LINKS OF PLANE CURVE SINGULARITIES ARE L-SPACE LINKS

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ABSTRACT. We prove that a sufficiently large surgery on any algebraic link is an L-space. For torus links we give a complete classification of integer surgery coefficients providing L-spaces.

Definition 1. A 3-manifold Y is called an L-space, if it is a rational homology sphere and its Heegaard-Floer homology has minimal possible rank: rk $\widehat{HF}(Y) = |H_1(Y, \mathbb{Z})|$.

We refer the reader to [15, 16, 17] for definitions of Heegaard-Floer homology, and to [6, 14] for the detailed discussion of the properties of L-spaces. Let $K \subset S^3$ be the embedded link of a complex plane curve singularity with r components: $K = K_1 \cup \cdots \cup K_r$. The following theorem is the main result of this note.

Theorem 2. *Every algebraic link $K \subset S^3$ is an L–space link. This means that an integral surgery of S^3 along the link components K_i with all coefficients sufficiently large provides an L–space.*

For $r = 1$ the result was proved by Hedden [6, Theorem 1.10]. Our proof is of different nature, is extremely short, and provides the argument uniformly for any r. It is based on some facts from theory of normal surface singularities and plumbed 3-manifolds. For plumbing calculus (which modifies the possible graph representations of the same 3-manifold) see [12]. We will need the following facts:

(a) A connected negative definite plumbing graph is called a "smooth graph" if by blowing down consecutively (-1)–vertices we can blow down the graph to the empty graph. Such a graph represents S^3. When we resolve plane curve singularities and we blow up $(\mathbb{C}^2, 0)$ in several infinitely near points we obtain such a graph.

(b) A surface singularity is rational if its geometric genus is zero. This property can be verified at the level of its negative definite plumbing graph (such graphs are called "rational graphs"), see [1, 2, 9].

Blowing up a rational graph we get a rational graph. Any subgraph of a rational graph is rational (use e.g. Laufer's criterion [9]). Since smooth graphs are rational, subgraphs of smooth graphs are rational (they are called "sandwiched" graphs, see [18]). The following remark describes two useful consequences of Laufer's criterion.

Remark 3. (cf. [18, Rmk 2.3, Prop 2.4]) For each vertex v of the negative definite plumbing graph Γ define $w(v) = -(E_v, E_v)$, where E_v is the corresponding curve. Let $\gamma(v)$ denote the valency of v in Γ.

(a) If Γ is rational, then $w(v) \geq \gamma(v) - 1$ for all v.

(b) If $w(v) \geq \gamma(v)$ for all v, then the Γ is rational (and, in fact, sandwiched). Such graphs are called minimal rational.

A key ingredient in the proof of Theorem 2 is the following result:

Theorem 4. [11, Thms 6.3, 8.3] *A 3–manifold plumbed from a rational graph is an L–space.*

Let $C = C_1 \cup \ldots \cup C_r \subset \mathbb{C}^2$ be the plane curve singularity, corresponding to K, so that the link of the irreducible component C_i is the knot $K_i \subset S^3$. The possible minimal embedded resolution (plumbing) graphs of complex plane curve singularities are well–known. We will represent such a graph in a schematic way by the graph Γ_K shown in Figure 1, emphasizing only those exceptional curves, say $E_1, \ldots, E_n$ which intersect one or more strict transform components. The strict transform components are encoded by arrowheads. The number of arrowheads supported by E_i is a_i, and the self-intersection number of E_i is b_i ($1 \le i \le n$). Definitely, some of b_i's are equal to -1.
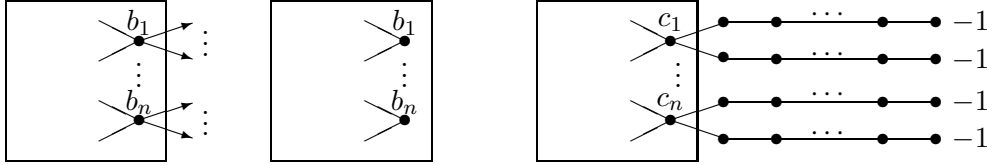

FIGURE 1. Graphs Γ_K, Γ_0 and $\widetilde{\Gamma_0}$.

If we delete the arrowheads, we get a smooth graph Γ_0 (the second graph in Figure 1). If we blow up Γ_0 few times (starting with E_i's and continuing with the newly created (-1) vertices), we obtain another graph $\widetilde{\Gamma_0}$, which is again smooth. This is the third graph in Figure 1. The unmarked vertices are (-2)–vertices and $c_i = b_i - a_i$. The length of the newly created legs can be different: the number of (-2)-vertices in the (i,j)th leg is $(k_{ij} - 1)$, where $k_{ij} \ge 1$ for all $1 \le i \le n$, $1 \le j \le a_i$.

