SOME LIMIT THEOREMS FOR HEIGHTS OF RANDOM WALKS ON A SPIDER

Dedicated to the memory of Marc Yor

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Abstract
A simple symmetric random walk is considered on a spider that is a collection of half lines (we call them legs) joined at the origin. We establish a strong approximation of this random walk by the so-called Brownian spider. Transition probabilities are studied, and for a fixed number of legs we investigate how high the walker and the Brownian motion can go on the legs in n steps. The heights on the legs are also investigated when the number of legs goes to infinity.

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1 Introduction

Paraphrasing Harrison and Shepp [15], in 1965 Itô and McKean ([18], Section 4.2, Problem 1) introduced a simple but intriguing diffusion process that they called skew Brownian motion, that was revisited by Walsh [29] in 1978. Walsh introduced it as a Brownian motion with excursions around zero in random directions on the plane. The random directions are values of a random variable in $[0, 2\pi)$ that are independent for different excursions with a constant value during each excursion. This "definition" can be made precise as, e.g., in Barlow, Pitman and Yor [3]. This motion is now called Walsh's Brownian motion.

Following Barlow et al. [4] and Example 1 in Evans and Sowers [13], we consider a version of Walsh's Brownian motion which lives on N semi-axes on the plane, called legs from now on, that are joined at the origin, the so-called Brownian spider, or Walsh's spider. Loosely speaking, this motion performs a regular Brownian motion on each one of the legs and, when it arrives to the origin, it continues its motion on any of the N legs with a given probability. Thus, one can construct the Brownian spider by independently putting the excursions from zero of a standard Brownian motion on the j-th leg of the spider with probability p_j, $j = 1, 2, \ldots, N$ with $\sum_{j=1}^{N} p_j = 1$. For a formal definition of the Brownian spider along these lines we refer to Section 2, (2.1). In the special case of $p_j = 1/N$, $j = 1, 2, \ldots, N$, Papanicolaou et al. [21] studied the exit time of this motion from specific sets and introduced a generalized arc-sine law as well, concerning the time spent globally on the legs. This question was further investigated in the elegant paper of Vakeroudis and Yor [28].

A natural discrete counterpart of this motion is that of random walks on a spider, i.e., replacing the Brownian motions with simple symmetric random walks on the legs. Hajri [14] studied such discrete versions as approximations of the Brownian spider, proving their weak convergence to the latter in a more general context of discrete approximations that are related to Walsh's Brownian motion. As to the weak convergence in hand, he showed that it can be deduced from the special case of $N = 2$ converging to a skew Brownian motion. Harrison and Shepp [15] reviewed the construction of a skew Brownian motion from its scale and speed measure and considered it to be a solution of a particular stochastic equation. Completing a random walk result of [15], Cherny et al. [5] concluded weak convergence of skew random walk to skew Brownian motion. For further discussions and references we refer to Lejay [20].

For the sake of studying random walks on the just mentioned spider, we proceed with concrete definitions in this regard.

Put $\mathbf{SP}(N) = (V_N, E_N)$, where, with $i = \sqrt{-1}$,

$$V_N = \left\{ v_N(r,j) = r \exp\left(\frac{2\pi i j}{N}\right), \quad r = 0, 1, ..., \quad j = 1, ..., N \right\} \tag{1.1}$$

is the set of vertices of $\mathbf{SP}(N)$, and

$$E_N = \{e_N(r,j) = (v_N(r,j), v_N(r+1,j)), \quad r = 0, 1, ..., \quad j = 1, ..., N\},$$

is the set of edges of $\mathbf{SP}(N)$. We will call the graph $\mathbf{SP}(N)$ a spider with N legs. The vertex

$$v_N(0) := v_N(0,1) = v_N(0,2) = ... = v_N(0,N)$$

is called the body of the spider, and $\{v_N(1,j), v_N(2,j), ...\}$ is the j-th leg of the spider. When the number of legs N is fixed, we will suppress it in the notation and, instead of $v_N(r,j)$ or $v_N(0)$, we will simply write $v(r,j)$ or $v(0) = \mathbf{0}$, whenever convenient.

In this paper we consider a random walk $\mathbf{S}_n,\ n = 1, 2 \ldots$, on $\mathbf{SP}(N)$ that starts from the body of the spider, i.e., $\mathbf{S}_0 = v_N(0) = \mathbf{0}$, with the following transition probabilities:

$$\mathbf{P}(\mathbf{S}_{n+1} = v_N(1,j)|\mathbf{S}_n = v_N(0)) = p_j, \quad j = 1, ..., N,$$

with

$$\sum_{j=1}^{N} p_j = 1,$$

and, for $r = 1, ..., \quad j = 1, ..., N$,

$$\mathbf{P}(\mathbf{S}_{n+1} = v_N(r+1,j)|\mathbf{S}_n = v_N(r,j)) = \mathbf{P}(\mathbf{S}_{n+1} = v_N(r-1,j)|\mathbf{S}_n = v_N(r,j)) = \frac{1}{2}.$$

The random walk $\mathbf{S}_n$ on spider $\mathbf{SP}(N)$ can be constructed from a simple symmetric random walk $S(n),\ n = 0, 1, \ldots$ on the line as follows. Consider the absolute value $|S(n)|,\ n = 1, 2, \ldots$, that consists of infinitely many excursions from zero, denoted by $G_1, G_2, \ldots$. Put these excursions, independently of each other, on leg j of the spider with probability $p_j,\ j = 1, 2, \ldots, N$. Thus we obtain the first n steps of the spider walk $\mathbf{S}.$, as above, from the first n steps of the random walk $S(\cdot)$.

We denote the Brownian spider on $\mathbf{SP}(N)$, as described in the second paragraph above, by $\mathbf{B}(t),\ t \geq 0$, that also starts from the body of the spider, i.e., $\mathbf{B}(0) = v_N(0)$.

In his book Révész [22] discussed the spider walk above in the case when $p_j = 1/N$, and the number of legs of the spider goes to infinity. In our just introduced definitions, we followed the latter book but allow the walker to select the legs with possibly unequal probabilities. In particular, one can construct this spider walk by independently putting the excursions from zero of a simple symmetric random walk on the j-th leg of the spider with probability p_j as above. Hence, in what follows, we will frequently make use of arguments in terms of the usual simple symmetric random walk on the line. In view of this, in the sequel, $\mathbf{S}_n$ will stand for spider walk, and $S(n)$ for a simple symmetric random walk on the line with respective probabilities denoted by $\mathbf{P}$ and P.

In our Section 2 we establish a strong invariance principle for approximating the spider walk $\mathbf{S}_n$ by the Brownian spider $\mathbf{B}(n)$, keeping N fixed. In Section 3 we investigate the transition probabilities, while in Section 4 we discuss how high the random walk can go on a spider with N legs, where N is still fixed. The last section, Section 5, is devoted to studying the probability that the walk goes up to certain heights simultaneously on all legs when the number of legs are increasing.

2 Strong approximations

The Brownian spider can be constructed from a standard Brownian motion $\{B(t),\, t \geq 0\}$ on the line as follows. The process $\{|B(t)|,\, t \geq 0\}$ has a countable number of excursions from zero, and let $J_1, J_2, \ldots$ denote a fixed enumeration of its excursion intervals away from zero. Then, for any $t > 0$ for which $B(t) \neq 0$, we have that $t \in J_m$ for one of the values of $m = 1, 2, \ldots$.

