

Stability in the Erdős–Gallai Theorem on cycles and paths

Zoltán Füredi*

Alexandr Kostochka†

Jacques Verstraëte‡

July 19, 2015. Revised on May 9, 2016

Dedicated to the memory of G. N. Kopylov

Abstract

The Erdős–Gallai Theorem states that for $k \geq 2$, every graph of average degree more than $k - 2$ contains a k -vertex path. This result is a consequence of a stronger result of Kopylov: if k is odd, $k = 2t + 1 \geq 5$, $n \geq (5t - 3)/2$, and G is an n -vertex 2-connected graph with at least $h(n, k, t) := \binom{k-t}{2} + t(n - k + t)$ edges, then G contains a cycle of length at least k unless $G = H_{n,k,t} := K_n - E(K_{n-t})$.

In this paper we prove a stability version of the Erdős–Gallai Theorem: we show that for all $n \geq 3t > 3$, and $k \in \{2t+1, 2t+2\}$, every n -vertex 2-connected graph G with $e(G) > h(n, k, t-1)$ either contains a cycle of length at least k or contains a set of t vertices whose removal gives a star forest. In particular, if $k = 2t+1 \neq 7$, we show $G \subseteq H_{n,k,t}$. The lower bound $e(G) > h(n, k, t-1)$ in these results is tight and is smaller than Kopylov's bound $h(n, k, t)$ by a term of $n - t - O(1)$.

Mathematics Subject Classification: 05C35, 05C38.

Keywords: Turán problem, cycles, paths.

1 Introduction

A cornerstone of extremal combinatorics is the study of Turán-type problems for graphs. One of the fundamental questions in extremal graph theory is to determine the maximum number of edges in an n -vertex graph with no k -vertex path. According to [10], this problem was posed by Turán. A solution to the problem was obtained by Erdős and Gallai [7]:

Theorem 1.1 (Erdős and Gallai [7]). *Let G be an n -vertex graph with more than $\frac{1}{2}(k-2)n$ edges, $k \geq 2$. Then G contains a k -vertex path P_k .*

This result is best possible for n divisible by $k-1$, due to the n -vertex graph whose components are cliques of order $k-1$. To obtain Theorem 1.1, Erdős and Gallai observed that if H is an n -vertex graph without a k -vertex path P_k , then adding a new vertex and joining it to all other vertices we have a graph H' on $n+1$ vertices $e(H) + n$ edges and containing no cycle C_{k+1} or longer. Then Theorem 1.1 is a consequence of the following:

*Alfréd Rényi Institute of Mathematics, Hungary E-mail: zfuredi@gmail.com. Research supported in part by the Hungarian National Science Foundation OTKA 104343, by the Simons Foundation Collaboration Grant 317487, and by the European Research Council Advanced Investigators Grant 267195.

†University of Illinois at Urbana–Champaign, Urbana, IL 61801 and Sobolev Institute of Mathematics, Novosibirsk 630090, Russia. E-mail: kostochk@math.uiuc.edu. Research of this author is supported in part by NSF grants DMS-1266016 and DMS-1600592 and by Grant NSH 1939.2014.1 of the President of Russia for Leading Scientific Schools.

‡Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0112, USA. E-mail: jverstra@math.ucsd.edu. Research supported by NSF Grant DMS-1101489.

Theorem 1.2 (Erdős and Gallai [7]). *Let G be an n -vertex graph with more than $\frac{1}{2}(k-1)(n-1)$ edges, $k \geq 3$. Then G contains a cycle of length at least k .*

This result is best possible for $n-1$ divisible by $k-2$, due to any n -vertex graph where each block is a clique of order $k-1$. Let $\text{ex}(n, P_k)$ be the maximum number of edges in an n -vertex graph with no k -vertex path; Theorem 1.1 shows $\text{ex}(n, P_k) \leq \frac{1}{2}(k-2)n$ with equality for n divisible by $k-1$. Several proofs and sharpenings of the Erdős-Gallai theorem were obtained by Woodall [16], Lewin [12], Faudree and Schelp [8, 9] and Kopylov [11] – see [10] for further details. The strongest version was proved by Kopylov [11]. To describe his result, we require the following graphs. Suppose that $n \geq k$, $(k/2) > a \geq 1$. Define the n -vertex graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three sets A, B, C such that $|A| = a$, $|B| = n - k + a$ and $|C| = k - 2a$ and the edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$. Let

$$h(n, k, a) := e(H_{n,k,a}) = \binom{k-a}{2} + a(n-k+a).$$

Theorem 1.3 (Kopylov [11]). *Let $n \geq k \geq 5$ and $t = \lfloor \frac{k-1}{2} \rfloor$. If G is an n -vertex 2-connected graph with no cycle of length at least k , then*

$$e(G) \leq \max \{h(n, k, 2), h(n, k, t)\} \quad (1)$$

with equality only if $G = H_{n,k,2}$ or $G = H_{n,k,t}$.

In this paper, we prove a stability version of Theorems 1.1 and 1.3. A *star forest* is a vertex-disjoint union of stars.

Theorem 1.4. *Let $t \geq 2$ and $n \geq 3t$ and $k \in \{2t+1, 2t+2\}$. Let G be a 2-connected n -vertex graph containing no cycle of length at least k . Then $e(G) \leq h(n, k, t-1)$ unless*

- (a) $k = 2t+1$, $k \neq 7$, and $G \subseteq H_{n,k,t}$ or
- (b) $k = 2t+2$ or $k = 7$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most t .

This result is best possible in the following sense. Note that $H_{n,k,t-1}$ contains no cycle of length at least k , is not a subgraph of $H_{n,k,t}$, and $H_{n,2t+2,t-1} - A$ has a cycle for every $A \subseteq V(H_{n,2t+2,t-1})$ with $|A| = t$. Thus the claim of Theorem 1.4 does not hold for $G = H_{n,k,t-1}$. Therefore the condition $e(G) \leq h(n, k, t-1)$ in Theorem 1.4 is best possible. Since

$$h(n, 2t+2, t) = \binom{t}{2} + t(n-t) + 1 = h(n, 2t+1, t) + 1$$

and

$$h(n, 2t+2, t-1) = \binom{t}{2} + (t-1)(n-t) + 6 = h(n, 2t+1, t-1) + 3,$$

the difference between Kopylov's bound and the bound in Theorem 1.4 is

$$h(n, k, t) - h(n, k, t-1) = \begin{cases} n-t-3 & \text{if } k = 2t+1 \\ n-t-5 & \text{if } k = 2t+2. \end{cases} \quad (2)$$

It is interesting that for a fixed k , the difference in (2) divided by $h(n, k, t)$ does not tend to 0 when $n \rightarrow \infty$.

Theorem 1.4 yields the following cleaner claim for 3-connected graphs.

Corollary 1.5. *Let $k \geq 11$, $t = \lfloor \frac{k-1}{2} \rfloor$, and $n \geq \frac{3k}{2}$. If G is an n -vertex 3-connected graph with no cycle of length at least k , then $e(G) \leq h(n, k, t-1)$ unless $G \subseteq H_{n,k,t}$.*

In the same way that Theorem 1.2 implies Theorem 1.1, Theorem 1.4 applies to give a stability theorem for paths:

Theorem 1.6. *Let $t \geq 2$ and $n \geq 3t-1$ and $k \in \{2t, 2t+1\}$, and let G be a connected n -vertex graph containing no k -vertex path. Then $e(G) \leq h(n+1, k+1, t-1) - n$ unless*

- (a) $k = 2t$, $k \neq 6$, and $G \subseteq H_{n,k,t-1}$ or
- (b) $k = 2t+1$ or $k = 6$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most $t-1$.

Indeed, let G' be obtained from an n -vertex connected graph G with more than $h(n+1, k+1, t-1) - n$ edges by adding a vertex adjacent to all vertices in G . Then G' is 2-connected and G' has more than $h(n+1, k+1, t-1)$ edges. If G has no k -vertex path, then G' has no cycle of length at least $k+1$. By Theorem 1.4, G' satisfies (a) or (b) in Theorem 1.4, which means G satisfies (a) or (b) in Theorem 1.6. Repeating this argument, Corollary 1.5 implies the following.

Corollary 1.7. *Let $k \geq 11$, $t = \lfloor \frac{k-1}{2} \rfloor$, and $n \geq \frac{3k}{2}$. If G is an n -vertex 2-connected graph with no k -vertex paths, then $e(G) \leq h(n+1, k+1, t-1) - n$ unless $G \subseteq H_{n,k,t-1}$.*

Organization. The proof of Theorem 1.4 will use a number of classical results listed in Section 2 and some lemmas on contractions proved in Section 3. Then in Section 4 we describe several families of extremal graphs and state and prove a more technical Theorem 4.1, implying Theorem 1.4 for $k \geq 9$. Finally, in Section 5 we prove the analog of our technical Theorem 4.1 for $4 \leq k \leq 8$. In particular, we describe *all* 2-connected graphs with no cycles of length at least 6.

Notation. We use standard notation of graph theory. Given a simple graph $G = (V, E)$, the *neighborhood* of $v \in V$, i.e. the set of vertices adjacent to v , is denoted by $N_G(v)$ or $N(v)$ for short, and the *closed neighborhood* is $N[v] := N(v) \cup \{v\}$. The *degree* of vertex v is $d_G(v) := |N_G(v)|$. Given $A \subseteq V$ we also use $N_G(v, A)$ for $N(v) \cap A$, $d(v, A)$ for $|N(v) \cap A|$, and $N(A) := \bigcup_{v \in A} N(v) \setminus A$. For an edge xy in G , let $T_G(xy)$ denote the number of triangles containing xy and $T(G) := \min\{T_G(xy) : xy \in E\}$. The minimum degree of G is denoted by $\delta(G)$. For an edge xy in G , G/xy denotes the graph obtained from G by contracting xy . We frequently use $x * y$ for the new vertex. The length of the longest cycle in G is denoted by $c(G)$, and $e(G) := |E|$. Denote by K_n the complete n -vertex graph, and $K(A, B)$ the complete bipartite graph with parts A and B ($A \cap B = \emptyset$). Given vertex-disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the graph $G_1 + G_2$ has vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup E(K(V_1, V_2))$. If G is a graph, then \overline{G} denotes the complement of G and for a positive integer ℓ , ℓG denotes the graph consisting of ℓ components, each isomorphic to G . For disjoint sets $A, B \subseteq V(G)$, let $G(A, B)$ denote the bipartite graph with parts A and B consisting of all edges of G between A and B , and for $A \subseteq V(G)$, let $G[A]$ denote the subgraph induced by A .

2 Classical theorems

We require a number of theorems on long paths and cycles in dense graphs. The following is an extension to 2-connected graphs of the well-known fact that an n -vertex non-hamiltonian graph has at most $\binom{n-1}{2} + 1$ edges:

Theorem 2.1 (Erdős [6]). *Let $d \geq 1$ and $n > 2d$ be integers, and*

$$\ell_{n,d} := \max \left\{ \binom{n-d}{2} + d^2, \binom{\lceil \frac{n+1}{2} \rceil}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2 \right\}.$$

Then every n -vertex graph G with $\delta(G) \geq d$ and $e(G) > \ell_{n,d}$ is hamiltonian.