Lemma 5. *For $k_{ij} \ge 1$ the 3-manifold represented by the graph Γ shown in Figure 2 is an L-space.*
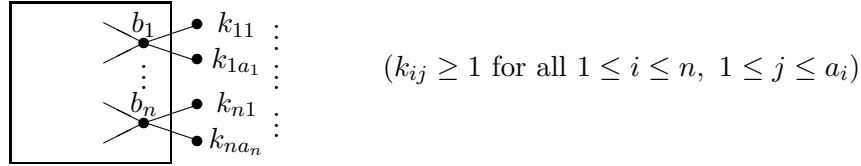

FIGURE 2. Graph Γ

Proof. By plumbing calculus (blowing up several times the most right edges) Γ is equivalent to the graph Γ' obtained from the smooth graph $\widetilde{\Gamma_0}$ by deleting the (-1) vertices from the end of the new legs. Hence Γ' is a subgraph of a smooth graph, thus it is rational. In particular, it represents an L–space. $\square$

Proof of Theorem 2: Suppose that the plane curve singularity, corresponding to K, is given by the equation $\{\prod_{i,j} f_{ij}(x,y) = 0\}$. Consider a positive surgery on S^3 along the link K, where the surgery coefficient for the component K_{ij} equals d_{ij}. It is known that $S^3_{\{d_{ij}\}_{ij}}(K)$ is a plumbed 3-manifold with the plumbing graph Γ, such that the parameters k_{ij} are defined by the equation $k_{ij} = d_{ij} - m_i$, where m_i are the multiplicities of the pullback of f_{ij} on the divisor E_i. For $d_{ij} > m_i$ the surgery space is an L-space by Lemma 5. $\square$

Remark 6. (a) Note that the above L-spaces are very special: they are given by connected negative definite sandwiched graphs (e.g, they are irreducible), so they definitely have even more additional properties.

(b) The above bound ($k_{ij} \geq 1$) is not optimal, usually one can find a collection of smaller numbers B_{ij}, B_{ij} smaller than 1, such that all the surgeries with coefficients $k_{ij} \geq B_{ij}$ provide L–spaces. But for the coefficients $B_{ij} = 1$ the proof is extremely transparent (and for smaller coefficients singularity theory is harder to apply).

(c) If $a_i = 1$ for all i then one can take $k_{i1} \geq 0$, and the surgery manifold is still an L–space. For example, if we take all of them zero then Γ is equivalent to a graph which is obtained from Γ by deleting the 0 vertices and the supporting E_i vertices. This graph is not connected, but each component is a subgraph of a smooth graph. Hence the corresponding 3-manifold is a connected sum of L–spaces, which is again an L-space. The proof of the general case $k_{i1} \geq 0$ (with $a_i = 1$) is a combination of this argument with the proof of Theorem 2.

(d) It is known that all algebraic knots (with one component) can be presented as iterated cables of the trivial knot. Hedden proved in [6] that the pq surgery of S^3 along an algebraic knot is an L-space, where p and q are the parameters of the last cabling. One can check that $pq = m_1$ in this case. See also [7] for a complete description of L-space surgeries of cable knots.

It turns out that the set of all surgeries on a link providing an L-space has an interesting structure. This is new phenomenon compared with the irreducible case, where (e.g. by [7]) a d-surgery on a knot K is an L-space iff $d \geq 2g(K) - 1$ (hence d runs in a half-line). The following theorem provides a description of L-space surgery coefficients for torus links.
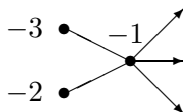

FIGURE 3. Plumbing graph for the singularity $x^6 + y^9 = 0$.

Theorem 7. *Let $p, q > 1$ be two coprime integers, $r \geq 1$ and $(d_1, \ldots, d_r) \in \mathbb{Z}^r$. Assume that $d_i \neq pq \pm 1$ for all i. Then $(d_1, \ldots, d_r)$–surgery on (pr, qr)–torus link is an L-space iff the surgery space is a rational homology sphere and one of the following conditions hold:*

(a) $d_i = pq$ *for some* i.
(b) $d_i > pq + 1$ *for all* i.
(c) $d_i < pq - 1$ *for all* i *and* $\max(d_i) \geq pq - p - q$.

Remark 8. It is easy to see that $d_i = d_j = pq$ for $i \neq j$ yields infinite H_1 for the surgery space, so in Theorem 7(a) $d_i = pq$ for exactly one i.

Remark 9. The surgeries with $d_i = pq \pm 1$ can be also analyzed by (rather long) case analysis. In particular:

(a) The $(pq \pm 1, d)$–surgery on $(2p, 2q)$–torus link is an L-space for all d.
(b) The $(pq+1, pq-1, d_3, \ldots, d_r)$–surgery on (pr, qr)–torus link coincides with $(d_3, \ldots, d_r)$ surgery on $(p(r-2), q(r-2))$–torus link. In particular, $(pq + 1, pq - 1, d)$–surgery on a $(3p, 3q)$–torus link is an L-space iff $d \geq pq - p - q$.