Extend the definition of $v_N(r,j)$ given in (1.1) to all positive values of r, i.e.,

$$v_N(r,j) = r \exp\left(\frac{2\pi i j}{N}\right). \quad r \geq 0, \quad j = 1, ..., N,$$

Thus, $v_N(r,j)$ is the j-th leg of the spider. Let κ_m, $m = 1, 2, \ldots$, be i.i.d. random variables, independent of B with

$$P(\kappa_m = j) = p_j, \quad j = 1, 2, \ldots, N.$$

We now construct the Brownian spider $\{\mathbf{B}(t),\, t \geq 0\}$ by putting the excursion whose interval is J_m, to leg κ_m on the spider $\mathbf{SP}(N)$. Hence we can define the Brownian spider as discussed in paragraph 2 of our Introduction by

$$\mathbf{B}(t) := \sum_{m=1}^{\infty} I\{t \in J_m\} v_N(|B(t)|, \kappa_m), \quad \text{if} \quad B(t) \neq 0, \tag{2.1}$$

and

$$\mathbf{B}(t) := v_N(0) = \mathbf{0}, \quad \text{if} \quad B(t) = 0,$$

where $I\{...\}$ is the indicator function.

This definition of the Brownian spider $\{\mathbf{B}(t),\, t \geq 0\}$ is an analogue of that of a skew Brownian motion given in Appuhamillage et al. [1]. In this regard, we may also refer to Revuz and Yor, Exercise 2.16, Chap XII in [23]. We note in passing that the Brownian spider with $N = 2$ is equivalent to the skew Brownian motion.

Moreover, define the distance on $\mathbf{SP}(N)$ by

$$|v_N(x,j) - v_N(y,j)| = |x - y|, \quad j = 1, \ldots, N$$

$$|v_N(x,j) - v_N(y,k)| = x + y, \quad j, k = 1, \ldots, N,\ j \neq k.$$

First we mention the weak convergence result of Hajri [14].

Theorem 2.1 *Let $\mathbf{S}(t)$, $t \geq 0$, be the linear interpolation of $\mathbf{S}_n$, $n = 0, 1, \ldots$. Then*

$$\left\{\frac{\mathbf{S}(nt)}{\sqrt{n}}, t \geq 0\right\} \to \{\mathbf{B}(t),\, t \geq 0\}$$

weakly on $C[0, \infty)$, as $n \to \infty$.

Our strong approximation result, that also contains Theorem 2.1, reads as follows.

Theorem 2.2 *On a rich enough probability space one can define a Brownian spider* $\{\mathbf{B}(t),\, t \geq 0\}$ *and a random walk* $\{\mathbf{S}_n,\, n = 0, 1, 2, \ldots\}$*, both on* $\mathbf{SP}(N)$*, and both selecting their legs with the same probabilities* p_j*,* $j = 1, 2, \ldots, N$ *so that, as* $n \to \infty$*, we have*

$$|\mathbf{S}_n - \mathbf{B}(n)| = O((n \log\log n)^{1/4} (\log n)^{1/2}) \quad a.s.$$

Proof. Start with a Skorokhod embedding for $B(\cdot)$ and $S(\cdot)$, i.e., define

$$\begin{aligned}
\tau_1 &= \inf\{t > 0 : |B(t)| = 1\}, \\
\tau_2 &= \inf\{t > \tau_1 : |B(t) - B(\tau_1)| = 1\}, \\
&\ldots \\
\tau_{i+1} &= \inf\{t > \tau_i : |B(t) - B(\tau_i)| = 1\}.
\end{aligned}$$

Then $\{S(n) := B(\tau_n),\, n = 1, 2, \ldots\}$ is a simple symmetric random walk on the line and, as $n \to \infty$, we have

$$|S(n) - B(n)| = |B(\tau_n) - B(n)| = O((n \log\log n)^{1/4} (\log n)^{1/2}) \quad a.s.$$

The latter Skorokhod embedding of B and S is a special case of Theorem 1.5 of Strassen [27], (cf. also Révész [22], Theorem 6.1 when S is a simple symmetric random walk).

Now construct $\mathbf{B}(t)$ from $B(t)$ as described above. It is clear from Skorokhod construction that an excursion of $S(\cdot)$ lies entirely within its corresponding excursion of $B(\cdot)$, so construct $\mathbf{S}_n$ by putting this excursion on the same leg as the corresponding excursion of $B(\cdot)$. We note in passing that small excursions of the underlying Brownian motion, namely those that do not reach 1, are not needed for the construction of $\{\mathbf{S}_n := \mathbf{B}(\tau_n),\, n = 1, 2, \ldots\}$, i.e., for that of a random walk on $\mathbf{SP}(N)$.

Consider now $\mathbf{B}(n)$ and $\mathbf{B}(\tau_n)$ when they are on the same leg. Then, as $n \to \infty$,

$$|\mathbf{S}_n - \mathbf{B}(n)| = |\mathbf{B}(\tau_n) - \mathbf{B}(n)| = |B(\tau_n) - B(n)| = O((n \log\log n)^{1/4} (\log n)^{1/2}) \quad a.s.$$

However, when $\mathbf{B}(n)$ and $\mathbf{B}(\tau_n)$ are on different legs, then

$$|\mathbf{S}_n - \mathbf{B}(n)| = |\mathbf{B}(\tau_n) - \mathbf{B}(n)| = |B(\tau_n)| + |B(n)|.$$

But in this case there is a point c_n between n and τ_n, where $B(c_n) = 0$, with $|n - c_n| \leq |n - \tau_n|$ and $|\tau_n - c_n| \leq |n - \tau_n|$. Since $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots$ is a sequence of random variables with mean 1 and variance 1 (cf., e.g., page 54 in Révész [22]), by the law of the iterated logarithm (LIL), as $n \to \infty$, we get that

$$|\tau_n - n| = O(n \log\log n)^{1/2} =: a_n.$$

Now applying the Wiener large increments result of Csörgő-Révész [10], (see also page 30 in [11]), as $n \to \infty$, we obtain

$$
\begin{aligned}
|B(n)| = |B(n) - B(c_n)| &\leq \sup_{0 \leq s \leq a_n} |B(n-s) - B(n)| + \sup_{0 \leq s \leq a_n} |B(n+s) - B(n)| \\
&= O((n \log\log n)^{1/4} (\log n)^{1/2}) \quad a.s.,
\end{aligned}
$$

and similarly

$$
|B(\tau_n)| = |B(\tau_n) - B(c_n)| = O((n \log\log n)^{1/4} (\log n)^{1/2}) \quad a.s.
$$

This completes the proof of the Theorem 2.2. $\Box$

3 Transition probabilities

We assume throughout that $\mathbf{S}_0 = \mathbf{0}$. Clearly, we have $\mathbf{P}(\mathbf{S}_{2n} = \mathbf{0}) = P(S(2n) = 0)$.

Theorem 3.1 *For $i \geq 1, j \geq 1$ integers*

(i) $\mathbf{P}(\mathbf{S}_{2n+2k} = v(2j, \ell) | \mathbf{S}_{2k} = \mathbf{0}) = 2p_\ell \, P(S(2n) = 2j), \quad j \leq n$

(ii) $\mathbf{P}(\mathbf{S}_{2k+2n} = v(2i, \ell^*) | \mathbf{S}_{2k} = v(2j, \ell)) = 2p_{\ell^*} P(S(2n) = 2(j+i)), \quad i + j \leq n, \quad \ell \neq \ell^*$

(iii) $\mathbf{P}(\mathbf{S}_{2n+2k} = v(2i, \ell) | \mathbf{S}_{2k} = v(2j, \ell))$
$= P(S(2n) = 2(j-i)) - (1 - 2p_\ell) P(S(2n) = 2(j+i)), \quad |i - j| \leq n.$

Proof: It is well-known that for the simple symmetric random walk we have

$$
P(S(2n) = 2k) = \frac{1}{2^{2n}} \binom{2n}{n+k}, \tag{3.1}
$$

and, for any integer $k \geq 1$, we have from the ballot theorem, that

$$
P(S(1) > 0,\ S(2) > 0, ..., S(2n-1) > 0,\ S(2n) = 2k) = \frac{k}{n} \frac{1}{2^{2n}} \binom{2n}{n+k}. \tag{3.2}
$$

Partitioning according to the time of the last return to the origin, using (3.2) and taking into account that the probability of the next step after the last return to zero in our context is p_ℓ instead of $1/2$, we get

$$
\begin{aligned}
&\mathbf{P}(\mathbf{S}_{2n} = v(2j, \ell)) \\
&= \sum_{m=0}^{n-1} \mathbf{P}(\mathbf{S}_{2m} = 0) 2p_\ell P(S(1) > 0, S(2) > 0, \ldots, S(2(n-m)-1) > 0, S(2(n-m)) = 2j) \\
&= \sum_{m=0}^{n-1} \mathbf{P}(\mathbf{S}_{2m} = 0) p_\ell \frac{j}{n-m} \frac{2}{2^{2(n-m)}} \binom{2n-2m}{n-m+j}
\end{aligned}
$$

$$= 2p_\ell \sum_{m=0}^{n-1} P(S(2m) = 0)\, P(S(1) \neq 0, S(2) \neq 0, \ldots, S(2(n-m)-1) \neq 0, S(2(n-m)) = 2j)$$
$$= 2p_\ell P(S(2n) = 2j),$$

which proves (i).