The bound on $\ell_{n,d}$ is sharp, due to the graphs $H_{n,n,2}$ and $H_{n,n,\lfloor (n-1)/2 \rfloor}$. Since $\delta(G) \geq 2$ for every 2-connected G , this has the following corollary.

Theorem 2.2 (Erdős [6]). *If $n \geq 5$ and G is an n -vertex 2-connected non-hamiltonian graph, then $e(G) \leq \binom{n-2}{2} + 4$, with equality only for $G = H_{n,n,2}$.*

It is well-known that every graph of minimum degree at least $d \geq 2$ contains a cycle of length at least $d + 1$. A stronger statement was proved by Dirac for 2-connected graphs:

Theorem 2.3 (Dirac [4]). *If G is 2-connected then $c(G) \geq \min\{n, 2\delta\}$.*

This theorem was strengthened as follows by Kopylov [11], based on ideas of Pósa [14]:

Theorem 2.4 (Kopylov [11]). *If G is 2-connected, P is an x, y -path of ℓ vertices, then $c(G) \geq \min\{\ell, d(x, P) + d(y, P)\}$.*

Theorem 2.5 (Chvátal [3]). *Let $n \geq 3$ and G be an n -vertex graph with vertex degrees $d_1 \leq d_2 \leq \dots \leq d_n$. If G is not hamiltonian, then there is some $i < n/2$ such that $d_i \leq i$ and $d_{n-i} < n - i$.*

The k -closure of a graph G is the unique smallest graph H of order $n := |V(G)|$ such that $G \subseteq H$ and $d_H(u) + d_H(v) < k$ for all $uv \notin E(H)$. The k -closure of G is denoted by $Cl_k(G)$, and can be obtained from G by a recursive procedure which consists of joining nonadjacent vertices with degree-sum at least k .

Theorem 2.6 (Bondy and Chvátal [1]). *If $Cl_n(G)$ is hamiltonian, then so is G . Therefore if $Cl_n(G) = K_n$, $n \geq 3$, then G is hamiltonian.*

Concerning long paths between prescribed vertices in a graph, Lovász [13] showed that if G is a 2-connected graph in which every vertex other than u and v has degree at least k , then there is a u, v -path of length at least $k + 1$. This result was strengthened by Enomoto. The following theorem immediately follows from Corollary 1 in [5]:

Theorem 2.7 (Enomoto [5]). *Let $5 \leq s \leq n$ and $\ell := 2(n-3)/(s-4)$. Suppose H is a 3-connected n -vertex graph with $d(x) + d(y) \geq s$ for all non-adjacent distinct $x, y \in V(H)$. Then for every distinct vertices x and y of H , there is an x, y -path of length at least $s - 2$. Moreover, if for some distinct $x, y \in V(H)$, there is no x, y -path of length at least $s - 1$, then either*

$$\overline{K_{s/2}} + \overline{K_{n-s/2}} \subseteq H \subseteq K_{s/2} + \overline{K_{n-s/2}}$$

or ℓ is an integer and

$$\overline{K_3} + \ell K_{s/2-2} \subseteq H \subseteq K_3 + \ell K_{s/2-2}.$$

A further strengthening of this result was given by Bondy and Jackson [2]. Finally, we require some results on cycles containing prescribed sets of edges. The following was proved by Pósa [15]:

Theorem 2.8 (Pósa [15]). *Let $n \geq 3$, $k < n$ and let G be an n -vertex graph such that*

$$d(u) + d(v) \geq n + k \quad \text{for every non-edge } uv \text{ in } G. \quad (3)$$

Then for every linear forest F with k edges contained in G , the graph G has a hamiltonian cycle containing all edges of F .

The analog of Pósa's Theorem for bipartite graphs below is a simple corollary of Theorem 7.3 in [17].

Theorem 2.9 (Zamani and West [17]). *Let $s \geq 3$ and K be a subgraph of the complete bipartite graph $K_{s,s}$ with partite sets A and B such that for every $x \in A$ and $y \in B$ with $xy \notin E(K)$, $d(x) + d(y) \geq s + 1 + i$. Then for every linear forest $F \subseteq K$ with at most $2i$ edges, there is a hamiltonian cycle in K containing all edges of F .*

We will use only the following partial case of Theorem 2.9.

Corollary 2.10. *Let $s \geq 4$, $1 \leq i \leq 2$ and K be a subgraph of $K_{s,s}$ with at least $s^2 - s + 2 + i$ edges. If $F \subseteq K$ is a linear forest with at most $2i$ edges and at most two components, then K has a hamiltonian cycle containing all edges of F .*

3 Lemmas on contractions

An essential part of the proof of Theorem 1.4 is to analyze contractions of edges in graphs. Specifically, we shall start with a graph G and contract edges according to some basic rules. Let us mention that the extensive use of contractions to prove the Erdős–Gallai Theorem was introduced by Lewin [12]. In this section, we present some basic structural lemmas on contractions.

Lemma 3.1. *Let $n \geq 4$ and let G be an n -vertex 2-connected graph. Let $v \in V(G)$ and $W(v) := \{w \in N(v) : N[v] \not\subseteq N[w]\}$. If $W(v) \neq \emptyset$, then there is $w \in W(v)$ such that G/vw is 2-connected.*

Proof. Let $w \in W(v)$, $G_w = G/vw$. Recall that $v * w$ is the vertex in G_w obtained by contracting v with w . Since G is 2-connected, G_w is connected. If $x \neq v * w$ is a cut vertex in G_w , then it is a cut vertex in G , a contradiction. So, the only cut vertex in G_w can be $v * w$. Thus, if the lemma does not hold, then for every $w \in W(v)$, $v * w$ is the unique cut vertex in G_w . This means that for every $w \in W(v)$, $\{v, w\}$ is a separating set in G .

Choose $w \in W(v)$ so that to minimize the order of a minimum component in $G - v - w$. Let C be the vertex set of such a component in $G - v - w$ and $C' = V(G) \setminus (C \cup \{v, w\})$. Since G is 2-connected, v has a neighbor $u \in C$ and a neighbor $u' \in C'$. Since $uu' \notin E(G)$, $u \in W(v)$. But the vertex set of every component of $G - v - u$ not containing w is contained in C . This contradicts the choice of w . \square

This lemma yields the following fact.

Lemma 3.2. *Let $n \geq 4$ and let G be an n -vertex 2-connected graph. For every $v \in V(G)$, there exists $w \in N(v)$ such that G/vw is 2-connected.*

Proof. If $W(v) \neq \emptyset$, this follows from Lemma 3.1. Suppose $W(v) = \emptyset$. This means $G[N(v)]$ is a clique. Then contracting any edge incident with v is equivalent to deleting v . Let $G' = G - v$. Since $d(v) \geq 2$ and $G[N(v)]$ is a clique, any cut vertex in G' is also a cut vertex in G . \square

For an edge xy in a graph H , let $T_H(xy)$ denote the number of triangles containing xy . Let $T(H) := \min\{T_H(xy) : xy \in E(H)\}$. When we contract an edge uv in a graph H , the degree of every $x \in V(H) \setminus \{u, v\}$ either does not change or decreases by 1. Also the degree of $u * v$ in H/uv is at least $\max\{d_H(u), d_H(v)\} - 1$. Thus

$$\delta(H/uv) \geq \delta(H) - 1 \text{ for every graph } H \text{ and } uv \in E(H). \quad (4)$$

Similarly,

$$T(H/uv) \geq T(H) - 1 \text{ for every graph } H \text{ and } uv \in E(H). \quad (5)$$

Suppose we contract edges of a 2-connected graph one at a step, choosing always an edge xy so that

- (i) the new graph is 2-connected and,
- (ii) xy is in the fewest triangles;
- (iii) the contracted edge xy is incident to a vertex of degree as small as possible up to (ii).

Lemma 3.3. *Let h be a positive integer. Suppose a 2-connected graph G is obtained from a 2-connected graph G' by contracting edge xy into $x * y$ using the above rules (i)–(iii). If G has at least h vertices of degree at most h , then either $G' = K_{h+2}$ or G' also has a vertex of degree at most h .*

Proof. Since G is 2-connected, $h \geq 2$. If G has a vertex of degree less than h , the lemma holds by (4). So, let A_j denote the set of vertices of degree exactly j in G , and assume $|A_h| \geq h$. Let $A'_h = A_h \setminus \{x * y\}$. Suppose the lemma does not hold. Then we have

$$\text{each } v \in A'_h \text{ has degree } h + 1 \text{ in } G' \text{ and is adjacent to both, } x \text{ and } y \text{ in } G'. \quad (6)$$

Case 1: $|A'_h| \geq h$. Then by (6), xy belongs to at least h triangles in which the third vertex is in A_h . So by (iii) and the symmetry between x and y , we may assume $d_{G'}(x) = h + 1$. This in turn yields $N_{G'}(x) = A_h \cup \{y\}$. Since G' is 2-connected each $v \in A'_h$ is not a cut vertex. Even more, xv is not a cut edge. Indeed, y is a common neighbor of all neighbors of x so all neighbors of x must be in the same component as y in $G' - x - v$. It follows that

$$\text{for every } v \in A'_h, G'/vx \text{ is 2-connected.} \quad (7)$$

If $uv \notin E(G)$ for some $u, v \in A_h$, then by (7) and (ii), we would contract the edge xu and not xy . Thus $G'[A'_h \cup \{x, y\}] = K_{h+2}$ and so either $G' = K_{h+2}$ or y is a cut vertex in G' , as claimed.

Case 2: $|A'_h| = h - 1$. Then $x * y \in A_h$. We obtain that $d_{G'}(x) = d_{G'}(y) = h + 1$ and $N_{G'}[x] = N_{G'}[y]$. So by (6), there is $z \in V(G)$ such that $N_{G'}[x] = N_{G'}[y] = A'_h \cup \{x, y, z\}$. Again (7) holds (for the

same reason that $N_{G'}[x] \subseteq N_{G'}[y]$. Thus similarly $vu \in E(G')$ for every $v \in A'_h$ and every $u \in A'_h \cup \{z\}$. Hence $G'[A'_h \cup \{x, y, z\}] = K_{h+2}$ and either $G' = K_{h+2}$ or z is a cut vertex in G' , as claimed. \square

Lemma 3.4. *Suppose that G is a 2-connected graph and C is a longest cycle in it. Then no two consecutive vertices of C form a separating set.*

Proof. Indeed, if for some i the set $\{v_i, v_{i+1}\}$ is separating, then let H_1 and H_2 be two components of $G - \{v_i, v_{i+1}\}$ such that $V(C) \cap V(H_1) \neq \emptyset$. Then $V(C) \setminus \{v_i, v_{i+1}\} \subseteq V(H_1)$. Let $x \in V(H_2)$. Since G is 2-connected, it contains two paths from x to $\{v_i, v_{i+1}\}$ that share only x . Since $\{v_i, v_{i+1}\}$ separates $V(H_2)$ from the rest, these paths are fully contained in $V(H_2) \cup \{v_i, v_{i+1}\}$. So adding these paths to $C - v_i v_{i+1}$ creates a cycle longer than C , a contradiction. \square

4 Proof of the main result, Theorem 1.4, for $k \geq 9$

In this section, we give a precise description of the extremal graphs for Theorem 1.4 for $k \geq 9$. The description for $k \leq 8$ is postponed to Section 5. For Theorem 1.4(a), when $k = 2t + 1$ and $t \neq 3$, these are simply subgraphs of the graphs $H_{n,k,t}$: recall that $H_{n,k,a}$ has a partition into three sets A, B, C such that $|A| = a$, $|B| = n - k + a$ and $|C| = k - 2a$ and the edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$. For Theorem 1.4(b), when $k = 2t + 2$ or $k = 7$, the extremal graphs G contain a set A of size at most t such that $G - A$ is a star forest. In this case a more detailed description is required.

