d -invariants $d(Y_G, \mathbf{s})$ and $d^L(G, \mathbf{s})$ are equal for every spin^c structure $\mathbf{s} \in \text{Spin}^c(Y_G)$. Furthermore, the difference $L^2 - K^2$ appearing in Theorem 3.9 can be identified with $c_1^2(\mathbf{u})$ of Theorem 2.6, since the restriction of L to G is equal to K , while on $G_{-k}(v_0) - G$ the cohomology class L is equal to $c_1(\mathbf{u})$. Therefore $c_1^2(\mathbf{u})$ is equal to $L^2 - K^2$, and the same reasoning shows that $c_1^2(\mathbf{u}') = L'^2 - K'^2$.

These arguments then verify Equation (4.2), implying (with the use of Equation (4.1)) that $a_i(\mathbf{s}) = a_i^L(\mathbf{s})$, $b_i(\mathbf{s}) = b_i^L(\mathbf{s})$, and hence $d_i(\mathbf{s}) = d_i^L(\mathbf{s})$ for all $i \in i_{\mathbf{s}} + \mathbb{Z}$, concluding the proof of the proposition. \square

Remark 4.2 *The assumption that v_0 is a leaf is used only through the fact that $G_{-k}(v_0)$ is almost rational (and therefore the proof of Theorem 3.7 is applicable). Therefore the above identification of $a_i(s)$ with $a_i^L(\mathbf{s})$ (and consequently $b_i(\mathbf{s}) = b_i^L(\mathbf{s})$ and $d_i(\mathbf{s}) = d_i^L(\mathbf{s})$) follows by the same argument provided the framings of $G = \Gamma_{v_0} - v_0$ imply that $G_{-k}(v_0)$ is almost rational for $k \in \mathbb{N}$ large enough.*

From this point, a purely algebraic argument about doubly filtered chain complexes will imply that (at least in the special case when v_0 is a leaf) the doubly filtered chain complexes in the two theories are filtered chain homotopic. Although the algebraic considerations here are closely related to the algebraic lemmas from [19], for the sake of completeness we prefer to give a self-contained treatment here.

To state the result, we recall some notations from [19]. Suppose that (C, j, A) is a doubly filtered chain complex over $\mathbb{F}[U^{-1}, U]$, and denote the associated bigraded complex by the same triple. (In our application the grading j will be an integer, while A will be an element of $i_C + \mathbb{Z}$ with some fixed rational number $i_C \in [0, 1)$.) Assume that (C, j, A) has the property that $(j(Ux), A(Ux)) = (j(x) - 1, A(x) - 1)$ for a bihomogeneous element $x \in C$. In our arguments only special types of chain complexes will appear. The relevant properties are spelled out in the following definition.

Definition 4.3 *A doubly-filtered complex (C, j, A) over $\mathbb{F}[U^{-1}, U]$ with Maslov grading M is of knot Floer homology type if:*

- *The differential drops the Maslov grading by 1.*
- *Multiplication by U drops the Maslov grading by 2.*
- *Multiplication by U drops both j and A by one.*
- *C is filtered chain homotopy equivalent to a finitely generated, free chain complex over $\mathbb{F}[U^{-1}, U]$.*

View C as an $\mathbb{F}[U]$ -module. We specify subcomplexes of C by specifying domains in the plane: For $R \subset \mathbb{R}^2$ the symbol $C(R)$ denotes the subvector space of C generated by those bihomogeneous elements which have bigradings in R . If R satisfies the property that for $(x_1, y_1) \in R$ and $x_2 \leq x_1, y_2 \leq y_1$ we also have $(x_2, y_2) \in R$, then $C(R)$ is a subcomplex of C . The subcomplex $C\{(x, y) \in \mathbb{R}^2 \mid x \leq 0, y \leq i\}$ is denoted by A_i . The subcomplex C^- is defined to be $C(\{(x, y) \mid x \leq 0\})$, and we get the quotient complex \widehat{C} of C^- by taking $U = 0$ in the $\mathbb{F}[U]$ -module C^- . Since $H_*(C)$ is finitely generated

(as an $\mathbb{F}[U^{-1}, U]$ -module), it follows that $H_*(\widehat{C})$ is a finite dimensional vector space over \mathbb{F} .