For $\ell \neq \ell^*$, partitioning again according to the last visit to the origin, we arrive at

$$\mathbf{P}(\mathbf{S}_{2k+2n} = v(2i, \ell^*) | \mathbf{S}_{2k} = v(2j, \ell))$$
$$= \sum_{m=i}^{n-j} P(S(0) = 2j, S(2n-2m) = 0)\, 2p_{\ell^*}\, P(S(0) = 0, S(1) > 0, \ldots, S(2m-1) > 0, S(2m) = 2i)$$
$$= \frac{1}{2^{2n}} 2p_{\ell^*} \sum_{m=i}^{n-j} \binom{2n-2m}{n-m+j} \binom{2m}{m+i} \frac{i}{m} = 2p_{\ell^*} P(S(2n) = 2(j+i)),$$

which proves (ii).

Finally to prove (iii), observe that any path from $v(2j, \ell)$ to $v(2i, \ell)$ either crosses the origin or not. For the transition with crossing the origin we have

$$\mathbf{P}(\mathbf{S}_{2k+2n} = v(2i, \ell) | \mathbf{S}_{2k} = v(2j, \ell), \mathbf{S}_{2k+2m} = \mathbf{0} \text{ for some } j \leq m \leq n-i)$$
$$= \mathbf{P}(\mathbf{S}_{2k+2n} = v(2i, \ell^*) | \mathbf{S}_{2k} = v(2j, \ell))$$

with the understanding that here leg ℓ^* is actually leg ℓ, and hence the probability gained in (ii) should be used with $p_{\ell^*} = p_\ell$. In the case when the transition happens without crossing the origin, the corresponding probability can be calculated just like for a simple symmetric walk, using the reflection principle. Thus

$$\mathbf{P}(\mathbf{S}_{2k+2n} = v(2i, \ell) | \mathbf{S}_{2n} = v(2j, \ell))$$
$$= 2p_\ell P(S(2n) = 2(j+i)) + P(S(2n) = 2(j-i)) - P(S(2n) = 2(j+i))$$
$$= P(S(2n) = 2(j-i)) - (1 - 2\,p_\ell)\, P(S(2n) = 2(j-i)).$$

proving (iii). □

Recall that by the local central limit theorem, as $n \to \infty$, we have

$$P(S(2n) = 2k) \sim \frac{1}{\sqrt{\pi n}} e^{-\frac{k^2}{n}}$$

if $k/\sqrt{n}$ is bounded. Hence, via Theorem 3.1, we obtain the corresponding limit theorem for transition probabilities as follows.

Theorem 3.2

(i) $\lim_{n\to\infty}\sqrt{n}\mathbf{P}(\mathbf{S}_{2[nt]+2k}=v(2[y\sqrt{n}],\ell)|\mathbf{S}_{2k}=\mathbf{0})=\frac{2p_\ell}{\sqrt{\pi t}}e^{-y^2/t},$

(ii) $\lim_{n\to\infty}\sqrt{n}\mathbf{P}(\mathbf{S}_{2[nt]+2k}=v(2[y\sqrt{n}],\ell^*)|\mathbf{S}_{2k}=v(2[x\sqrt{n}],\ell))=\frac{2p_{\ell^*}}{\sqrt{\pi t}}e^{-(x+y)^2/t},\quad \ell\neq\ell^*,$

(iii) $\lim_{n\to\infty}\sqrt{n}\mathbf{P}(\mathbf{S}_{2[nt]+2k}=v(2[y\sqrt{n}],\ell)|\mathbf{S}_{2k}=v(2[x\sqrt{n}],\ell))$
$=\frac{1}{\sqrt{\pi t}}e^{-(x-y)^2/t}-\frac{1-2p_\ell}{\sqrt{\pi t}}e^{-(x+y)^2/t}.$

The transition density for Brownian spider in the case of $N=2$, i.e., for a skew Brownian motion, is given in equations (3) and (4) in Walsh [29] (see also (2.2) in Appuhamillage et al. [1]). The transition density for Brownian spider in the case of $p_j=1/N$, $j=1,2,\ldots,N$, is given in Papanicolaou et al. [21]. For general p_j, via Walsh, it can be given as follows. Define the transition density $p(t,v(x,\ell),v(y,\ell^*))$ as

$$\mathbf{P}(\mathbf{B}(t+s)\in v(dy,\ell^*)|\mathbf{B}(s)=v(x,\ell))=p(t,v(x,\ell),v(y,\ell^*))dy.$$

As a consequence of Theorem 3.2, we can conclude the following Brownian spider transition density analogue.

Corollary 3.1

$$\begin{aligned}
p(t,v(0),v(y,\ell))&=\frac{2p_\ell}{\sqrt{2\pi t}}e^{-y^2/2t},\\
p(t,v(x,\ell),v(y,\ell^*))&=\frac{2p_{\ell^*}}{\sqrt{2\pi t}}e^{-(x+y)^2/2t},\quad \ell\neq\ell^*,\\
p(t,v(x,\ell),v(y,\ell))&=\frac{1}{\sqrt{2\pi t}}e^{-(x-y)^2/2t}-\frac{1-2p_\ell}{\sqrt{2\pi t}}e^{-(x+y)^2/2t}.
\end{aligned}$$

4 Brownian and random walk heights on spider

One of the natural questions to ask is how high does the walker go up on the legs of the spider. Let $H(j,n)$ denote the highest point reached by the random walk on leg j of the spider in n steps. Formally, let

$$\xi(v(r,j),n):=\#\{k:0<k\leq n,\,\mathbf{S}_k=v(r,j)\}\tag{4.1}$$

and define

$$H(j,n)=\max\{r:\,\xi(v(r,j),n)\geq 1\}.$$

Let

$$H_M(n)=\max_{1\leq j\leq N}H(j,n),\qquad H_m(n)=\min_{1\leq j\leq N}H(j,n).$$

Similarly, let $\mathbf{H}(j,t)$ be the highest point reached by the Brownian spider $\mathbf{B}(\cdot)$ on leg j by time t. Put

$$\mathbf{H}_M(t) = \max_{1\le j\le N} \mathbf{H}(j,t), \qquad \mathbf{H}_m(t) = \min_{1\le j\le N} \mathbf{H}(j,t).$$

Note that for fixed j the distribution of $H(j,n)$ and $\mathbf{H}(j,t)$ can be reduced to the case $N=2$, which is equivalent to skew Brownian motion and skew random walk. This can be done by keeping the $j-$th leg as a new leg 1, and unite all the other legs into leg 2. Then the distribution of heights $H(j,n)$ and $\mathbf{H}(j,n)$ are equal to the distribution of the maximum of skew random walk and maximum of skew Brownian motion, respectively. The latter one is given in Appuhamillage and Sheldon [2]. Using this result, we obtain the distribution of $\mathbf{H}(j,n)$, that also gives the limiting distribution of $H(j,n)$ as follows.