Classes $\mathcal{G}_i(n, k)$ for $i \leq 3$. Let $\mathcal{G}_1(n, k) := \{H_{n,k,t}\}$. Each $G \in \mathcal{G}_2(n, k)$ is defined by a partition $V(G) = A \cup B \cup J$, $|A| = t$ and a pair $a_1 \in A$, $b_1 \in B$ such that $G[A] = K_t$, $G[B]$ is the empty graph, $G(A, B)$ is a complete bipartite graph and for every $c \in J$ one has $N(c) = \{a_1, b_1\}$. Every member of $G \in \mathcal{G}_3(n, k)$ is defined by a partition $V(G) = A \cup B \cup J$, $|A| = t$ such that $G[A] = K_t$, $G(A, B)$ is a complete bipartite graph, and

- $G[J]$ has more than one component
- all components of $G[J]$ are stars with at least two vertices each
- there is a 2-element subset A' of A such that $N(J) \cap (A \cup B) = A'$
- for every component S of $G[J]$ with at least 3 vertices, all leaves of S are adjacent to the same vertex $a(S)$ in A' .

The class $\mathcal{G}_4(n, k)$ is empty unless $k = 10$. Each member of $\mathcal{G}_4(n, 10)$ has a 3-vertex set A such that $G[A] = K_3$ and $G - A$ is a star forest such that if a component S of $G - A$ has more than two vertices then all its leaves are adjacent to the same vertex $a(S)$ in A . These classes are illustrated below:

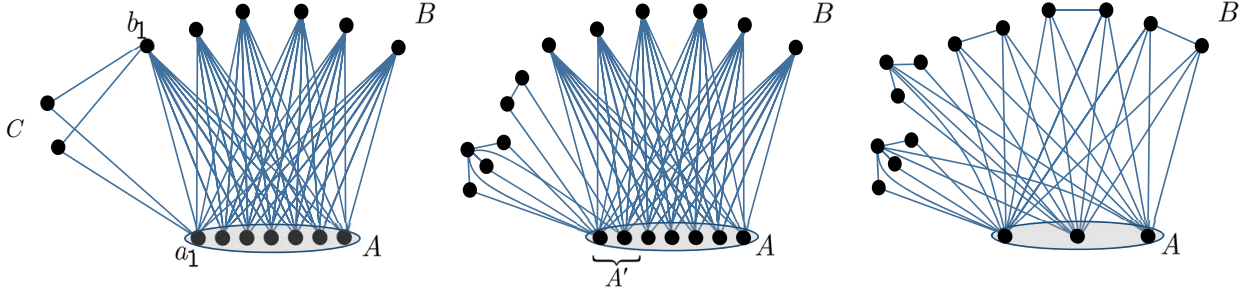


Figure 1: Classes $\mathcal{G}_2(n, k)$, $\mathcal{G}_3(n, k)$ and $\mathcal{G}_4(n, 10)$.

Statement of main theorem. Having defined the classes $\mathcal{G}_i(n, k)$ for $i \leq 4$, we now state a theorem which implies Theorem 1.4 for $k \geq 9$ and shows that the extremal graphs are the graphs in the classes $\mathcal{G}_i(n, k)$:

Theorem 4.1. (Main Theorem) *Let $k \geq 9$, $n \geq \frac{3k}{2}$ and $t = \lfloor \frac{k-1}{2} \rfloor$. Let G be an n -vertex 2-connected graph with no cycle of length at least k . Then $e(G) \leq h(n, k, t-1)$ or G is a subgraph of a graph in $\mathcal{G}(n, k)$, where*

- (1) *if k is odd, then $\mathcal{G}(n, k) := \mathcal{G}_1(n, k) = \{H_{n, k, t}\}$;*
- (2) *if k is even and $k \neq 10$, then $\mathcal{G}(n, k) := \mathcal{G}_1(n, k) \cup \mathcal{G}_2(n, k) \cup \mathcal{G}_3(n, k)$;*
- (3) *if $k = 10$, then $\mathcal{G}(n, k) := \mathcal{G}_1(n, 10) \cup \mathcal{G}_2(n, 10) \cup \mathcal{G}_3(n, 10) \cup \mathcal{G}_4(n, 10)$.*

We prove this theorem in this section. We also observe that if $k \geq 11$, then the only graph in the classes $\mathcal{G}_i(n, k)$ that is 3-connected is $H_{n, k, t}$. Therefore Theorem 4.1 implies Corollary 1.5.

The idea of the proof is to take a graph G satisfying the conditions of the theorem with $c(G) < k$, and to contract edges while preserving the average degree and 2-connectivity of G . A key fact is that if a graph contains a cycle of length at least k and is obtained from another graph by contracting edges, then that other graph also contains a cycle of length at least k . The process terminates with an m -vertex graph G_m such that G_m is 2-connected, $m \geq k$, and if $m > k$ then G_m has minimum degree at least $t-1$. If $m > k$, then we apply Theorem 2.7 to show that G_m is a dense subgraph of $H_{m, k, t}$. If $m = k$, then we apply Theorems 2.1, 2.2, 2.5, and 2.6 to show that G_m is a dense subgraph of $H_{k, k, t}$. Using this, we show that G_m contains a dense nice subgraph. Analyzing contractions, we then show that G itself contains a dense nice subgraph. Finally, we show that every dense n -vertex graph containing a dense nice subgraph but not containing a cycle of length at least k must be a subgraph of a graph in one of the classes described in Theorem 4.1.

4.1 Basic Procedure

Let k, n be positive integers with $n \geq k$. Let G be an n -vertex 2-connected graph with $c(G) < k$ and $e(G) \geq h(n, k, t-1) + 1$. We denote G as G_n and run the following procedure.

Basic Procedure. At the beginning of each round, for some $j : k \leq j \leq n$, we have a j -vertex 2-connected graph G_j with $e(G_j) \geq h(j, k, t-1) + 1$.

- (R1) If $j = k$, then we stop.
- (R2) If there is an edge xy with $T_{G_j}(xy) \leq t - 2$ such that G_j/xy is 2-connected, choose one such edge so that
 - (i) $T_{G_j}(xy)$ is minimum, and subject to this
 - (ii) xy is incident to a vertex of minimum possible degree.
 Then obtain G_{j-1} by contracting xy .
- (R3) If (R2) does not hold, $j \geq k + t - 1$ and there is $uv \in E(G_j)$ such that $G_j - u - v$ has at least 3 components and one of the components, say H_1 is a K_{t-1} , then let $G_{j-t+1} = G_j - V(H_1)$.
- (R4) If neither (R2) nor (R3) occurs, then we stop.

Remark 1. By construction, every obtained G_j is 2-connected and has $c(G_j) < k$. Let us check that

$$e(G_j) \geq h(j, k, t - 1) + 1 \quad (8)$$

for all $m \leq j \leq n$. For $j = n$, (8) holds by assumption. Suppose $j > m$ and (8) holds. If we apply (R2) to G_j , then the number of edges decreases by at most $t - 1$, and $(h(j, k, t - 1) + 1) - (h(j - 1, k, t - 1) + 1) = t - 1$. If we apply (R2) to G_j , then the number of edges decreases by at most $\binom{t+1}{2} - 1$, and $(h(j, k, t - 1) + 1) - (h(j - (t - 1), k, t - 1) + 1) = (t - 1)^2$. But for $k \geq 9$, $(t - 1)^2 \geq \binom{t+1}{2} - 1$. Thus every step of the basic procedure preserves (8).

Let G_m denote the graph with which the procedure terminates.

Remark 2. Note that if the rule (R3) applies for some G_j , then $\delta(G_j) \geq t$ and the set $\{u, v\}$ is still separating in G_{j-t+1} , thus $T_{G_{j-t+1}}(xy) \geq t - 1$ for every edge xy such that G_{j-t+1}/xy is 2-connected. In particular, $\delta(G_{j-t+1}) \geq t$. So (R2) does not apply after any application of (R3) and $\delta(G_m) \geq t$.

4.2 The structure of G_m

In the next two subsections, we prove Proposition 4.2 below, considering the cases $m = k$ and $m > k$ separately. Let F_4 be the graph obtained from $K_{3,6}$ by adding three independent edges in the part of size six. In this section we usually suppose that $n \geq 3t$, $t \geq 4$, although many steps work for smaller values as well.

Proposition 4.2. *The graph G_m satisfies the following properties:*

- (1) $G_m \subseteq H_{m,k,t}$ or
- (2) $m > k = 10$ and $G_m \supseteq F_4$.

4.2.1 The case $m = k$

If G_k is hamiltonian, then $c(G) \geq k$, a contradiction. So G_k is not hamiltonian.