Definition 4.4 A complex of knot Floer homology type (C, j, A) is said to be of L -space type if the homologies of the subcomplexes

$$A_i = C(\{x \in C \mid j(x) \leq 0, A(x) \leq i\})$$

are all isomorphic to $\mathbb{F}[U]$. For such a chain complex the sequences $\{a_i\}, \{b_i\}$ and $\{d_i\}$ can be defined exactly as in Theorems 2.3 and 3.7.

If (C, j, A) is of L -space type, then the embeddings $\psi_i: A_i \rightarrow A_{i+1}$ and $\phi_{i+1}: A_{i+1} \rightarrow A_i$ (we get by composing the multiplication by U map with the obvious embedding) induce maps on the homologies which are multiplications by U^{d_i} and $U^{e_{i+1}}$ respectively. Since the composition $\phi_{i+1} \circ \psi_i: A_i \rightarrow A_i$ is multiplication by U , so $d_i + e_{i+1} = 1$ and hence from $d_i, e_{i+1} \geq 0$ it follows that $d_i, e_{i+1} \in \{0, 1\}$.

Next we define a model complex of L -space type. For that matter fix rational numbers $\{q, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_{n+1}\}$ with the following properties:

- $\alpha_i \equiv \alpha_j \pmod{2\mathbb{Z}}$ for all $i, j \in 1, \dots, n$,
- $\beta_i \equiv \beta_j \pmod{2\mathbb{Z}}$ for all $i, j \in 1, \dots, n+1$,
- $\alpha_1 \equiv \beta_1 + 1 \pmod{2\mathbb{Z}}$, and
- $\beta_i > \alpha_i > \beta_{i+1}$ for $i = 1, \dots, n$.

The model complex of L -space type $C(q, \alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_{n+1})$ is defined as follows. The generators (over $\mathbb{F}[U^{-1}, U]$) of the complex are $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_{n+1}\}$. The Alexander and Maslov values of these generators are defined as follows:

- for $k = 1, \dots, n$, $j(x_k) = 0$,
- for $k = 1, \dots, n$, $A(x_k) = \alpha_k$,
- for $k = 1, \dots, n$ $M(x_k) = q - 2 \sum_{\ell=1}^k (\beta_\ell - \alpha_\ell) + 1$;
- for $k = 1, \dots, n+1$, $j(y_k) = 0$,
- for $k = 1, \dots, n+1$, $A(y_k) = \beta_k$, and
- for $k = 1, \dots, n+1$ $M(y_k) = q - 2 \sum_{\ell=1}^{k-1} (\beta_\ell - \alpha_\ell)$. (In particular, $M(y_1) = q$.)

We equip $C(q, \alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_{n+1})$ with the differential

$$\begin{aligned} \partial x_k &= U^{\beta_k - \alpha_k} y_k + y_{k+1} \\ \partial y_\ell &= 0 \end{aligned}$$

for $k = 1, \dots, n$ and $\ell = 1, \dots, n + 1$. It is easy to check that

Lemma 4.5 *The complex $C(q, \alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_{n+1})$ is a chain complex of L -space type. \square*

Definition 4.6 *A complex of knot Floer homology type is called minimal if the differential on its associated graded complex vanishes.*

Lemma 4.7 *Every chain complex of knot Floer homology type is filtered chain homotopy equivalent to a minimal one.*

Proof We can assume without loss of generality that C is finitely generated. Now, the homology of the associated graded object C' is a module of free $\mathbb{F}[U^{-1}, U]$ -modules. Let C' be the chain complex gotten from C by taking the homology of its associated graded object, equipped with its induced differential. It is easy to see that C' is chain homotopy equivalent to C . \square

After these preparations, we are now ready to describe the proposition which will be the key ingredient in identifying knot Floer homology and knot lattice homology.