Theorem 4.1

$$\lim_{n\to\infty} \mathbf{P}(H(j,n) < y\sqrt{n}) = \mathbf{P}(\mathbf{H}(j,t) < y\sqrt{t}) = 2p_j \sum_{k=1}^{\infty} (1-2p_j)^{k-1}(2\Phi((2k-1)y)-1), \tag{4.2}$$

where Φ is the standard normal distribution function.

Clearly, $H_M(n)$ and $\mathbf{H}_M(t)$ are equal to the maximum of a simple symmetric walk $S(n)$ and of a standard Brownian motion, respectively. So the law of the iterated logarithm (LIL) and the so called other LIL of Chung [6] continue to hold for these processes.

Theorem 4.2

$$\limsup_{n\to\infty} \frac{H_M(n)}{\sqrt{2n\log\log n}} = \limsup_{t\to\infty} \frac{\mathbf{H}_M(t)}{\sqrt{2t\log\log t}} = 1 \quad a.s.$$

$$\liminf_{n\to\infty} \left(\frac{\log\log n}{n}\right)^{1/2} H_M(n) = \liminf_{t\to\infty} \left(\frac{\log\log t}{t}\right)^{1/2} \mathbf{H}_M(t) = \frac{\pi}{\sqrt{8}} \quad a.s.$$

However, it is a much more interesting question to seek the maximal height which can be reached on all legs simultaneously. To be more precise, we are to describe what one can say about $H_m(n)$ and $\mathbf{H}_m(t)$. For limsup and Hirsch-type liminf of these processes, we will prove the following respective results.

Theorem 4.3

$$\limsup_{n\to\infty} \frac{H_m(n)}{\sqrt{2n\log\log n}} = \limsup_{t\to\infty} \frac{\mathbf{H}_m(t)}{\sqrt{2t\log\log t}} = \frac{1}{2N-1} \quad a.s. \tag{4.3}$$

Let $g(t)$, $t\ge 1$, be a nonincreasing function. Then

$$\liminf_{n\to\infty} \frac{H_m(n)}{n^{1/2}g(n)} = \liminf_{t\to\infty} \frac{\mathbf{H}_m(t)}{t^{1/2}g(t)} = 0 \quad or \quad \infty \tag{4.4}$$

according as $\int_1^\infty g(t)\,dt/t$ diverges or converges.

Proof. By the strong approximation given in Section 2, it suffices to prove this theorem either for $H_m(n)$ or for $\mathbf{H}_m(t)$. Denote by

$$M_1(t) \geq M_2(t) \geq \ldots \geq M_k(t) \geq \ldots$$

the ranked heights of excursions of a standard Brownian motion on the line up to time t, including the height of a possible incomplete excursion at the end. It is shown in Csáki and Hu [8] it is shown for fixed k that

$$\limsup_{t\to\infty} \frac{M_k(t)}{\sqrt{2t\log\log t}} = \frac{1}{2k-1} \quad a.s. \tag{4.5}$$

and that, for a nonincreasing function $g(t)$,

$$\liminf_{t\to\infty} \frac{M_k(t)}{t^{1/2}g(t)} = 0 \quad or \quad \infty$$

according as $\int_1^\infty g(t)\,dt/t$ diverges or converges.

It is clear that for the Brownian spider, constructed from the standard Brownian motion as in (2.1), we have $\mathbf{H}_m(t) \leq M_N(t)$, whenever $M_1(t)$, $M_2(t)$, $\ldots, M_N(t)$ are on different legs. Consequently,

$$\limsup_{t\to\infty} \frac{\mathbf{H}_m(t)}{\sqrt{2t\log\log t}} \leq \limsup_{t\to\infty} \frac{M_N(t)}{\sqrt{2t\log\log t}} = \frac{1}{2N-1}$$

and

$$\liminf_{t\to\infty} \frac{\mathbf{H}_m(t)}{t^{1/2}g(t)} \leq \liminf_{t\to\infty} \frac{M_N(t)}{t^{1/2}g(t)} = 0,$$

provided $\int_1^\infty g(t)\,dt/t$ diverges.

To show the lower bound in (4.3), let the events A_n and C_n be defined by

$$A_n = \left\{ M_N(n) \geq (1-\varepsilon)\frac{\sqrt{2n\log\log n}}{2N-1} \right\},$$

$$C_n = \{M_1(n),\, M_2(n),\, \ldots, M_N(n) \text{ are on different legs}\}.$$

Then, in view of (4.5), $P(\limsup_n A_n) =: P(A_n\ i.o.) = 1$ and, furthemore, $P(C_n) \geq p_1 p_2 \ldots p_N = c > 0$.

For the next lemma we refer to Klass [19].

Lemma 4.1 *Let $\{A_n\}_{n\geq 1}$ be an arbitrary sequence of events such that $P(A_n\ i.o.) = 1$. Let $\{C_n\}_{n\geq 1}$ be another arbitrary sequence of events that is independent of $\{A_n\}_{n\geq 1}$ and assume that for some $n_0 > 0$, $P(C_n) \geq c > 0$, for all $n > n_0$. Then we have $P(A_n C_n\ i.o.) \geq c$.*

Applying this, we get $P(A_n\, C_n\ i.o.) > 0$ with A_n, C_n as above. From the $0-1$ law we have also $P(A_n\, C_n\ i.o.) = 1$. This implies

$$\mathbf{P}\left(\mathbf{H}_m(n) \geq (1-\varepsilon)\frac{\sqrt{2n\log\log n}}{2N-1}\quad i.o.\right) = 1,$$

with arbitrary $0 < \varepsilon < 1$. Hence we have the lower bound in (4.3).

Now we turn to the convergence part of the liminf result. From the limit distribution of $\mathbf{H}(j,t)$ given in (4.2), for small y we have $2\Phi((2k-1)y) - 1 \leq c(2k-1)y$, hence

$$\mathbf{P}(\mathbf{H}(j,t) < y\sqrt{t}) \leq 2cp_j y \sum_{k=1}^{\infty}(1-2p_j)^{k-1}(2k-1) =: c_j y,$$

as $y \to 0$, with some positive constant c_j. It is easy to see that this implies that for any constant $C > 0$ we have

$$\mathbf{P}(\mathbf{H}_m(t) < Cy\sqrt{t}) < c'y \tag{4.6}$$

as well with some positive constant c'. By Theorem 2.1, or by the strong approximation result of Theorem 2.2, we also have

$$\mathbf{P}(H_m(n) < Cy\sqrt{n}) < c'y. \tag{4.7}$$

We prove the convergence part of the liminf in (4.4) just like that in Hirsch [16]. Suppose that $g(n)$ is nonincreasing and

$$\sum_{n=1}^{\infty}\frac{g(n)}{n} < \infty.$$

Then

$$\sum_{n=1}^{\infty} g(2^n) < \infty \tag{4.8}$$

as well. Consequently, from (4.7) we conclude

$$\mathbf{P}\left(\frac{H_m(2^n)}{2^{n/2}} \leq 2C\, g(2^n)\right) \leq 2c'\, g(2^n). \tag{4.9}$$

By (4.8) and the Borel-Cantelli lemma, we arrive at

$$H_m(2^n) \geq 2C\, 2^{n/2} g(2^n) \quad a.s. \tag{4.10}$$

for $n \geq n_0$ with some n_0. For an arbitrary ℓ, on selecting k_ℓ such that

$$2^{k_\ell} < \ell < 2\, 2^{k_\ell},$$

we have

$$H_m(\ell) \geq H_m\left(2^{k_\ell}\right) > 2C\, 2^{k_\ell/2}\, g\left(2^{k_\ell}\right) \geq C\sqrt{\ell}g(\ell) \quad a.s.$$

Since C is arbitrary,

$$\lim_{n\to\infty} \frac{H_m(n)}{\sqrt{n}g(n)} = \infty \quad a.s.$$

The convergence part for liminf in (4.4) is proved. This also completes the proof of Theorem 4.3. $\Box$

Recall the definitions of $H(j,n)$ and $\mathbf{H}(j,t)$, i.e., the respective maximum heights of the random walk on spider and Brownian spider on leg j up to time n and t, respectively.