By Theorem 2.5, for every non-hamiltonian n -vertex graph G with vertex degrees $d_1 \leq d_2 \leq \dots \leq d_n$, we define

$$r(G) := \min\{i : d_i \leq i \text{ and } d_{n-i} < n - i\}.$$

Lemma 4.3. *Let $t \geq 4$, $n \geq 3t$. If the vertex degrees of G_k are $d_1 \leq d_2 \leq \dots \leq d_k$, then $r(G_k) = t$.*

Proof for $k = 2t + 2$. Note that $r(G_k) \leq t$ since $r(G) < n/2$ (see Theorem 2.5). Suppose $r := r(G_k) \leq t - 1$. Then by Remark 2, Rule (R3) never applied, and G_k was obtained from G by a sequence of $n - m$ edge contractions according (R2). We may assume that for all $m \leq j < n$, graph G_j was obtained from G_{j+1} by contracting edge $x_j y_j$. Then conditions for (R2) imply

$$T_{G_j}(x_{j-1} y_{j-1}) \leq t - 2 \quad \text{for every } m + 1 \leq j \leq n. \quad (9)$$

By Lemma 3.3, $\delta(G_{m+1}) \leq r$. This together with (9) and (4) yield that for every $m < j \leq n$,

$$\delta(G_j) \leq r + j - m - 1 \text{ and so } T_{G_j}(x_{j-1} y_{j-1}) \leq \min\{r + j - m - 2, t - 2\}. \quad (10)$$

Contracting edge $x_{j-1} y_{j-1}$ in G_j , we lose $T_{G_j}(x_{j-1} y_{j-1}) + 1$ edges. Since $e(G) \geq h(n, k, t - 1) + 1$, by (5) we obtain,

$$\begin{aligned} e(G_k) &\geq h(n, k, t - 1) + 1 - \sum_{j=m+1}^n \min\{t - 1, r + j - m - 1\} \\ &= \binom{t+3}{2} + (t-1)(n-t-3) + 1 - \sum_{j=m+1}^n \min\{t-1, r+j-m-1\} \\ &= \binom{t+3}{2} + (t-1)(n-t-3) + 1 - (t-1)(n-m) + \sum_{j=m+1}^n \max\{0, m+t-r-j\} \\ &= \frac{3t^2+t+10}{2} + \sum_{j=m+1}^n \max\{0, 3t+2-r-j\}. \end{aligned} \quad (11)$$

Since $n \geq 3t$, $\{\max\{0, 3t+2-r-j\} : m+1 \leq j \leq n\} = \{0, 1, 2, \dots, t-1-r\}$. Therefore

$$e(G_k) \geq \frac{3t^2+t+10}{2} + \sum_{i=1}^{t-1-r} i = \frac{3t^2+t+10}{2} + \binom{t-r}{2}. \quad (12)$$

On the other hand, by the definition of r , G_m has at most r^2 edges incident with the r vertices of the smallest degrees and at most $\binom{m-r}{2}$ other edges. Thus $e(G_m) \leq r^2 + \binom{2t+2-r}{2}$. Hence

$$\frac{3t^2+t+10}{2} + \binom{t-r}{2} \leq r^2 + \binom{2t+2-r}{2}. \quad (13)$$

Expanding the binomial terms in (13) and regrouping we get

$$t(r-3) \leq r^2 - 2r - 4. \quad (14)$$

If $r = 3$, then the left hand side of (14) is 0 and the right hand side is -1 , a contradiction. If $r \geq 4$, then dividing both sides of (14) by $r-3$ we get $t \leq r+1-1/(r-3)$, which yields $r \geq t$, as claimed.

So suppose $r = 2$ and let v_1, v_2 be two vertices of degree 2 in G_k . Then by (12), the graph

$H = G_k - v_1 - v_2$ has at least

$$\frac{3t^2 + t + 10}{2} + \binom{t-2}{2} - 2(2) = 2t^2 - 2t + 4$$

edges. So the complement of H has at most $t - 4$ edges and thus, for $u, w \in V(H)$:

$$d_H(u) + d_H(w) \geq 2(2t - 1) - (t - 4) - 1 = 3t + 1 = |V(H)| + t + 1.$$

Hence by Theorem 2.8,

for each linear forest $F \subseteq H$ with $e(F) \leq t + 1$, H has a spanning cycle containing $E(F)$. (15)

If $N(v_i) = \{u_i, w_i\}$ for $i = 1, 2$ and $v_1 v_2 \in E(G_k)$, say $u_1 = v_2$ and $u_2 = v_1$, then by (15), graph $H' = H + w_1 w_2$ has a spanning cycle containing $w_1 w_2$, and this cycle yields a hamiltonian cycle in G_k , a contradiction. So $v_1 v_2 \notin E(G_k)$. Similarly, if $N(v_1) \neq N(v_2)$, then by (15), graph $H'' = H + u_1 w_1 + u_2 w_2$ has a spanning cycle containing $u_1 w_1$ and $u_2 w_2$. Note $w_1 \neq w_2$ since H is 2-connected. Again this yields a hamiltonian cycle in G_k . Thus we may assume $N(v_1) = N(v_2) = \{u, w\}$. Let

$$H_0 = H + uw \text{ if } uw \notin E(G) \text{ and } H_0 = H \text{ otherwise.} \quad (16)$$

If $x_m * y_m \notin N[v_1] \cup N[v_2]$, then $T_{G_{m+1}}(x_m y_m) \leq 1$ (since $T_{G_{m+1}}(v_1 u_1) \leq 1$) and G_{m+1} contains vertices v_1 and v_2 of degree 2. So by Lemma 3.3 for $h = 2$, G_{m+2} also has a vertex of degree 2. Thus by (4) for $r = 2$ instead of (10) we have for every $m + 2 \leq j \leq n$,

$$\delta(G_j) \leq \min\{j - m, t - 1\} \text{ and so } T_{G_j}(x_{j-1} y_{j-1}) \leq \min\{j - m - 1, t - 2\}. \quad (17)$$

Plugging (17) instead of (10) into (11) for $r = 2$, we will instead of (13) get the stronger inequality

$$\frac{3t^2 + t + 10}{2} + (t - 3) + \binom{t-2}{2} \leq 2^2 + \binom{2t+2-2}{2}. \quad (18)$$

Thus instead of (14) we have for $r = 2$ the stronger inequality $t(2 - 3) + (t - 3) \leq 2^2 - 4 - 4$, which does not hold. This contradiction implies $x_m * y_m \in N[v_1] \cup N[v_2]$. By symmetry we have two cases.

Case 1: $x_m * y_m = v_1$. As above, graph H_0 has a spanning cycle C containing uw . If

$$x_m u, y_m w \in E(G_{m+1}), \quad (19)$$

then C extends to a k -cycle in G_{m+1} by replacing uw with path u, x_m, y_m, w . A similar situation holds if

$$x_m w, y_m u \in E(G_{m+1}). \quad (20)$$

But by degree conditions each of x_m, y_m has a neighbor in $\{u, w\}$. By definition, each of u, w has a neighbor in $\{x_m, y_m\}$. So at least one of (19) and (20) holds.

Case 2: $x_m * y_m = u$. If $d_{G_{m+1}}(v_1) = d_{G_{m+1}}(v_2) = 2$, then as before we get (18) instead of (14) and get a contradiction. So by symmetry we may assume that v_1 is adjacent to both x_m and y_m in G_{m+1} . Since G_m is 2-connected, vertex w does not separate $\{v_1, v_2, u\}$ from the rest of the graph.

Thus by symmetry we may assume that y_m has a neighbor $z \in V(G_{m+1}) \setminus \{x_m, v_1, v_2, w\}$. Again by (15), graph H_0 defined by (16) has a spanning cycle containing edges uw and uz , and again this cycle yields a k -cycle in G_{m+1} (using path w, v_1, x_m, y_m, z), a contradiction.

Proof for $k = 2t + 1$. We repeat the argument for $k = 2t + 2$, but instead of (12) and (13), we get

$$\frac{3t^2 - t + 6}{2} + \binom{t-r}{2} \leq e(G_k) \leq r^2 + \binom{2t+1-r}{2}.$$

Expanding the binomial terms and regrouping, similarly to (14), we get

$$t(r-2) \leq r^2 - r - 3.$$

The analysis of this inequality is simpler than that of (14): If $r = 2$, then the left hand side is 0 and the right hand side is -1 , while if $r \geq 3$, then dividing both sides by $r-2$ we get $t \leq r+1-1/(r-2)$, which yields $r \geq t$, as claimed. \square

Lemma 4.4. *Under the conditions of Lemma 4.3, G_k is a subgraph of the graph $H_{k,k,t}$.*

Proof for $k = 2t + 2$. By Lemma 4.3, $r(G_k) = t$. Let G' be the k -closure of G_k and $d'_1 \leq d'_2 \leq \dots \leq d'_k$ be the vertex degrees in G' . By the definition of the k -closure,

$$d(u) + d(v) \leq k - 1 \quad \text{for every non-edge } uv \text{ in } G'. \quad (21)$$

Since $d'_i \geq d_i$ for every i and G' is also non-hamiltonian, $r(G') \geq r(G_k) = t$. Since $r(G') \leq t$ from $r(G) < n/2$, $r(G') = t$. Let $V(G') = \{v_1, \dots, v_k\}$ where $d_{G'}(v_i) = d'_i$ for all i . By the definition of $r(G')$, on the one hand $d'_t \leq t$ and $d'_{k-t} \leq k - t - 1 = t + 1$, on the other hand either $d'_{t-1} > t - 1$ or $d'_{k-(t-1)} \geq k - (t - 1) = t + 3$. In any case, $d'_{t+3} \geq t$. Summarizing,

$$d'_{t+3} \geq t, \quad d'_t \leq t \text{ and } d'_{t+1} \leq d'_{t+2} \leq t + 1. \quad (22)$$

Let $B = \{v_1, \dots, v_{t+2}\}$ and $A = V(G') \setminus B$. If $d'_{t+4} \leq t + 2$, then

$$\sum_{i=1}^k d'_i \leq (t|B| + 2) + (t+2)2 + (2t+1)(t-2) = 3t^2 + t + 4,$$

a contradiction to $e(G_k) \geq h(k, k, t-1) + 1$. Thus $d'_{t+4} \geq t + 3$, and by (21) and (22), $G'[A] = K_t$. In summary,

$$d'_{t+4} \geq t + 3 \quad \text{and} \quad G'[A] = K_t. \quad (23)$$

Suppose that there are distinct $v_{i_1}, v_{i_2} \in B$ and distinct $v_{j_1}, v_{j_2} \in A$ such that $v_{i_1}v_{j_1}$ and $v_{i_2}v_{j_2}$ are non-edges in G' . Then by (21) and (22),

$$\begin{aligned} \sum_{i=1}^{2t+2} d'_i &\leq (2t+1)2 + t(|B| - 2) + 2 + (2t+1)(|A| - 2) \\ &= 4t + 2 + t^2 + 2 + 2t^2 - 3t - 2 = 3t^2 + t + 2. \end{aligned}$$

This contradicts $e(G_k) > h(k, k, t-1)$. So, some v_j is incident with all non-edges of G' connecting

A with B .

Case 1: $j \leq t + 2$, i.e. $v_j \in B$. Then each $v \in B - v_j$ has t neighbors in A . Thus each $v \in B \setminus \{v_j, v_{t+1}, v_{t+2}\}$ has no neighbors in B , and each of v_{t+1}, v_{t+2} has at most one neighbor in B . If each of v_{t+1}, v_{t+2} is adjacent to v_j , then G' has a hamiltonian cycle using edges $v_{t+1}v_j$ and v_jv_{t+2} . Otherwise $G'[B]$ has at most one edge, as claimed.

Case 2: $j \geq t + 3$, i.e. $v_j \in A$. Together with (23), this yields that G' contains $K_{t-1, t+3}$ with partite sets $A \setminus \{v_j\}$ and $B \cup \{v_j\}$. In particular, all pairs of vertices in $A \setminus \{v_j\}$ are adjacent. So, G' is obtained from $K_{2t+2} - E(K_{t+3})$ by adding at least $e(G') - \binom{2t+2}{2} + \binom{t+3}{2} \geq 7$ edges. If $G'[B \cup \{v_j\}]$ contains a linear forest with four edges, then G' has a hamiltonian cycle. So suppose

$$G'[B \cup \{v_j\}] \text{ contains no linear forests with four edges,} \quad (24)$$

Case 2.1: $G'[B \cup \{v_j\}]$ contains a cycle C . By (24), $|C| \leq 4$ and if $|C| = 4$, then each other edge in $G'[B \cup \{v_j\}]$ has both ends in $V(C)$. Thus $G'[B \cup \{v_j\}]$ has at most 6 edges, a contradiction. So suppose $C = (x, y, z)$. If no other edge is incident with $V(C)$, then the set of the remaining at least four edges in $G'[B \cup \{v_j\}]$ contains a linear forest with two edges, a contradiction to (24). Thus we may assume that $G'[B \cup \{v_j\}]$ has an edge xu where $u \notin \{y, z\}$. Then by (24) and the fact that $G'[B \cup \{v_j\}]$ contains no 4-cycles, none of u, y, z is incident with other edges. On the other hand, if $G'[B \cup \{v_j\}]$ has an edge not incident with $V(C)$, this would contradict (24). Hence $G'[B \cup \{v_j\} \setminus \{x\}]$ has only the edge yz , as claimed.