Proof. The plumbing graph of the surgery space can be described as above, with $k_i = d_i - pq$ (see Figure 3 for the case of $(6, 9)$ torus link with three components, see also [8] for more details). It is star-shaped, so the surgery space $Y := S_d^3(L)$ is Seifert fibered. By [13], the intersection matrix for the Neumann normal form is negative definite either for Y or

for $-Y$. Any negative definite Seifert graph is almost rational in the sense of [11], so it represents an L-space if and only if it is rational.

If $d_i = pq$ then $k_i = 0$ and the surgery space is a connected sum of lens spaces. From now on we will assume that $k_i \notin \{-1, 0, 1\}$ for all i.

If $k_i > 1$ for all i, then we get an L-space by Lemma 5. Suppose that k_i have different signs, for example, $k_1 > 0$ and $k_2 < 0$. To obtain the normal form of Y, one needs to blow up the vertices with positive k_i, and there are at most $r - 1$ of them, so the self-intersection of the central vertex in the normal form is greater than or equal to $-1 - (r - 1) = -r$, while its valency is $r + 2$. To obtain the normal form of $-Y$, one reverses all signs and blows up all positive vertices (at most $r + 1$ of them), so the self-intersection of the central vertex is greater than or equal to $1 - (r + 1) = -r$. If Y is an L-space, either Y or $-Y$ should be a negative definite rational graph, what contradicts Remark 3(a).

Finally, suppose that $k_i < 0$ for all i. In this case we can use a theorem of Lisca and Stipsicz [10] describing Seifert fibered L-spaces. Suppose that $1 \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{r+2} \geq 0$ are the Seifert invariants of singular fibers, and the central vertex has self-intersection (-1). Then the corresponding 3-manifold is an L-space if and only if there are no coprime integers (l, m) such that

$$(9.1) \qquad m\alpha_1 < l < m(1 - \alpha_2) \text{ and } m\alpha_i < 1 \text{ for } i \geq 3.$$

One can check that in our situation α_1 and $1 - \alpha_2$ are two neighboring fractions (in the sense of Farey series) with denominators p and q (that is, $\frac{a}{p}$ and $\frac{b}{q}$ with $aq - bp = \pm 1$), and $\alpha_i = 1/|k_{i-2}|$ for $i \geq 3$.

It is well known (see e.g. [3, Section 4.5]) that the fraction $\frac{a+b}{p+q}$ has the least possible denominator among rational numbers between $\frac{a}{p}$ and $\frac{b}{q}$. Therefore if $k_i \geq -p - q$ for some i then (9.1) cannot be satisfied and Y is an L-space; if $k_i < -p - q$ for all i then $(l, m) = (a + b, p + q)$ satisfies (9.1) and Y is not an L-space. $\qquad \square$

Example 10. For $r = 1$ the conditions of Theorem 7 are equivalent to the inequality $d_1 \geq pq - p - q = 2g(T(p, q)) - 1$, which also follows from [7].

Example 11. Consider the case $r = 2$. The set of all pairs (d_1, d_2) providing an L-space is shown in grey in Figure 4. The point (pq, pq) (marked by a star) provides a surgery with infinite first homology and should be excluded.

Theorem 2 can be reinterpreted as follows: start with a negative definite non-minimal plumbing graph Γ_0 of S^3, put some arrows on the vertices, and regard them as link components of K in S^3. Then the surgery manifold with all sufficiently large surgery coefficients is an L–space.

The proof works totally unmodified if we start with any non–minimal rational graph of a 3–manifold M (instead of S^3). If we wish to have natural (well–defined) surgery properties, it is convenient to consider only the case when M is an integral homology sphere. On the other hand, this is rather restrictive for (negative definite) rational graphs, there are only two possibilities: $M = S^3$ (case treated above), and $M = \Sigma(2, 3, 5)$, the Poincaré 3–sphere oriented as the link of the surface singularity $\{x^2 + y^3 + z^5 = 0\}$. Nevertheless, in this case, by identical proof we obtain:

Theorem 12. *Consider any negative definite non-minimal plumbing graph of $\Sigma = \Sigma(2, 3, 5)$, put some arrows on the vertices, and regard them as link components of K in Σ. Then the surgery manifold $\Sigma(K)$ for all sufficiently large surgery coefficients is an L–space.*