Our Theorem 4.3. tells us how high could the random walker on a spider, and Brownian spider, respectively, go up simultaneously on each leg. Now we ask the following question: if we select N non-negative numbers, as heights, under what conditions is it possible that the random walker can go up that high on each leg. The same question can be asked for Brownian spider. Introducing the notations $\mathbb{R}_+^N$ for the set of vectors with non-negative components in N-dimensional Euclidean space $\mathbb{R}^N$, i.e.,

$$\mathbb{R}_+^N := \{(a(1),\ldots,a(N)) \in \mathbb{R}^N,\ a(1) \geq 0,\ldots,a(N) \geq 0\},$$

our answer is the following.

Theorem 4.4 *The set of vectors*

$$\left(\frac{H(1,n)}{\sqrt{2n\log\log n}},\ldots,\frac{H(N,n)}{\sqrt{2n\log\log n}}\right), \quad n \geq 3 \tag{4.11}$$

and

$$\left(\frac{\mathbf{H}(1,t)}{\sqrt{2t\log\log t}},\ldots,\frac{\mathbf{H}(N,t)}{\sqrt{2t\log\log t}}\right), \quad t \geq 3 \tag{4.12}$$

are almost surely relatively compact in $\mathbb{R}_+^N$ and their respective sets of limit points, as $n \to \infty$ and $t \to \infty$, are given by

$$\left\{(a(1),\ldots,a(N)) \in \mathbb{R}_+^N :\ \mathcal{A}(N) := 2\sum_{j=1}^N a(j) - \max_{1\leq j\leq N} a(j) \leq 1\right\}. \tag{4.13}$$

For the case $N = 2$ equivalent statements are given in Csáki and Hu [9], Theorem 1.2, and Révész [22], Theorem 5.6. For the proof we will use the celebrated functional law of the iterated logarithm of Strassen [26]. By our strong invariance principle, it suffices to prove Theorem 4.4 for random walk on spider. Let $\mathcal{S}$ be the Strassen class of functions, i.e., $\mathcal{S} \subset C([0,1],\mathbb{R})$ is the class of absolutely continuous functions (with respect to the Lebesgue measure) on $[0,1]$ for which

$$f(0) = 0 \qquad \text{and} \qquad I(f) = \int_0^1 \dot{f}^2(x)dx \leq 1. \tag{4.14}$$

Consider the continuous versions of the random walk process $\{S(nx);\ 0 \leq x \leq 1\}_{n=1}^\infty$ defined by linear interpolation from the simple symmetric random walk $\{S(n)\}_{n=0}^\infty$.

Theorem 4.5 [26] *The sequence of random functions*

$$\left\{ \frac{S(nx)}{(2n\log\log n)^{1/2}};\ 0 \le x \le 1 \right\}_{n\ge 3},$$

as $n \to \infty$, is almost surely relatively compact in the space $C([0,1])$ and the set of its limit points is the class of functions $\mathcal{S}$.

Proof of Theorem 4.4. Recall the construction of spider walk from simple symmetric random walk as described in Section 1. If $a(j) = 0$ for all $j = 1, 2, \ldots, N$, then consider the function $f(x) = 0$, $0 \le x \le 1$. It is obvious that this function is in $\mathcal{S}$, so almost surely there is a subsequence n_k for which

$$\lim_{k\to\infty} \sup_{0\le x\le 1} \frac{|S(n_k x)|}{\sqrt{2n_k \log\log n_k}} = 0.$$

This is also true for the maximums of all excursions. Consequently,

$$\lim_{k\to\infty} \frac{H(j, n_k)}{\sqrt{2n_k \log\log n_k}} = 0, \quad j = 1, 2, \ldots, N$$

i.e., $(0, \ldots, 0)$ is almost surely a limit point of (4.11) .

Now assume that there are L strictly positive elements among $a(j)$, $j = 1, 2, \ldots, N$, denoted by $a(r_1), a(r_2), \ldots, a(r_L)$, and let $a(r_L) = \max_{1\le j\le N} a(j)$ so that we have

$$\mathcal{A}(N) = 2\sum_{i=1}^{L-1} a(r_i) + a(r_L) = 2\sum_{j=1}^{N} a(j) - \max_{1\le j\le N} a(j) \le 1. \tag{4.15}$$

We show that $(a(1), \ldots, a(N))$ is almost surely a limit point of (4.11). Construct a piecewise linear function $f(\cdot)$ as follows. Let

$$x_\ell = 2(a(r_1) + \ldots + a(r_{\ell-1})) + a(r_\ell), \quad \ell = 1, 2, \ldots, L$$

and

$$f(0) = 0, \quad f(x_\ell) = (-1)^{\ell-1} a(r_k), \quad \ell = 1, 2, \ldots, L, \quad f(1) = f(x_L) = (-1)^{L-1} a(r_L)$$

and let $f(\cdot)$ be linear in between.

It is easy to see that $f(\cdot)$ is absolutely continuous and $I(f) = 2\sum_{i=1}^{L-1} a(r_i) + a(r_L) \le 1$, consequently $f(\cdot) \in \mathcal{S}$. It follows that almost surely there exists a subsequence n_k such that for the largest L excursion heights $M(r_i, n_k)$ of $S(n_k)$ we have

$$\lim_{k\to\infty} \frac{M(r_i, n_k)}{\sqrt{2n_k \log\log n_k}} = a(r_i) \quad a.s.,$$

for $i = 1, 2, \ldots, L$ and, if $M(n_k)$ is another excursion maximum, then we have

$$\lim_{k\to\infty} \frac{M(n_k)}{\sqrt{2n_k \log\log n_k}} = 0.$$

For the simple symmetric random walk with n steps, define the event

$$A_n = \left\{\text{there are excursion heights } M(j,n) \text{ such that } \left|\frac{M(j,n)}{\sqrt{2n\log\log n}} - a(j)\right| \le \varepsilon,\ j = 1, 2, \ldots, N\right\}$$

Then $P(A_n\, i.o.) = 1$.

Let C_n be the event that on constructing spider walk from a simple symmetric random walk, the excursion with height $M(j,n)$ falls to leg j for all $j = 1, 2, \ldots, N$. $P(C_n) = p_1 \ldots p_N > 0$, hence, by Lemma 4.1, $P(A_nC_n\, i.o.) > 0$, and by the 0-1 law this probability is 1. Consequently, $(a(1), a(2), \ldots, a(N))$ is almost surely a limit point of (4.11).

To conclude the only if part, assume that $(a(1), \ldots, a(N))$ is a limit point of (4.11), i.e., there exists a subsequence n_k such that

$$\lim_{k\to\infty} \frac{H(j,n_k)}{\sqrt{2n_k\log\log n_k}} = a(j), \quad j = 1, 2, \ldots, N.$$

Then there exist excursion heights $M(j,n)$ of the random walk $S(i)$, $i = 1, 2, \ldots$, and a subsequence n_k, $k = 1, 2, \ldots$ for which

$$\lim_{k\to\infty} \frac{M(j,n_k)}{\sqrt{2n_k\log\log n_k}} = a(j), \quad j = 1, 2, \ldots, N,$$

and a function $f(\cdot) \in \mathcal{S}$ such that

$$|f(x_{2\ell-1})| = a(r_\ell),\ f(x_{2\ell-2}) = 0,\ \ell = 1, \ldots, L,\ |f(1)| \le a(r_L),$$

where, as before, $a(r_1), \ldots, a(r_L)$ are the strictly positive terms among $a(1), \ldots, a(N)$, and $x_0 = 0 < x_1 < x_2 < \ldots < x_{2L-1} \le 1$.