Case 2.2: $G'[B \cup \{v_j\}]$ is a forest. By (24), there is $x \in B \cup \{v_j\}$ of degree at least 3 in $G'[B \cup \{v_j\}]$. If there is another vertex y of degree at least 3 in $G'[B \cup \{v_j\}]$, then we can choose two edges incident with x and two edges incident with y that together form a linear forest with four edges. So $G'[B \cup \{v_j\} \setminus \{x\}]$ is a linear forest, call it F , and thus has at most 3 edges. Each edge of F has at most one end adjacent to x and the degree of x in $G'[B \cup \{v_j\}]$ is at least four. So if F has exactly $m \in \{2, 3\}$ edges, then we can choose $4 - m$ edges incident with x so that together with F they form a linear forest. And if F has at most one edge, then the lemma holds.

Proof for $k = 2t + 1$. The proof is almost identical to the case $k = 2t + 2$. By Lemma 4.3, $r(G_k) = t$. Let G' be the k -closure of G_k and $d'_1 \leq d'_2 \leq \dots \leq d'_k$ be the vertex degrees in G' . As in (21), we have

$$d(u) + d(v) \leq k - 1 = 2t \quad \text{for every non-edge } uv \text{ in } G'. \quad (25)$$

As in the proof in the case $k = 2t + 2$, $r(G') = t$. Let $V(G') = \{v_1, \dots, v_k\}$ where $d_{G'}(v_i) = d'_i$ for all i . Instead of (22), we get the stronger claim

$$d'_{t+2} \geq t \text{ and } d'_t \leq d'_{t+1} = t. \quad (26)$$

Let $B = \{v_1, \dots, v_{t+1}\}$ and $A = V(G') \setminus B$. If $d'_{t+3} \leq t + 1$, then

$$\sum_{i=1}^{2t+1} d'_i \leq t|B| + (t+1)2 + (2t)(t-2) = 3t^2 - t + 2 \leq h(k, k, t-1),$$

a contradiction. Thus,

$$d'_{t+3} \geq t + 2 \text{ so by (25) and (26), } G'[A] = K_t. \quad (27)$$

If there are distinct $v_{i_1}, v_{i_2} \in B$ and distinct $v_{j_1}, v_{j_2} \in A$ such that $v_{i_1}v_{j_1}$ and $v_{i_2}v_{j_2}$ are non-edges in G' , then by (25) and (26),

$$\sum_{i=1}^k d'_i \leq (2t)2 + t(|B| - 2) + (2t)(|A| - 2) = 4t + t^2 - t + 2t^2 - 4t = 3t^2 - t \leq h(k, k, t - 1),$$

a contradiction. So, some v_j is incident with all non-edges of G' connecting A with B .

Case 1: $j \leq t + 1$, i.e. $v_j \in B$. Then each $v \in B - v_j$ has t neighbors in A . Thus by (26), each $v \in B - v_j$ has no neighbors in B , hence B is independent, as claimed.

Case 2: $j \geq t + 2$, i.e. $v_j \in A$. Together with (27), this yields that $G' - v_j$ contains $K_{t-1, t+2}$ with partite sets $A \setminus \{v_j\}$ and $B \cup \{v_j\}$. In particular, each vertex in $A \setminus \{v_j\}$ is all-adjacent. So, G' is obtained from $K_k - E(K_{t+2})$ by adding at least four edges. If $G'[B \cup \{v_j\}]$ contains a linear forest with three edges, then G' has a hamiltonian cycle. Every graph with at least four edges not containing a linear forest with three edges is a star plus isolated vertices. And if $G'[B \cup \{v_j\}]$ is a star plus isolated vertices, then $G' \subseteq H_{k, k, t}$. \square

4.2.2 The case $m > k$.

Lemma 4.5. *Let $m > k \geq 9$.*

- (1) *If $k \neq 10$, then $G_m \subseteq H_{m, k, t}$.*
- (2) *If $k = 10$ then $G_m \subseteq H_{m, k, t}$ or $G_m \supseteq F_4$.*

Proof for $k = 2t + 2$. G_m is an m -vertex 2-connected graph with $c(G_m) \leq 2t + 1$ satisfying $e(G) \geq h(n, k, t - 1) + 1$. Since (R2) is not applicable,

$$T_{G_m}(xy) \geq t - 1 \text{ for every non-separating edge } xy. \quad (28)$$

By Lemmas 3.2 and 3.1, (28) implies

$$\delta(G_m) \geq t \text{ and for each } v \in V(G_m) \text{ with } d(v) = t, G_m[N(v)] = K_{t+1}. \quad (29)$$

Let $C = (v_1, \dots, v_q)$ be a longest cycle in G_m . Since $\delta(G_m) \geq t$, Dirac's Theorem (Theorem 2.3) yields $q \geq 2t$. Obviously, $q \leq 2t + 1$.

By (28) and Lemma 3.4, each edge of C is in at least $t - 1$ triangles. By the maximality of C , the third vertex of each such triangle is in $V(C)$. So

$$\text{the minimum degree of } G_m[V(C)] \text{ is at least } t. \quad (30)$$

We now prove that

$$G_m[V(C)] \text{ is 3-connected.} \quad (31)$$

Indeed, assume (31) fails and $G_m[V(C)]$ has a separating set S of size 2. By symmetry, we may assume that $S = \{v_1, v_j\}$ and that $j \leq \lfloor q/2 \rfloor + 1 \leq t + 1$. Then by (30), $j = t + 1$ and $G_m[\{v_1, \dots, v_{t+1}\}] = K_{t+1}$. In particular,

$$v_1 v_{t+1} \in E(G_m). \quad (32)$$

Let $H_1 = G_m[\{v_1, \dots, v_{t+1}\}]$ and $H_2 = G_m[\{v_{t+1}, \dots, v_q, v_1\}]$. Similarly to H_1 , graph H_2 is either K_{t+1} (when $q = 2t$) or is obtained from K_{t+2} by deleting some matching (when $q = 2t + 1$).

Concerning almost complete graphs we need the following statement which is an easy consequence of Theorem 2.8 (or one can prove it directly).

$$\begin{aligned} & \text{For } p \geq 6 \text{ and for any matching } M \subseteq K_p, \text{ every two edges of } K_p - M \\ & \text{are in a common hamiltonian cycle of } K_p - M. \end{aligned} \quad (33)$$

Since G_m is 2-connected, each component F of $G_m - V(C)$ has at least two neighbors, say $y(F)$ and $y'(F)$, in C . If at least one of them, say $y'(F)$, is not in $S = \{v_1, v_{t+1}\}$, then we can construct a cycle longer than C as follows.

If $y(F) \in V(H_1) \setminus \{v_1, v_{t+1}\}$ and $y'(F) \in V(H_2) \setminus \{v_1, v_{t+1}\}$, then $H_1 - v_{t+1}$ has a hamiltonian $v_1, y(F)$ -path P_1 (recall that $H_1 - v_{t+1}$ is a complete graph), and H_2 has a hamiltonian $v_1, y'(F)$ -path P_2 , by (33) and since $k \geq 4$. So $P_1 \cup P_2$ and a $y(F), y'(F)$ -path through F form a longer than C cycle in G_m .

If both, $y(F)$ and $y'(F)$ are in the same H_j , then we let H'_j be the graph obtained from H_j by adding the edge $y(F)y'(F)$. Recall that by (32), $v_1 v_{t+1} \in E(H_j)$. If we have a hamiltonian cycle C' in H'_j containing $y(F)y'(F)$ and $v_1 v_{t+1}$, then let P be the v_1, v_{t+1} -path obtained from C' by deleting edge $v_1 v_{t+1}$ and replacing edge $y(F)y'(F)$ with a $y(F), y'(F)$ -path P' through F , and then replace in C the v_1, v_{t+1} -path through $V(H_j)$ with the longer path P . There is such a C' if $|V(H_j)| \geq 6$ by (33), and also if $|V(H_j)| = 5$ because in the latter case $|V(H_j)| = t + 1$ with $t = 4$ and it is a complete graph.

Thus every component F of $G_m - V(C)$ is adjacent only to S , and S is a separating set in G_m . In particular, $H_1 - S = K_{t-1}$ and $H_2 - S$ are components of $G_m - S$. So, if $m \geq 3t + 1$, then Rule (R3) is applicable, contradicting the definition of G_m . Hence $2t + 2 \leq m \leq 3t$. On the other hand, by (29), every component of $G_m - S$ has at least $t - 1$ vertices, and so $m - q \geq t - 1$. Therefore, $3t - 1 \leq m \leq 3t$.

If $m = 3t - 1$, then $q = 2t$, $H_2 = K_{t+1}$ and $H_3 := G_m - (V(C) - S) = K_{t+1}$. Hence

$$\begin{aligned} e(G_m) - h(m, k, t - 1) - 1 &= 3 \binom{t+1}{2} - 2 - h(3t - 1, k, t - 1) - 1 \\ &= \frac{3t^2 + 3t - 4}{2} - \frac{5t^2 - 7t + 16}{2} = -t^2 + 5t - 10 < 0. \end{aligned}$$

Similarly, if $m = 3t$, then the component sizes of $G_m - S$ are $t, t-1, t-1$. Thus in this case

$$\begin{aligned} e(G_m) - h(m, k, t-1) - 1 &\leq t^2 + t + \binom{t+2}{2} - 2 - h(3t, k, t-1) - 1 \\ &= \frac{3t^2 + 5t}{2} - 1 - \frac{5t^2 - 5t + 14}{2} = -t^2 + 5t - 8 < 0. \end{aligned}$$

These contradictions prove (31).

So by (31) and Theorem 2.7 for $n = q$, $s = 2t$ and $H = G_m[V(C)]$, one of three cases below holds:

Case 1: $\overline{K_t} + \overline{K_{q-t}} \subseteq G_m[V(C)] \subseteq K_t + \overline{K_{q-t}}$. Let B be the independent set of size $q-t$ in $G_m[V(C)]$ and $A = V(C) \setminus B$. In this case, since $G_m[V(C)]$ has hamiltonian cycle C and an independent set B of size $q-t$, we need $q = 2t$.