Proposition 4.8 *Let (C, j, A) be a filtered chain complex of L -space type. Then, (C, j, A) is homotopy equivalent to one of the model complexes of L -space type $C(q, \alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_{n+1})$. Indeed, the parameters which determine the model are uniquely specified by:*

- $2n + 1 = \#\{i \mid d_i \neq d_{i-1}\}$
- Consider the sequence $\gamma_1, \dots, \gamma_{2n+1}$, arranged in decreasing order, for which $d_{\gamma_i} \neq d_{\gamma_{i-1}}$. For $i = 1, \dots, n + 1$, define $\beta_i = \gamma_{2i-1}$, and for $i = 1, \dots, n$, define $\alpha_i = \gamma_{2i}$.
- Define q to be the Maslov grading of the generator of $H_*(C(H)) = H_*(C^-) \cong \mathbb{F}[U]$.

Proof By Lemma 4.7, we can assume without loss of generality that C is minimal. Our aim is to show that this minimal C is equal to the model $C(q, \alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_{n+1})$.

Let us concentrate on the chain complex (C, j, A) . The group at bigrading $(0, i)$ can be determined by an iterated mapping cone; i.e. the following 2×2 -complex:

$$\begin{array}{ccc} A_{i+1} & \xrightarrow{\phi_{i+1}} & A_i \\ \psi_i \uparrow & & \uparrow \psi_{i-1} \\ A_i & \xrightarrow{\phi_i} & A_{i-1} \end{array}$$

There are two natural projection maps from this iterated mapping cone which we will consider. The first one is the projection to the mapping cone $A_i \xrightarrow{\psi_i} A_{i+1}$. This gives a model for the connecting differential from $C(\{j = 0, A = i\}) \rightarrow C(\{j = 0, A < i\})$. The other projection maps to $A_i \xrightarrow{\phi_i} A_{i-1}$. This gives a model for the connecting differential from $C(\{j = 0, A = i\}) \rightarrow C(\{j < 0, A = i\})$. (Here, again, ϕ_i is defined as the composition of multiplication by U followed by the natural embedding.)

We can take the homology of the 2×2 complex to get a new associated graded object:

$$\begin{array}{ccc} H_*(A_{i+1}) & \xrightarrow{(\phi_{i+1})_*} & H_*(A_i) \\ (\psi_i)_* \uparrow & & \uparrow (\psi_{i-1})_* \\ H_*(A_i) & \xrightarrow{(\phi_i)_*} & H_*(A_{i-1}). \end{array}$$

Using the fact that the composites $(\phi_{i+1})_* \circ (\psi_i)_* = U$ and $(\psi_{i-1})_* \circ (\phi_i)_* = U$, we can rewrite the above square as:

$$\begin{array}{ccc} H_*(A_{i+1}) \cong \mathbb{F}[U] & \xrightarrow{U^{1-d_i}} & \mathbb{F}[U] \cong H_*(A_i) \\ U^{d_i} \uparrow & & \uparrow U^{d_{i-1}} \\ H_*(A_i) \cong \mathbb{F}[U] & \xrightarrow{U^{1-d_{i-1}}} & \mathbb{F}[U] \cong H_*(A_{i-1}). \end{array}$$

It is not hard to see that the homology of this square is the trivial group if $d_{i-1} = d_i$ and is equal to \mathbb{F} if $d_{i-1} \neq d_i$. Indeed, those results imply that the same is true for the group $H_*(C(\{j = 0, A = i\}))$.

Since $d_i, d_{i-1} \in \{0, 1\}$, there are two subcases where the homology is non-trivial. The first subcase is where $d_{i-1} = 1$ and $d_i = 0$, in which case the

mapping cone looks like:

$$\begin{array}{ccc} H_*(A_{i+1}) \cong \mathbb{F}[U] & \xrightarrow{U} & \mathbb{F}[U] \cong H_*(A_i) \\ 1 \uparrow & & \uparrow U \\ H_*(A_i) \cong \mathbb{F}[U] & \xrightarrow{1} & \mathbb{F}[U] \cong H_*(A_{i-1}). \end{array}$$