We use the following result (cf. Riesz and Sz.-Nagy [24], p. 75, or Shorack and Wellner [25], p. 79).

Lemma 4.2 *$f(\cdot) \in \mathcal{S}$ if and only if $f(0) = 0$ and for every partition $0 = x_0 < x_1 < \ldots < x_m = 1$ we have*

$$\sum_{i=1}^{m} \frac{(f(x_i) - f(x_{i-1}))^2}{x_i - x_{i-1}} \le 1. \tag{4.16}$$

This lemma yields

$$\sum_{\ell=1}^{L-1} a^2(r_\ell)\left(\frac{1}{x_{2\ell-1} - x_{2\ell-2}} + \frac{1}{x_{2\ell} - x_{2\ell-1}}\right) + \frac{a^2(r_L)}{1 - x_{2L-2}}$$

$$\leq \sum_{\ell=1}^{L-1} a^2(r_\ell)\left(\frac{1}{x_{2\ell-1}-x_{2\ell-2}}+\frac{1}{x_{2\ell}-x_{2\ell-1}}\right)+\frac{a^2(r_L)}{x_{2L-1}-x_{2L-2}}+\frac{(f(1)-a(r_L))^2}{1-x_{2L-1}} \leq 1. \tag{4.17}$$

In case $x_{2L-1}=1$ we take the last term in (4.17) to be equal to zero.

The summation of (4) is of the form

$$g(z_1,z_2,\ldots,z_{2L-1}) := \sum_{i=1}^{2L-1}\frac{b_i^2}{z_i},$$

where $z_i>0$ with $\sum_{i=1}^{2L-1} z_i = 1$. We want to show that (4.17) implies that $\mathcal{A}(N)\leq 1$. To this end, we first calculate the minimum of $g(z_1,z_2,\ldots,z_{2L-1})$.

To find the values of z_i such that the function g takes its minimum, we have to solve a conditional extreme value problem by the Lagrange multiplier method, i.e., minimize

$$g(z_1,\ldots,z_{2L-1})+\lambda(z_1+\ldots+z_{2L-1}-1).$$

So we have to solve the equations

$$\frac{b_i^2}{z_i^2}=\lambda, \quad i=1,\ldots,2L-1.$$

Its solution is $z_i = b_i/(\sum_{i=1}^{2L-1} b_i)$, $i=1,\ldots,L$, i.e., the minimum value of g is $(\sum_{i=1}^{2L-1} b_i)^2$. Having $g\leq 1$, by (4.16)-(4.17) we conclude that $\sum_{i=1}^{2L-1} b_i \leq 1$. Consequently, for $\mathcal{A}$ as in (4.15), we obtain

$$\mathcal{A}(N) = 2\sum_{j=1}^{N} a(j) - \max_{1\leq j\leq N} a(j) \leq 2\sum_{i=1}^{L-1} a(r_i) + a(r_L) \leq 1.$$

This completes the proof of Theorem 4.4. □

It is worthwhile to give the following corollaries.

Corollary 4.1 *Let $M_1(n)\geq M_2(n)\geq\ldots$ be the ranked heights of excursions of a simple symmetric random walk up to time n. Then for finite N we have*

$$\limsup_{n\to\infty}\frac{M_1(n)+2\sum_{i=2}^{N} M_i(n)}{\sqrt{2n\log\log n}} = 1$$

almost surely.

The same is true for Brownian motion.

Corollary 4.2 *Let $M_1(t)\geq M_2(t)\geq\ldots$ be the ranked heights of excursions of a standard Brownian motion up to time t. Then for finite N we have*

$$\limsup_{n\to\infty}\frac{M_1(t)+2\sum_{i=2}^{N} M_i(t)}{\sqrt{2t\log\log t}} = 1$$

almost surely.

5 Increasing number of legs

In this section we suppose that $p_1 = p_2 = \ldots = p_N = \dfrac{1}{N}$.

Let

$$\xi(v_N(r,j),n) := \#\{k : k \le n,\, \mathbf{S}_k = v_N(r,j)\},$$

i.e., $\xi(v_N(r,j),n)$ is the local time of $\mathbf{S}$ at time n and locus $r > 0$ on the leg j and, for $r = 0$, put

$$\zeta(n) := \#\{k : k \le n,\, \mathbf{S}_k = v_N(0)\} = \xi(v_N(0),n).$$

Define also the events

$$\begin{aligned} M(n,L) : &= \{\min_{1\le j\le N} \xi(v_N(L,j),n) \ge 1\} \\ A(n,L,k) : &= \{\min_{1\le j\le N} \xi(v_N(L,j),n) \ge k\}. \end{aligned}$$

Observe that the meaning of the event $M(n,L)$ is that in n steps the walker climbs up to at least L on each leg. The special case $M(n,1)$ means that in n steps each leg is visited at least once. $A(n,L,k)$ means that in n steps the walker visits each leg at height L at least k times.

We recall the main result from Révész [22], page 374:

Theorem 5.1 *For the* $\mathbf{SP}(N)$

$$\lim_{N\to\infty} \mathbf{P}(M([(N\log N)^2],1)) = \left(\frac{2}{\pi}\right)^{1/2} \int_1^\infty e^{-u^2/2}\,du = P(|Z|>1), \tag{5.1}$$

where Z is a standard normal random variable.

We also have the well-known result that for any $x > 0$

$$\lim_{n\to\infty} P\left(\frac{\xi(0,n)}{\sqrt{n}} \le x\right) = P(|Z| \le x). \tag{5.2}$$

In this section we summerize what we can say about $M(n,L)$. Our main result is as follows.

Theorem 5.2 *For any integer $L \le \frac{N}{\log N}$, for the $\mathbf{SP}(N)$ we have*

$$\lim_{N\to\infty} \mathbf{P}(M([(cLN\log N)^2],L)) = P\left(|Z| > \frac{1}{c}\right). \tag{5.3}$$

To formulate in words, the theorem above gives the limiting probability of the event that, as $N \to \infty$, in $[(cLN\log N)^2]$ steps the walker arrives at least to height L on each of the N legs at least once.

The next two results are natural companions of the latter one.

Theorem 5.3 *For any integer $L \leq \frac{N}{\log N}$, and any sequence $f(N) \uparrow \infty$, for the* $\mathbf{SP}(N)$ *we have*

$$\lim_{N\to\infty} \mathbf{P}(M([(f(N)LN\log N)^2], L)) = 1. \tag{5.4}$$

Theorem 5.4 *For any integer $L \leq \frac{N}{\log N}$, and any sequence $f(N) \downarrow 0$, for the* $\mathbf{SP}(N)$ *we have*

$$\lim_{N\to\infty} \mathbf{P}(M([(f(N)LN\log N)^2], L)) = 0. \tag{5.5}$$

Furthermore we have

Theorem 5.5 *For any integer $L \leq \frac{N}{\log N}$ and any fixed integer $k \geq 1$, for the* $\mathbf{SP}(N)$ *we have*

$$\lim_{N\to\infty} \mathbf{P}(A([(cLN\log N)^2], L, k)) = P\left(|Z| > \frac{1}{c}\right). \tag{5.6}$$

Theorem 5.6 *For any integer $L \leq \frac{N}{\log N}$, and any fixed integer $k \geq 1$, and any sequence $f(N) \uparrow \infty$, for the* $\mathbf{SP}(N)$ *we have*

$$\lim_{N\to\infty} \mathbf{P}(A([(f(N)LN\log N)^2], L, k)) = 1. \tag{5.7}$$

Remark. In the above five theorems the $L \leq \frac{N}{\log N}$ condition is a technical one, which may be eliminated. So we ask the following questions.
Question 1: *Determine for each $0 \leq p \leq 1$, the function $g(N, L, p)$ such that for the* $\mathbf{SP}(N)$

$$\lim_{N\to\infty} \mathbf{P}(M([g(N, L, p)], L)) = p$$

should hold.