Suppose that $G_m - V(C)$ has a component D with at least two vertices. By Menger's Theorem, there are two fully disjoint paths, say P_1 and P_2 , connecting some two distinct vertices, say u and v , of D with two distinct vertices, say x and y , of C . Since $G_m[V(C)]$ contains $K_{t,t}$, it has an x, y -path with at least $2t-1$ vertices. This path together with P_1, P_2 and a u, v -path in D form a cycle of length at least $2t+1$, a contradiction to the maximality of C . Thus each component of $G_m - V(C)$ is a single vertex and is adjacent either only to vertices in A or only to vertices in B . Moreover, by (29), each such vertex has degree exactly t , and thus its neighborhood is a complete graph. Since B is independent, each $v \in V(G_m) - C$ is adjacent only to vertices in A . Thus $G_m = K_m - E(K_{m-t}) = H_{m,k-1,t} \subseteq H_{m,k,t}$.

Case 2: $\overline{K_3} + \ell K_{t-2} \subseteq G_m[V(C)] \subseteq K_3 + \ell K_{t-2}$, where $\ell = 2(q-3)/(2t-4)$. Again, since $G_m[V(C)]$ has hamiltonian cycle C and a separating set of size 3 (call this set A), $\ell \leq 3$. If $\ell \leq 2$, then $q \leq 3 + 2(t-2) < 2t$, a contradiction. Thus, $\ell = 3$ and $q = 3 + 3(t-2) = 3t-3$. Since $2t \leq q \leq 2t+1$, we get $t \in \{3, 4\}$. Since $t \geq 4$ by assumption, we obtain that $t = 4$ and $F_4 \subseteq G_m$.

Case 3: For every two distinct $x, y \in V(C)$, the graph $G_m[V(C)]$ contains an x, y -path with at least $2t$ vertices. Let $W = V(G_m) - V(C)$. Repeating the argument of the second paragraph of Case 1, we obtain that in our case

$$\text{each component of } G_m[W] \text{ is a singleton and so } N(w) \subseteq V(C) \text{ for each } w \in W. \quad (34)$$

Since no $w \in W$ is adjacent to two consecutive vertices of C (by the maximality of C) and $q \leq 2t+1$, by (29),

$$d_{G_m}(w) = t \text{ for every } w \in W. \quad (35)$$

Fix some $w_1 \in W$. Then we may relabel the vertices of C so that $N_{G_m}(w_1) = \{v_1, v_3, v_5, \dots, v_{2t-1}\}$. By (29), this also yields $G_m[\{v_1, v_3, \dots, v_{2t-1}\}] = K_t$ and thus $d_{G_m}(v_i) \geq t+1$ for all $i \in \{1, 3, \dots, 2t-1\}$. In particular,

$$d_{G_m}(v) \geq t+1 \text{ for every } v \in N_{G_m}(w_1). \quad (36)$$

Then for every $j \in \{2, 4, \dots, 2t-2\}$ (and for $j = 2t$ in the case $q = 2t$) we can replace v_j with w_1 in C and obtain another longest cycle. By (35) and (34), this yields $d_{G_m}(v_j) = t$ and

$$N_{G_m}(v_j) \subseteq V(C) \text{ for all } j \in \{2, 4, \dots, 2t-2\} \text{ (and for } j = 2t \text{ in the case } q = 2t). \quad (37)$$

Case 3.1: $q = 2t$. Switching the roles of w_1 with v_j together with (36) yields

$$N_{G_m}(v_j) = \{v_1, v_3, v_5, \dots, v_{2t-1}\} \text{ for all } j = 2, 4, \dots, 2t. \quad (38)$$

By (35) and (38), $N_{G_m}(w) = \{v_1, v_3, v_5, \dots, v_{2t-1}\}$ for all $w \in V(G_m) - \{v_1, v_3, v_5, \dots, v_{2t-1}\}$. This means $G_m \subseteq H_{m,2t+2,t}$, as claimed.

Case 3.2: $q = 2t + 1$. Since $m \geq 2t + 3$, there is $w_2 \in W - w_1$. By (37), vertex w_2 is not adjacent to v_j for $j \in \{2, 4, \dots, 2t - 2\}$. Suppose that w_2 is adjacent to v_{2t} or v_{2t+1} , say $w_2 v_{2t} \in E(G_m)$. Then by the maximality of C , $w_2 v_{2t+1}, w_2 v_{2t-1} \notin E(G_m)$. So the only possible t -element set of neighbors of w_2 is $\{v_1, v_3, \dots, v_{2t-3}, v_{2t}\}$. But then G_m has the $(2t + 2)$ -cycle $(w_2, v_3, v_4, v_5, \dots, v_{2t-1}, w_1, v_1, v_{2t+1}, v_{2t}, w_2)$, a contradiction. Thus

$$N_{G_m}(w) = \{v_1, v_3, v_5, \dots, v_{2t-1}\} \text{ for all } w \in W. \quad (39)$$

Since we can replace in C any v_j for $j \in \{2, 4, \dots, 2t - 2\}$ with w_1 , (39) yields $N_{G_m}(v_j) = \{v_1, v_3, v_5, \dots, v_{2t-1}\}$ for all $j = 2, 4, \dots, 2t - 2$. It follows that $\{v_1, v_3, v_5, \dots, v_{2t-1}\}$ covers all edges in G_m apart from edge $v_{2t} v_{2t+1}$. This means $G_m \subseteq H_{m,2t+2,t}$, as claimed.

Proof for $k = 2t + 1$. Similarly to the proof for $k = 2t + 2$, we have (28) and (29). Let $C = (v_1, \dots, v_q)$ be a longest cycle in G_m . Since $\delta(G_m) \geq t$, by Theorem 2.3, $q \geq 2t$; so $c(G_m) < k$ yields $q = 2t$. Then repeating the argument for $k = 2t + 2$, we obtain (30) and finally (31). So by Theorem 2.7 for $n = s = 2t$ and $H = G_m[V(C)]$, one of three cases below holds:

Case 1: $\overline{K_t} + \overline{K_t} \subseteq G_m[V(C)] \subseteq K_t + \overline{K_t}$. As in the proof for $k = 2t + 2$, we derive $G_m = K_m - E(K_{m-t}) = H_{m,k,t}$.

Case 2: $\overline{K_3} + \ell K_{t-2} \subseteq G_m[V(C)] \subseteq K_3 + \ell K_{t-2}$, where $\ell = 2(2t - 3)/(2t - 4)$. Again, since $G_m[V(C)]$ has hamiltonian cycle C and a separating set of size three (call this set A), $\ell \leq 3$. Since $t \geq 4$, $\ell \neq 3$. If $\ell \leq 2$, then $q \leq 3 + 2(t - 2) < 2t$, a contradiction.

Case 3: For every two distinct $x, y \in V(C)$, graph $G_m[V(C)]$ contains a hamiltonian x, y -path. Then for any component H of $G_m - V(C)$, let x and y be neighbors of H in $V(C)$. By the case, $G_m[V(C)]$ contains a $2t$ -vertex path, say P . Then P together with an x, y -path through H forms a cycle with at least k vertices, a contradiction. But since $m > k$, such a component H does exist.

□

4.3 Subgraphs of G_m

In this section, we define classes of graphs which we shall show are subgraphs of G_m , and these subgraphs will have the important property that they have many long paths and are preserved by the reverse of the contraction process in the Basic Procedure.

For a graph F and a nonnegative integer s , we denote by $\mathcal{K}^{-s}(F)$ the family of graphs obtained from F by deleting at most s edges.

Let $F_0 = F_0(t)$ denote the complete bipartite graph $K_{t,t+1}$ with partite sets A and B where $|A| = t$ and $|B| = t + 1$. Let $\mathcal{F}_0 := \mathcal{K}^{-t+3}(F_0)$, i.e., the family of subgraphs of $K_{t,t+1}$ with at least

$t(t+1) - t + 3$ edges.

Let $F_1 = F_1(t)$ denote the complete bipartite graph $K_{t,t+2}$ with partite sets A and B where $|A| = t$ and $|B| = t + 2$. Let $\mathcal{F}_1 := \mathcal{K}^{-t+4}(F_1)$, i.e., the family of subgraphs of $K_{t,t+2}$ with at least $t(t+2) - t + 4$ edges.

Let \mathcal{F}_2 denote the family of graphs obtained from a graph in $\mathcal{K}^{-t+4}(F_1)$ by subdividing an edge a_1b_1 with a new vertex c_1 , where $a_1 \in A$ and $b_1 \in B$. Note that any member $H \in \mathcal{F}_2$ has at least $|A||B| - (t-3)$ edges between A and B and the pair a_1b_1 is not an edge.

Let $F_3 = F_3(t, t')$ denote the complete bipartite graph $K_{t, t'}$ with partite sets A and B where $|A| = t$ and $|B| = t'$. Take a graph from $\mathcal{K}^{-t+4}(F_3)$, select two non-empty subsets $A_1, A_2 \subseteq A$ with $|A_1 \cup A_2| \geq 3$ such that $A_1 \cap A_2 = \emptyset$ if $\min\{|A_1|, |A_2|\} = 1$, add two vertices c_1 and c_2 , join them to each other and add the edges from c_i to the elements of A_i , ($i = 1, 2$). The class of obtained graphs is denoted by $\mathcal{F}(A, B, A_1, A_2)$. The family \mathcal{F}_3 consists of these graphs when $|A| = |B| = t$, $|A_1| = |A_2| = 2$ and $A_1 \cap A_2 = \emptyset$. In particular, $\mathcal{F}_3(4)$ consists of exactly one graph, call it $F_3(4)$.

Recall that F_4 is a 9-vertex graph with vertex set $A \cup B$, $A = \{a_1, a_2, a_3\}$, $B := \{b_1, b_2, \dots, b_6\}$ and edges of the complete bipartite graph $K(A, B)$ and three extra edges b_1b_2 , b_3b_4 , and b_5b_6 . Define F'_4 as the (only) member of $\mathcal{F}(A, B, A_1, A_2)$ where $|A| = |B| = t = 4$, $A_1 = A_2$, and $|A_i| = 3$. Let $\mathcal{F}_4 := \{F_4, F'_4\}$, which is defined only for $t = 4$.

In this subsection we will prove two useful properties of graphs in $\mathcal{F}_0 \cup \dots \cup \mathcal{F}_4$: First we show that G_m contains one of them (Proposition 4.6) and then show that such graphs have long paths with given end-vertices (Lemma 4.8).

Proposition 4.6. *Let $k \geq 9$. If k is odd, then G_m contains a member of \mathcal{F}_0 , and if k is even then G_m contains a member of $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_4$.*

Proof. By Proposition 4.2, $G_m \subseteq H_{m,k,t}$ or $m > k = 10$ and $F_4 \subseteq G_m$. In the latter case, the proof is complete. So assume $G_m \subseteq H_{m,k,t}$ and A, B, C are as in the definition of $H_{m,k,t}$. First suppose k is even and $C = \{c_1, c_2\}$. If $m = k$ then by (2),

$$e(H_{m,k,t}) - e(G_m) \leq h(m, k, t) - h(m, k, t-1) - 1 = t - 4,$$

i.e. $G_m \in \mathcal{K}^{-t+4}(H_{m,k,t})$. Since $F_1(t) \subseteq H_{m,k,t}$, G_m contains a subgraph in \mathcal{F}_1 . If $m > k$ then by (R2) and Lemma 3.2, we have $\delta(G_m) \geq t$. So, each $v \in B$ is adjacent to every $u \in A$ and each of c_1, c_2 has at least $t-1$ neighbors in A . Since $|B \cup \{c_1\}| \geq m - t - 1 \geq t + 2$, G_m contains a member of $\mathcal{K}^{-1}(F_1(t))$. Thus G_m contains a member of \mathcal{F}_1 unless $t = 4$, $m = 2t + 3$ and c_1 has a nonneighbor $x \in A$. But then $c_1c_2 \in E(G_m)$, and so G_m contains either $F_3(4)$ or F'_4 .