In this case, we immediately see that both $H_*(C\{j = 0, A < i\}) = H_*(C\{j < 0, A = i\}) = 0$. Moreover, we see that the homology of $C(\{j \leq 0, A = i\})$ is supported in $j = 0$. This means that, under the map in the long exact sequence of homologies induced by the short exact sequence

$$0 \longrightarrow C(\{j < 0, A = i\}) \longrightarrow C(\{j \leq 0, A = i\}) \longrightarrow C(\{j = 0, A = i\}) \longrightarrow 0,$$

the group $H_*(C(\{j \leq 0, A = i\}))$ maps isomorphically onto $H_*(C(\{j = 0, A = i\}))$.

The second subcase is where $d_{i-1} = 0$ and $d_i = 1$, in which case the mapping cone looks like:

$$\begin{array}{ccc} H_*(A_{i+1}) & \xrightarrow{1} & H_*(A_i) \\ U \uparrow & & \uparrow 1 \\ H_*(A_i) & \xrightarrow{U} & H_*(A_{i-1}), \end{array}$$

and in this case we have that $H(C\{j = 0, A < i\}) = \mathbb{F}$, and indeed, the connecting homomorphism associated to the short exact sequence

$$0 \longrightarrow C(\{j = 0, A < i\}) \longrightarrow C(\{j = 0, A \leq i\}) \longrightarrow C(\{j = 0, A = i\}) \longrightarrow 0$$

induces an isomorphism $H_*(C(\{j = 0, A = i\})) \cong H_*(C(\{j = 0, A < i\}))$, and $H_*(C(\{j = 0, A \leq i\})) = 0$.

Notice first of all that for i large enough the embedding $A_i \rightarrow A_{i+1}$ induces isomorphism on homologies, hence for i large enough $d_i = 0$. Similarly, for small enough (negative, large in absolute value) i we get $e_{i+1} = 0$ and so $d_i = 1$. This shows that whenever $d_i = 0$ and $d_{i-1} = 1$, the corresponding i is equal to some β_k , while for i with $d_i = 0$ and $d_{i-1} = 0$ we get α_k . The above homological computations then show that $U^{\beta_k - \alpha_k} y_k$ and y_{k+1} both appear with non-zero multiplicity in ∂x_k , and indeed

$$\partial x_k = y_{k+1} + U^{\beta_k - \alpha_k} y_k + L_k, \quad (4.3)$$

where L_k is of lower order in the following ways:

- any non-zero term $U^j y_m$ in L_k with $A(U^j y_m) = A(x_k)$ has $j > \beta_k - \alpha_k$, and

- any non-zero term y_m (with $j = 0$) in L_k has $A(y_m) = \beta_m < \beta_{k+1}$.

Equation (4.3) (and the convention that Maslov grading drops by 1 under the differential and by 2 under U -multiplication) suffices to identify the Maslov gradings of the generators of C with the corresponding generators of the model.

As an $\mathbb{F}[U]$ -module, C splits into two summands (which are switched by the differential): the part in Maslov grading $q \pmod{2}$ (more concretely, the part generated over $\mathbb{F}[U]$ by the y_k 's), and the part in grading $1 + q \pmod{2}$ (i.e., the part generated by the x_k 's). We call the first part *the part in even parity*, and the second *the part in odd parity*.

Equation (4.3) immediately implies that ∂ is injective on the part with odd parity, since by Equation (4.3) $\partial x_k = y_{k+1}$ plus terms with higher U -powers in them (i.e. with $j > 0$). We claim that this property ensures that $\partial y_k = 0$. Indeed, observe that ∂y_k is a cycle in $C(\{j < 0, A < A(y_k)\})$, with odd parity. Since the differential is injective on the part with odd parity, we conclude that $\partial y_k = 0$.