Question 2: *Determine for each $0 \leq p \leq 1$, the function $g^*(N, L, p)$ such that for the* $\mathbf{SP}(N)$

$$\lim_{N\to\infty} \mathbf{P}(A([g^*(N, L, p)], L, k)) = p$$

should hold.

Having assumed in this section that $p_1 = p_2 = \cdot = p_N = 1/N$, in this setup we can make use of the famous Erdős-Rényi [12] coupon collector theorem:

Theorem 5.7 *Suppose that there are N urns given, and that $N\log N + (m-1)N\log\log N + Nx$ balls are placed in these urns one after the other, independently and equally likely, i.e., with equal probability $1/N$. Then, for every real x, the probability that each urn will contain at least m balls converges to*

$$\exp\left(-\frac{1}{(m-1)!}\exp(-x)\right), \tag{5.8}$$

as $N \to \infty$.

It is worthwhile to spell out the most important special case $m = 1$, as follows.

Theorem 5.8 *Suppose that there are N urns given, and that $N \log N + Nx$ balls are placed in these urns one after the other, independently and equally likely, i.e., with probability $1/N$. Then, for every real x, the probability that each urn will contain at least one ball converges to*

$$\exp\left(-\exp(-x)\right), \tag{5.9}$$

as $N \to \infty$.

We will also need the following Hoeffding [17] inequality.

Lemma 5.1 *Let $a_i \le X_i \le b_i \quad (i = 1, 2, ...k)$ be independent random variables and $S_k = \sum_{i=1}^{k} X_i$. Then for every $x > 0$*

$$P(|S_k - E(S_k)| \ge kx) \le 2 \exp\left(-\frac{2k^2x^2}{\sum_{i=1}^{k}(b_i - a_i)^2}\right). \tag{5.10}$$

We will use the above inequality in the following special case:

Let $X_1, X_2, ...X_j$ i.i.d. Bernoulli random variables, then for $j \le k$

$$P(|S_j - E(S_j)| \ge kx) \le 2 \exp\left(-2kx^2\right). \tag{5.11}$$

To see this, it is enough to observe that for $j \le k$ we might take $X_{j+1} = X_{j+2} = ... = X_k = 0$, then $\sum_{i=1}^{k}(b_i - a_i)^2 = j$.

We begin the proofs with some notations. Let $\{S(n)\}_{n=0}^{\infty}$ be a simple symmetric one-dimensional random walk and let

$$\begin{aligned}
\xi(0, n) &= \#\{k : 1 \le k < n,\ S(k) = 0\}, \\
\zeta(L, n) &:= \#\{k : 1 \le k < n,\ S(k) = 0 \text{ and } |S(k+i)|\ \ i = 1, 2, ... \text{hits } L \text{ before returning to } 0\}, \\
\rho(0) &:= 0 \ \text{ and } \rho(m) := \min\{k : k > \rho(m-1),\ S(k) = 0\},
\end{aligned}$$

i.e., here, $\zeta(L, n)$ is the number of excursions of the simple symmetric random walk $S(\cdot)$ reaching $|L|$ before time n.

Thus $\xi(0, \rho_m) = m$. Also observe that $\xi(0, n) = \zeta(1, n)$.

Furthermore, $E(\zeta(L, \rho(i))) = i/L$, on account of

$$\zeta(L, \rho(i)) = \sum_{k=1}^{i}(\zeta(L, \rho(k)) - \zeta(L, \rho(k-1))$$

being a sum of independent identically distributed Bernoulli random variables with mean $1/L$. This observation enables us to apply later on the Hoeffding inequality as above.

Define also

$$H(n) := \rho(\xi(0, n) + 1),$$

i.e., the time of the first return to zero after n steps.

Lemma 5.2

$$|\zeta(L, H(n)) - \zeta(L, n)| \leq 1 \quad a.s. \tag{5.12}$$

$$|\xi(0, H(n)) - \xi(0, n)| \leq 1 \quad a.s. \tag{5.13}$$

Proof: The two statements in hand amounts to observations in view of the respective definitions of the entities therein.

The next lemma concludes that the probability that the number of excursions reaching $|L|$ before time n and the number of excursions occuring before n divided by L are too far apart is small.

Lemma 5.3

$$P\left(|\zeta(L, n) - L^{-1}\xi(0, n)| \geq 4n^{1/4}(\log n)^{3/4}\right) \leq \frac{2}{n}$$

for n big enough.

Proof: Let

$$D(n) = |\zeta(L, H(n)) - L^{-1}\xi(0, H(n))| = |\zeta(L, \rho(\xi(0, n) + 1)) - L^{-1}\xi(0, \rho(\xi(0, n) + 1))|.$$

As

$$|\zeta(L, n) - L^{-1}\xi(0, n)| \leq |\zeta(L, H(n)) - L^{-1}\xi(0, H(n))| + 2,$$

for n big enough, we get

$$\begin{aligned}
&P(|\zeta(L, n) - L^{-1}\xi(0, n)| \geq 4n^{1/4}(\log n)^{3/4}) \\
&\leq P(|\zeta(L, H(n)) - L^{-1}\xi(0, H(n))| \geq 3n^{1/4}(\log n)^{3/4}) = \\
&= P(|D(n)| \geq 3\, n^{1/4}(\log n)^{3/4},\, \xi(0, n) \geq 2n^{1/2}(\log n)^{1/2}) + \\
&+ P(|D(n)| \geq 3\, n^{1/4}(\log n)^{3/4},\, \xi(0, n) < 2n^{1/2}(\log n)^{1/2}) = \\
&= I + II.
\end{aligned}$$

For n big enough, on account of Lemma 2.2 in Csáki and Földes, [7] we have

$$I \leq P(\xi(0, n) \geq 2n^{1/2}(\log n)^{1/2}) \leq \frac{C}{n^{2(1-\varepsilon)}} \leq \frac{1}{n},$$

whith an appropriate constant $C > 0$ and arbitrary $\varepsilon \in (0, 1/2)$. Furthermore,

$$\begin{aligned}
II &\leq \sum_{i=1}^{2n^{1/2}(\log n)^{1/2}} P(|\zeta(L, \rho(i)) - L^{-1}\xi(0, \rho(i))| > 3n^{1/4}(\log n)^{3/4},\, \xi(0, n) = i) \\
&\leq \sum_{i=1}^{2n^{1/2}(\log n)^{1/2}} P(|\zeta(L, \rho(i)) - L^{-1}i| > 3\, n^{1/4}(\log n)^{3/4})
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{2n^{1/2}(\log n)^{1/2}} 2\exp(-9\,\log n) \\
&\leq 4n^{1/2}(\log n)^{1/2}\exp(-9\,\log n) \leq \exp(-2\log n) = \frac{1}{n^2} < \frac{1}{n},
\end{aligned}
$$

where we applied Hoeffding inequality (5.11) with $k = 2\,n^{1/2}(\log n)^{1/2}$ and $x = \dfrac{3}{2}\left(\dfrac{\log n}{n}\right)^{1/4}$. $\square$

Proof of Theorem 5.2 The proof follows the basic ideas of Theorem 5.1. Suppose that the walker makes $n = [(cLN\log N)^2]$ steps on $\mathbf{SP}(N)$. This walk can be modelled in the following way. We consider the absolute value of $S(n)$, where $S(n)$ is a simple symmetric random walk on the line. Then we get positive excursions which we throw in N urns (the legs of the spider) with equal probability. We will use Lemma 5.2 to estimate the number of tall (at least L high) excursions, which are randomly placed in the N urns, and then apply Theorem 5.8. To follow this plan, let

$$
\begin{aligned}
\mu &= N\log N \\
B_n^- &= \{\zeta(L,n) \leq (1-2\epsilon)\mu\} \\
B_n &= \{(1-2\epsilon)\mu < \zeta(L,n) < (1+2\epsilon)\mu\} \\
B_n^+ &= \{\zeta(L,n) \geq (1+2\epsilon)\mu\}.
\end{aligned}
$$