Similarly, if k is odd and $m = k$, then by (2), $G_m \in \mathcal{K}^{-t+3}(H_{m,k,t})$. Thus, since $H_{m,k,t} \supseteq F_0(t)$, G_m contains a subgraph in \mathcal{F}_0 . If k is odd and $m > k$ then by (R2) we have $\delta(G_m) \geq t$. So, each $v \in V(G_m) - A$ is adjacent to every $u \in A$. Hence G_m contains $K_{t, m-t}$. \square

In order to prove Lemma 4.8, we will use Corollary 2.10 and the following implication of it.

Lemma 4.7. *Let $t \geq 4$ and $H \in \mathcal{F}(A, B, A_1, A_2)$ with $|B| \geq t-1$, $|A| = t$. Let P be a path $a_1c_1c_2a_2$ and L be a subtree of H with $|E(L)| \leq 2$ such that $P \cup L$ form a linear forest. Then*

$$H \text{ has a cycle } C \text{ of length } 2t + 1 \text{ containing } P \cup L. \quad (40)$$

Proof. Choose some $B' \subseteq B$ with $|B'| = t - 1$ such that $B \cap V(L) \subseteq B'$. Let Q be the bipartite graph whose t -element partite sets are A and $B' \cup \{c\}$ where c is a new vertex, and the edge set consists of $H[A \cup B']$ and all edges joining c to A . By the conditions of the lemma, the set $E' := E(L) \cup \{a_1c, ca_2\}$ forms a linear forest in Q . Since Q misses at most $t - 4$ edges connecting A with $B' \cup \{c\}$, by Corollary 2.10 with $s = t$ and $i = 2$, Q has a hamiltonian cycle C' containing E' . Then the $(2t + 1)$ -cycle C in H obtained from C' by replacing path a_1ca_2 with P satisfies (40). \square

Lemma 4.8. *Let $H \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \dots \cup \mathcal{F}_4$ and $x, y \in V(H)$.*

- (a) *H contains an x, y -path of length at least $2t - 2$;*
- (b) *if H does not contain an x, y -path of length at least $2t - 1$, then*
 - (b0) *$H \in \mathcal{F}_0$ and $\{x, y\} \subseteq A$, or*
 - (b1) *$H \in \mathcal{F}_1$ and $\{x, y\} \subseteq A$, or*
 - (b2) *$H = F_4 \in \mathcal{F}_4$ and $\{x, y\} \subseteq A$;*
- (c) *if H does not contain an x, y -path of length at least $2t$, then*
 - (c0) *$H \in \mathcal{F}_0$, or*
 - (c1) *$H \in \mathcal{F}_1$ and at least one of x, y is in A , or*
 - (c2) *$H \in \mathcal{F}_2$ and either $\{x, y\} \subseteq A$ or $\{x, y\} = \{a_1, b_1\}$, or*
 - (c3) *$H \in \mathcal{F}_3$ and $\{x, y\} \subseteq A$, or*
 - (c4) *$H \in \mathcal{F}_4$ and $\{x, y\} \subseteq A$.*

Proof. The statements concerning $H \in \mathcal{F}_0 \cup \mathcal{F}_1$ are the easiest. Using Corollary 2.10 (or just using induction on t) it is easy to prove a bit more. Suppose that $H \in \mathcal{K}_{t,t+1}^{-(t-2)}(A, B)$, $t \geq 2$. Then every pair $x, y \in A \cup B$ is joined by a path of maximum possible length. This means that every pair of vertices $b_1, b_2 \in B$ is joined by a path of length $2t$, every pair $a \in A$, $b \in B$ is joined by a path of length $2t - 1$, and every pair $a_1, a_2 \in A$ is joined by a path of length $2t - 2$. For example, the proof for $H \in \mathcal{F}_0$, $a \in A$ and $b \in B$ is as follows. Consider H' obtained from H by adding edge ab if $ab \notin E(H)$ and deleting any $b' \in B - b$. Then by Corollary 2.10, H' has a hamiltonian cycle containing ab , which yields an a, b -path in H of length $2t - 1$.

The cycle $(b_1b_2a_1b_3b_4a_2b_5b_6a_3b_1)$ and path $b_1b_2a_1b_3a_2b_4a_3b_5b_6$ in F_4 prove (b2) and the part of (c4) related to F_4 .

Suppose now that $H \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{F_4'\}$; even in a more general setting suppose that $H \in \mathcal{F}(A, B, A_1, A_2)$ with $|B| = |A| = t$, $|A_1 \cup A_2| \geq 3$, $|A_2| \geq |A_1| \geq 1$ (and in case of $|A_1| = 1$ one has $A_1 \cap A_2 = \emptyset$). We prove the statements in reverse order, first (c2) and (c3), then (b), finally (a). When we comment below "Case BC" or "Case AA", this means that we consider paths from B to C or from A to A , respectively.

By Lemma 4.7, we already knew that c_1c_2 is contained in a cycle of length $2t + 1$ so these two vertices are joined by a path of length $2t$ (Case CC). If $b \in B$, and $a_i \in A_i$, then the almost complete bipartite subgraph $H[A \cup B]$ contains a b, a_i -path of length $2t - 1$, so b and c_{3-i} is joined in H by a path of length $2t + 1$ (Case BC). Concerning $b_1, b_2 \in B$ we can define H^+ by adding an extra vertex a_{t+1} to A and joining it to each vertex of B . Applying Lemma 4.7 to H^+ (with $t + 1$ in place of t) we get that it has a cycle C_{2t+3} through $b_1a_{t+1}b_2$. This cycle gives a b_1, b_2 -path of length $2t + 1$ in H (Case BB). In case of $x \in A$, $y \in A$ the high edge density of H implies that x and y have a common neighbor $b \in B$. One can find a path $P = a_1c_1c_2a_2$ such that P and xb_y

form a linear forest. Then Lemma 4.7 yields a cycle C_{2t+1} through all these edges. Leaving out b one gets an x, y -path of length $2t - 1$ in H (Case AA). In case of $x \in A, y \in B$ maybe we have to add the edge xy to obtain a cycle C_{2t+1} through it by Lemma 4.7. This yields an x, y -path of length $2t$ (Case AB). Finally, if $x \in A, y = c_i$ one uses a path c_i, c_{3-i}, x' and an x, x' -path of length $2t - 2$ in $A \cup B$ to get an x, y -path of length $2t$, if this can be done. If such an $x' \neq x$ does not exist, then $x = a_1 \in A_1, |A_1| = 1$, and $y = c_2$. This is the case described in (c2) (Case AC). \square

4.4 Reversing contraction

The aim of this section is to prove Lemma 4.9 below on preserving certain subgraphs during the reverse of the Basic Procedure.

Lemma 4.9 (Main lemma on contraction). *Let $k \geq 9$ and suppose F and F' are 2-connected graphs such that $F = F'/xy$ and $c(F') < k$.*

If k is even and F contains a subgraph $H \in \mathcal{F}_1 \cup \dots \cup \mathcal{F}_4$, then F' has a subgraph $H' \in \mathcal{F}_1 \cup \dots \cup \mathcal{F}_4$.

If k is odd and F contains a subgraph $H \in \mathcal{F}_0$, then F' has a subgraph $H' \in \mathcal{F}_0$.

Proof for k even. Case 1. $H \in \mathcal{F}_1$. Let $u = x * y$. If $u \notin V(H)$ then $H \subseteq F'$ and we are done. In case of $u \in A$ consider the sets $X := N_{F'}(x) \cap B$ and $Y := N_{F'}(y) \cap B$. If $X = X \cup Y$ then F' restricted to $(A \setminus \{u\}) \cup \{x\} \cup B$ contains a copy of H . If $X = X \cup Y \setminus \{y'\}$ for $y' \in V(H')$, then F' restricted to $(A \setminus \{u\}) \cup \{x\} \cup B \cup \{y\}$ contains a copy of a graph from \mathcal{F}_2 (with $a_1 := x, b_1 := y'$, and $c_1 := y$). We proceed in the same way if $Y = X \cup Y$ or if $|Y| = |X \cup Y| - 1$. In the remaining case $|X \setminus Y| \geq 2$ and $|Y \setminus X| \geq 2$, so one can choose five distinct elements b_0, x_1, x_2, y_1, y_2 from B such that $\{x_1, x_2\} \subseteq X \setminus Y$ and $\{y_1, y_2\} \subseteq Y \setminus X$. Then the bipartite subgraph Q_0 of F' generated by the sets $A \setminus \{u\} \cup \{x, y\}$ and $B \setminus \{b_0\}$ contains the linear forest L consisting of the paths $x_1 x x_2$ and $y_1 y y_2$. If we define the graph Q by adding to Q_0 all edges joining x and y to $B \setminus \{b_0\}$, then Q has at least $(t+1)^2 - (t-4)$ edges. So by Corollary 2.10 for $s = t+1$ and $i = 2$, Q has a hamiltonian cycle C_{2t+2} containing all edges of L , and this cycle also appears in F' , contradicting $c(F') < k$.

In case of $u \in B$ consider the sets $X := N_{F'}(x) \cap A$ and $Y := N_{F'}(y) \cap A$. If $|X \setminus Y| \leq 1$ or $|Y \setminus X| \leq 1$, then we proceed as above and find a subgraph H' of F either isomorphic to H or belonging to \mathcal{F}_2 . If $|X \setminus Y| \geq 2$ and $|Y \setminus X| \geq 2$, then we have four distinct elements x_1, x_2, y_1, y_2 in A such that $\{x_1, x_2\} \subseteq X \setminus Y$ and $\{y_1, y_2\} \subseteq Y \setminus X$. Then F' contains a member of \mathcal{F}_3 with $(c_1, c_2) = (x, y)$, $A_1 := \{x_1, x_2\}$, and $A_2 := \{y_1, y_2\}$.

Case 2. $H \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{F'_4\}$. The proof in this case follows from two claims. We say that the graph H has the Property (W_ℓ) if the following holds.

(W $_\ell$) *For all $z \in V(H)$ there exists $w \in N(z)$ such that for all $w' \in N(z) \setminus \{w\}$, the graph H has a cycle C_ℓ containing the path wzw' .*

Claim 1. *Suppose that the graph F contains a subgraph H satisfying Property (W_ℓ) , and $c(F') \leq \ell$. Then F' has a subgraph H' isomorphic to H .*

Let $z = x * y$ and $V = V(F) - z = V(F') - x - y$. If $V(H) \subseteq V$, then there is nothing to prove.