It remains to verify that ∂x_k has exactly the two stated term, i.e. that $L_k = 0$. This follows from a more careful look at the Maslov gradings. Specifically, according to the grading formulas, it is easy to see that if $k < \ell$, then $M(y_k) < M(y_\ell)$ and $M(U^{\beta_k} y_k) < M(U^{\beta_\ell} y_\ell)$. Now, consider a possible term $U^t y_m$ in ∂x_k . If $m > k + 1$, then $M(U^t y_m) < M(y_m) < M(y_{k+1}) = M(x_k) - 1$, so the Maslov degree of $U^t y_m$ is too small for it to appear in ∂x_k . If $m < k$, then observe that

$$A(U^t y_m) = \beta_m - t \leq A(x_k) = \alpha_k,$$

which in turn implies

$$\begin{aligned} M(U^t y_m) &= M(U^{t-\beta_m+\alpha_k} U^{\beta_m-\alpha_k} y_m) \\ &\leq 2(\alpha_k - \beta_m - t) + M(U^{\beta_m-\alpha_k} y_m) \\ &< M(U^{\beta_k-\alpha_k} y_k) = M(x_k) - 1. \end{aligned}$$

Thus, the terms with $m < k$ and with Alexander grading $\leq A(x_k)$ also have too small Maslov grading to appear in ∂x_k . This line of reasoning ensures that ∂x_k consists of the two stated terms, i.e. $L_k = 0$. This completes the identification of (C, j, A) with the model complex $C(q, \alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_{n+1})$. \square

The combination of Proposition 4.1 and 4.8 then readily implies the following

Corollary 4.9 *Suppose that $G = \Gamma_{v_0} - v_0$ is a rational tree. If $v_0 \in \Gamma_{v_0}$ is a leaf then the doubly filtered chain complex $(\text{CF}^\infty(Y_G, \mathbf{s}), j, A)$ in Heegaard*

Floer homology and $(\mathbb{CF}^\infty(G, \mathbf{s}), j, A)$ in lattice homology are filtered chain homotopic for any spin^c structure $\mathbf{s} \in \text{Spin}^c(Y_G)$.

Proof By Theorems 2.3 and 3.7 both the knot Floer complex of (Y_G, K_{v_0}) and the knot lattice complex of Γ_{v_0} are of L -space type. According to Proposition 4.1 the corresponding sequences $\{d_i(\mathbf{s})\}$ and $\{d_i^L(\mathbf{s})\}$ are equal, hence the model complexes determined by these sequences are equal for the two cases. Since by Proposition 4.8 the complexes are filtered chain homotopic to the models described by the statement of the proposition, the statement of the corollary follows at once. \square

Proof of Theorem 1.2 Corollary 4.9 verifies the statement of the theorem in case v_0 is a leaf. Consider now the case when v_0 is not a leaf. This means that Γ_{v_0} is the connected sum of a number of trees/forest, all with distinguished vertices, and in which the distinguished vertices are leaves. For those graphs Corollary 4.9 verifies the isomorphism between the two theories, and for Γ_{v_0} then the connected sum formulae of Subsections 2.1 and 3.1 imply the statement of the theorem. \square

Proof of Theorem 1.5 Consider the graph Γ_w we get from G by erasing the framing of w . In order to show that $\mathbb{HF}^-(G)$ and $\text{HF}^-(Y_G)$ are isomorphic, it is enough to show that

- the doubly filtered chain complex in the knot lattice homology of the vertex w in Γ_w , and the doubly filtered chain complex in the knot Floer homology of the knot $K = K_w$ in $Y_{\Gamma_w - w}$ are filtered chain homotopic, and
- the maps N_n and N_n^L in the mapping cone constructions of the two theories are equal.

The first issue is exactly the content of Theorem 1.2. The second one follows from the fact that N_n (and similarly N_n^L) is a chain isomorphism between $C_i(\mathbf{s})$ and $B(\mathbf{s}_{[K]})$ (and between $C_i^L(\mathbf{s})$ and $B^L(\mathbf{s}_{v_0})$, resp.), and since the homologies of all these chain complexes are isomorphic to $\mathbb{F}[U]$, the maps N_n (and similarly N_n^L) are uniquely determined, hence are necessarily equal. This observation then concludes the proof of the theorem. \square

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