If we drop the integral homology restriction, we can start with any non–minimal rational graph (now, this family is really large), but we have to consider only those surgeries which

$$pq - p - q \qquad pq - 1 \qquad pq + 1$$
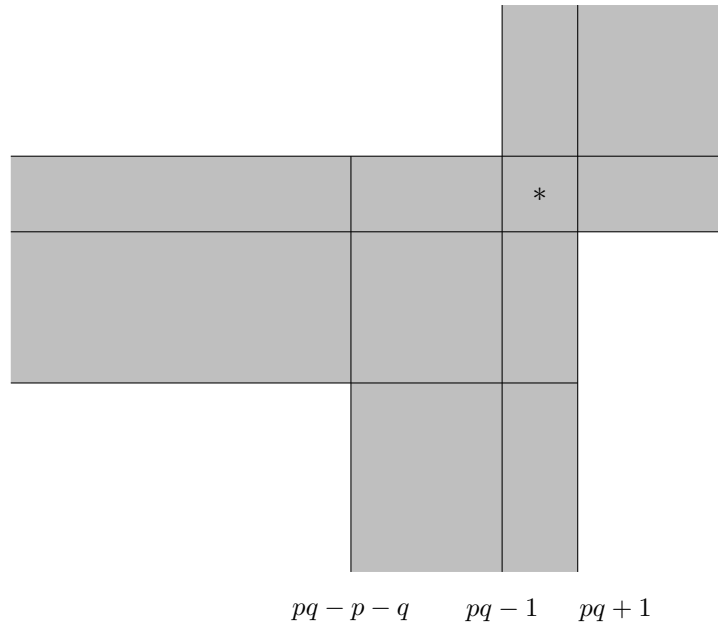
FIGURE 4. Possible L-space surgery coefficients for $(2p, 2q)$ torus link

can be realizes by the above construction (steps $\Gamma_0 \mapsto \widetilde{\Gamma_0} \mapsto \Gamma'$). For these surgeries the statement and the proof still holds.

ACKNOWLEDGMENTS

REFERENCES

[1] M. Artin. Some numerical criteria for contractibility of curves on algebraic surfaces. Amer. J. of Math. 84 (1962), 485–496.

[2] M. Artin. On isolated rational singularities of surfaces. Amer. J. of Math. 88 (1966), 129–136.

[3] R. L. Graham, D. E. Knuth, O. Patashnik. Concrete mathematics. A foundation for computer science. Second edition. Addison-Wesley Publishing Company, Reading, MA, 1994.

[4] J. Hanselman. Bordered Heegaard Floer homology and graph manifolds. arXiv:1310.6696

[5] J. Hanselman. `tree_manifolds_HFhat`, a Python program computing $\widehat{HF}$ for plumbing trees. Available at `http://math.columbia.edu/∼jhansel/graph_manifolds_program.html`

[6] M. Hedden. On knot Floer homology and cabling, II. Int. Math. Res. Not. IMRN 2009, no. 12, 2248–2274.

[7] J. Hom. A note on cabling and L-space surgeries. Algebraic and Geometric Topology **11** (2011) 219–223.

[8] T. Kadokami, M. Shimozawa. Dehn surgery along torus links. J. Knot Theory Ramifications **19** (2010), no. 4, 489–502.

[9] H.B. Laufer. On rational singularities, Amer. J. of Math. 94)1972), 597–608.

[10] P. Lisca, A. Stipsicz. Ozsváth-Szabó invariants and tight contact 3-manifolds. III. J. Symplectic Geom. **5** (2007), no. 4, 357–384.

[11] A. Némethi. On the Ozsváth-Szabó invariant of negative definite plumbed 3-manifolds, Geometry and Topology **9** (2005), 991–1042.

[12] W. Neumann. A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, Trans. of Amer. Math. Soc. **268** (2) (1981), 299–344.

[13] W. Neumann, F. Raymond. Seifert manifolds, plumbing, μ-invariant and orientation reversing maps. Algebraic and geometric topology (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977), pp. 163–196, Lecture Notes in Math., **664**, Springer, Berlin, 1978

[14] P. Ozsváth, Z. Szabó. On knot Floer homology and lens space surgeries. Topology **44** (2005), no.6, 1281–1300.

[15] P. Ozsváth, Z. Szabó. Holomorphic discs and knot invariants. Adv. Math., **186** (1), 2004, 58–116.

[16] P. Ozsváth, Z. Szabó. Holomorphic disks, link invariants and the multi-variable Alexander polynomial. Algebr. Geom. Topol. **8** (2008), no. 2, 615–692.

[17] P. Ozsváth, Z. Szabó. Holomorphic discs and topological invariants for closed three-manifolds. Ann. of Math. (2) **159** (2004), no. 3, 1027–1158.

[18] M. Spivakovsky. Sandwiched Singularities and Desingularization of Surfaces by Normalized Nash Transformations, Ann. of Math. (2) **131** (1990), no. 3, 411–491.

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