In this proof we put $n = [(cLN\log N)^2]$ everywhere, $[\cdot]$ being the integer part. Having

$$
\mathbf{P}(M(n,L)) = \mathbf{P}(M(n,L)|B_n^-)P(B_n^-) + \mathbf{P}(M(n,L),B_n) + \mathbf{P}(M(n,L)|B_n^+)P(B_n^+), \tag{5.14}
$$

observe that, by Theorem 5.8,

$$
\lim_{N\to\infty}\mathbf{P}(M(n,L)|B_n^-) = 0.
$$

Using Lemma 5.3, for n big enough, we have

$$
\begin{aligned}
P(B_n) &= P((1-2\epsilon)\mu \leq \zeta(L,n) \leq (1+2\epsilon)\mu) \\
&\leq P\left(\frac{\xi(0,n)}{L} \leq (1+3\epsilon)\mu\right) - \mathbf{P}\left(\frac{\xi(0,n)}{L} \leq (1-\epsilon)\mu\right) + \frac{4}{n},
\end{aligned}
$$

where we used that the condition $L\log N \leq N$ of the theorem ensures that

$$
\epsilon\mu \geq 4n^{1/4}(\log n)^{3/4} \tag{5.15}
$$

for large enough N. Consequently, for N big enough, by (5.2) we have

$$
P(B_n) \leq P\left(|Z| \leq \frac{(1+3\epsilon)}{c}\right) - P\left(|Z| \leq \frac{(1-\epsilon)}{c}\right) + o(1).
$$

Thus

$$\lim_{N\to\infty} P(B_n) \le P\left(|Z| \le \frac{(1+3\epsilon)}{c}\right) - P\left(|Z| \le \frac{(1-\epsilon)}{c}\right).$$

Again by Theorem 5.8

$$\lim_{N\to\infty} \mathbf{P}(M(n,L)|B_n^+) = 1,$$

and, by Lemma 5.3, if N is big enough and $L \log N \le N$, using (5.15) again, we have that

$$\begin{aligned} P(B_n^+) &= P(\zeta(L,n) \ge (1+2\epsilon)\mu) \\ &\ge P\left(\frac{\xi(0,n)}{L} \ge (1+3\epsilon)\mu\right) + \frac{2}{n}. \end{aligned}$$

Consequently, by (5.2),

$$\lim_{N\to\infty} P(B_n^+) \ge \mathbf{P}\left(|Z| \ge \frac{1+3\epsilon}{c}\right)$$

and, similarly,

$$\begin{aligned} P(B_n^+) &= P(\zeta(L,n) \ge (1+2\epsilon)\mu) \\ &\le P\left(\frac{\xi(0,n)}{L} \ge (1+\epsilon)\mu\right) + \frac{2}{n} \end{aligned}$$

and

$$\lim_{N\to\infty} P(B_n^+) \le P\left(|Z| \ge \frac{1+\epsilon}{c}\right).$$

Hence, by the above conclusions, we obtain

$$P\left(|Z| \ge \frac{1+3\epsilon}{c}\right) \le \lim_{N\to\infty} \mathbf{P}(M(n,L))$$

$$\le P\left(|Z| \le \frac{(1+3\epsilon)}{c}\right) - P\left(|Z| \le \frac{(1-\epsilon)}{c}\right) + P\left(|Z| \ge \frac{1+\epsilon}{c}\right),$$

for any small enough $\epsilon > 0$. Letting $\epsilon \to 0$, we finally get that

$$\lim_{N\to\infty} \mathbf{P}(M(n,L)) = P\left(|Z| \ge \frac{1}{c}\right). \qquad \square$$

Proof of Theorem 5.3 We use the notations of the previous theorem, with the sole exception that now $n = [(f(N)LN\log N)^2]$, with $f(N) \to \infty$. Observe that

$$\mathbf{P}(M(n,L)) \geq \mathbf{P}(M(n,L)|B_n^+)\mathbf{P}(B_n^+), \tag{5.16}$$

and, as above, we know that

$$\lim_{N\to\infty} \mathbf{P}(M(n,L)|B_n^+) = 1.$$

So we only have to show that

$$\lim_{N\to\infty} P(B_n^+) = 1.$$

Now, again by Lemma 5.3, with any $\epsilon > 0$, for N big enough we have

$$\begin{aligned} P(B_n^+) &= P(\zeta(L,n) \geq (1+2\epsilon)\mu) \geq P\left(\frac{\xi(0,n)}{L} \geq 4n^{1/4}(\log n)^{3/4} + (1+2\epsilon)N\log N\right) - \frac{2}{n} \\ &= P\left(\frac{\xi(0,n)}{n^{1/2}} \geq \frac{4n^{1/4}(\log n)^{3/4}}{N\log N f(N)} + \frac{(1+2\epsilon)N\log N}{N\log N f(N)}\right) - \frac{2}{n}. \end{aligned}$$

Furthermore, having the condition $L\log N \leq N$ and $f(N) \to \infty$, it is easy to see that

$$\lim_{N\to\infty} \left(\frac{4n^{1/4}(\log n)^{3/4}}{N\log N f(N)} + \frac{(1+2\epsilon)N\log N}{N\log N f(N)}\right) = 0.$$

Thus the limit of the above probability when $N \to \infty$ is $P(|Z| \geq 0) = 1$, on account of (5.2). This also proves our Theorem 5.3. □

Proof of Theorems 5.5 and 5.6 To prove these two theorems, it is enough to repeat the proof of Theorems 5.2 and 5.3, and apply Theorem 5.7 instead of Theorem 5.8.

Proof of Theorem 5.4 Notations are the same as in Theorem 5.3, except that now $f(N) \downarrow 0$ as $N \to \infty$. During the proof we suppose that $f(N)LN\log N \to \infty$, otherwise there is nothing to prove. Observe that

$$\mathbf{P}(M(n,L)) \leq \mathbf{P}(M(n,L)|B_n^-)P(B_n^-) + P(\overline{B_n^-}). \tag{5.17}$$

As we know from Theorem 5.7 that $\lim_{N\to\infty} \mathbf{P}(M(n,L)|B_n^-) = 0$, it is enough to prove that $\lim_{N\to\infty} P(\overline{B_n^-}) = 0$. We show that $\lim_{N\to\infty} P(B_n^-) = 1$. Using Lemma 5.3 and the condition $L\log N \leq N$, with any $0 < \epsilon < 1/2$, we have

$$\begin{aligned} P(B_n^-) &= P(\zeta(L,n) \leq (1-2\epsilon)\mu) \\ &\geq P\left(\frac{\xi(0,n)}{L} + 4n^{1/4}(\log n)^{3/4} \leq (1-2\epsilon)\mu\right) \end{aligned}$$

$$\geq P\left(\frac{\xi(0,n)}{\sqrt{n}} - \frac{2}{n} \leq \frac{(1-2\epsilon)}{f(N)} - \frac{4n^{1/4}(\log n)^{3/4}}{N \log N f(N)}\right) - \frac{2}{n}$$
$$\geq P\left(\frac{\xi(0,n)}{\sqrt{n}} \leq \frac{1}{f(N)}\left(1 - 2\epsilon - \frac{4f^{1/2}(N)(4\log N + 2\log f(N))^{3/4}}{\log N}\right)\right) - \frac{2}{n}.$$

Since, now $f(N) \to 0$, as $N \to \infty$, $\frac{1-2\epsilon}{f(N)} \to +\infty$, while the fraction next to $(1-2\epsilon)$ goes to 0. Consequently, by (5.2), we arrive at

$$\lim_{N\to\infty} P(B_n^-) = P(|Z| < +\infty) = 1. \qquad \square$$

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