Suppose that $z \in V(H) \subseteq V(F)$ and define $X := N_{F'}(x) \cap N_H(z)$ and $Y := N_{F'}(y) \cap N_H(z)$. Then $X \cup Y = N_H(z)$. Let $w \in N(z)$ be the vertex from the definition of the Property (W_ℓ) . Since $N_H(z) = X \cup Y$, we may assume by symmetry that $w \in X$.

We claim that $Y - w = \emptyset$. Indeed, suppose there is $w' \in Y - w$. By Property (W_ℓ) , H has a cycle C_ℓ containing the path wzw' . Then the path $C_\ell - z$ in F' together with the edges $w'y$, yx and xw forms a cycle of length $\ell + 1$, contradicting $c(F') \leq \ell$.

This implies that $N_{F'}(x)$ contains $N_H(z)$. So F' contains a copy of H with the vertex set $(V(H) \setminus \{z\}) \cup \{x\}$. \square

Claim 2. *If $H \in \mathcal{F}_2 \cup \mathcal{F}_3$ or $H = F'_4$, then H satisfies Property (W_{2t+1}) .*

We prove a bit more: every $H \in \mathcal{F}(A, B, A_1, A_2)$ with $|B| \geq t - 1$, $|A| = t$ satisfies (W_{2t+1}) . Indeed, for $z = c_i$ we can choose a $w := c_{3-i}$. For $z \in B$ we can choose a $w \in A$ arbitrarily. For $z \in A$ we can choose $w \in N(z) \subseteq B$ arbitrarily, except if $z \in A_i$ and $|A_i| = 1$. In this latter case we can use $w := c_i$. In each of these cases, given $L := wzw'$ one can find a path $P := a_1c_1c_2a_2$ such that $P \cup L$ is a linear forest. Then Lemma 4.7 yields that H has a cycle C_{2t+1} through wzw' .

Since each $H \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{F'_4\}$ belongs to such $\mathcal{F}(A, B, A_1, A_2)$, this completes the proof of Claim 2. \square

Case 3. $H = F_4$. Let $u = x * y$. By symmetry, we can consider only two cases: $u = a_1$ and $u = b_1$. First, suppose $u = a_1$ and $xb_1 \in E(F')$. Then since $c(F') \leq 9$, y is not adjacent to any of b_3, b_4, b_5, b_6 . Thus x is adjacent to all of them, and if $yb_2 \in E(F')$, then the cycle $(yb_2b_1a_2b_3b_4a_3b_5b_6xy)$ contradicts $c(F') \leq 9$. So $xb_2 \in E(F')$ and the subgraph of F' with vertex set $V(H) \setminus \{u\} \cup \{x\}$ contains F_4 .

Similarly, suppose $u = b_1$ and $xb_2 \in E(F')$. Then to avoid a 10-cycle in F' , y has no neighbors in A and thus x is adjacent to all of A . So, again the subgraph of F' with vertex set $V(H) \setminus \{u\} \cup \{x\}$ contains F_4 .

Proof for k odd. First we prove the following statement (41) which is true for every $t \geq 2$. Let $H \in \mathcal{K}^{-t+2}(K(A, B))$ with $|A| = t$, $|B| = t + 1$. Let P be a path of length two in H . Then

$$H \text{ has a cycle } C \text{ of length } 2t \text{ containing } P. \quad (41)$$

If every vertex of $B \setminus P$ is joined to all vertices of A , then one can find a C_{2t} through P directly. Otherwise, there is a vertex $v \in B \setminus P$ of degree at most $t - 1$, so $H \setminus \{v\}$ is a subgraph of $K_{t,t}$ with at least $t^2 - t + 3$ edges. Then the statement follows from Corollary 2.10 for $s = t$ and $i = 1$.

Now suppose that $H \in \mathcal{F}_0$, $H \subseteq F$, $F = F'/xy$, and H, F, F' satisfy the constraints of Lemma 4.9. Then (41) implies that H satisfies property (W_{2t}) . Thus by Claim 1, F' has a subgraph H' isomorphic to H . \square

4.5 Completing the proof of Theorem 4.1

Proof for k even. Proposition 4.6 and Lemma 4.9 imply that there is a subgraph H of $G = G_n$

such that $H \in \mathcal{F}_1 \cup \dots \cup \mathcal{F}_4$. Let $G' = G - V(H)$ and S_1, \dots, S_s be the components of G' . Each of S_i has at least two neighbors, say x_i and y_i in $V(H)$. Let ℓ_i denote the length of a longest x_i, y_i -path in $G[V(S_i) \cup \{x_i, y_i\}]$. Since $c(G) < k$, by Lemma 4.8(a) and (b),

$$\text{for all } i, \quad \ell_i \leq 3 \quad \text{and if } H \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{F'_4\}, \text{ then } \ell_i \leq 2. \quad (42)$$

Case 1: $H \in \mathcal{F}_3 \cup \{F'_4\}$. By (42), $\ell_i \leq 2$ for all i and all choices of x_i and y_i . Since G is 2-connected, this yields that each S_i is a singleton, say v_i . Moreover, Lemma 4.8(c3) and (c4) imply $N(v_i) \subseteq A$ for all i . So G is contained in a graph in $\mathcal{G}_1(n, k)$, and the only edge outside A is $c_1 c_2$.

Case 2: $H \in \mathcal{F}_2$. Again, by (42), $\ell_i \leq 2$ for all i and all choices of x_i and y_i . So again this yields that each S_i is a singleton, say v_i . But now Lemma 4.8(c2) implies that for all i , either $N(v_i) \subseteq A$ or $N(v_i) = \{a_1, b_1\}$. Thus G is contained in a graph in $\mathcal{G}_2(n, k)$, where the only possible star component of $G - A$ with at least three vertices is a star with center b_1 and c_1 a leaf.

Case 3: $H \in \mathcal{F}_1$. Suppose first that some x_i is in B . Then by Lemma 4.8(c3), $y_i \in A$ and by Lemma 4.8(b), $\ell_i = 2$. So, denoting the common neighbor of x_i and y_i in S_i by c_1 , we get Case 2. Thus it is enough to consider below only the situation when

$$N(S_i) \cap V(H) \subseteq A \text{ for every } i. \quad (43)$$

We consider three cases.

Case 3.1: For some $i \neq j$, $\ell_i \geq 3$ and $\ell_j \geq 3$, say $\ell_1 \geq 3$ and $\ell_2 \geq 3$. Then by (42), $\ell_1 = \ell_2 = 3$. For $i = 1, 2$, let (x_i, v_i, v'_i, y_i) denote an x_i, y_i -path of length three in $G[V(S_i) \cup \{x_i, y_i\}]$. Also, by (43), $x_1, y_1, x_2, y_2 \in A$. Suppose first that $\{x_1, y_1\} \neq \{x_2, y_2\}$. We proceed as in the beginning of the proof of Lemma 4.9. Choose a $(t - 2)$ -element subset $B' \subseteq B$ and add two new vertices b'_1 and b'_2 and join them to all vertices of A . Then the obtained bipartite graph H' has at least $t^2 - t + 4$ edges so there is a hamiltonian cycle C' containing the linear forest $x_1 b'_1 y_1 \cup x_2 b'_2 y_2$ by Corollary 2.10. This C' corresponds to a cycle of length k in G , a contradiction.

It follows that every component S_i with $\ell_i \geq 3$ has exactly two neighbors in $V(H)$ and these two neighbors, say x_1, y_1 , are the same for all such components; furthermore $x_1, y_1 \in A$. Furthermore, in order to have $\ell_i \leq 3$, all leaves of S_i have the same neighbor in A . Thus G is contained in a graph in $\mathcal{G}_3(n, k)$.

Case 3.2: There exists exactly one i with $\ell_i \geq 3$, say $\ell_1 \geq 3$. Then by (42), $\ell_1 = 3$. Let (x_1, v_1, v'_1, y_1) be an x_1, y_1 -path of length 3 in $G[V(S_1) \cup \{x_1, y_1\}]$. By (43), every other component S_i is a singleton, say v_i with $N(v_i) \subseteq A$. As in Case 3.2, in order to have $\ell_1 \leq 3$, S_1 should be a star, and if $S_1 \neq K_2, K_1$, then all leaves of S_1 are adjacent to the same vertex in A . Thus G is contained in a graph in $\mathcal{G}_1(n, k) \cup \mathcal{G}_2(n, k)$.

Case 3.3: $\ell_i \leq 2$ for all i . Here G is contained in a graph in $\mathcal{G}_1(n, k)$. Then each S_i is a singleton with all neighbors in A . It follows that $G - A$ is an independent set.

Case 4: $H = F_4$. By Lemma 4.8(c4), (43) holds. Together with (42), this yields that every component S of $G - A$ is a star and if $|S| \geq 3$, then all leaves of S have the same neighbor in A . It follows that $G \in \mathcal{G}_4(n, k)$.

Proof for k odd. By Proposition 4.6 and Lemma 4.9, G_n contains some $H \in \mathcal{F}_0$. Let $G' = G_n - H$ and S_1, \dots, S_s be the components of G' . Each of S_i has at least two neighbors, say x_i and y_i in $V(H)$. Let ℓ_i denote the length of a longest x_i, y_i -path in $G_n[V(S_i) \cup \{x_i, y_i\}]$. Since $c(G_n) \leq 2t$, by Lemma 4.8,

$$\text{for all } i, \quad \ell_i \leq 2 \quad \text{and} \quad \{x_i, y_i\} \subseteq A. \quad (44)$$

Then each S_i is a singleton with all neighbors in A . It follows that $G - A$ is an independent set. This completes the proof of Theorem 4.1 for k odd. \square

5 Proof of Theorem 1.4 for $k \leq 8$

Recall that Theorem 4.1 describes for $k \geq 9$ and $n \geq 3k/2$ the n -vertex 2-connected graphs with no cycle of length at least k and more than $h(n, k, t-1)$ edges. In this section, we will do the same for $4 \leq k \leq 8$ and $n \geq k$. We will use for this the classes $\mathcal{G}_i(n, k')$ defined in Section 4 and the notion of a J_3 -bridge. For $A \subseteq V(G)$ and $S \subseteq V(G) \setminus A$, S forms a J_3 -bridge of A with endpoints a_1, a_2 if $a_1, a_2 \in A$, $A' := \{a_1, a_2\}$ is a cutset of G , $G[S \cup A'] \cup \{a_1 a_2\}$ is a 2-connected graph, $G[S]$ is connected, and the length of the longest a_1, a_2 -path in $G[S \cup A']$ is three.

Furthermore, since the description (but not the proof) for $k = 8$ is more sophisticated, we will need four more special graph classes for $k = 8$: Each of the graph classes $\mathcal{G}_i(n, 8)$ ($5 \leq i \leq 8$) contains 2-connected n -vertex graphs G with $c(G) < 8$ and having a special vertex set $A = \{a_1, a_2, \dots, a_s\}$ with $G[A]$ being a complete graph and such that $G \setminus A$ consists of J_3 -bridges and isolated vertices having exactly two neighbors in A .

If $G \in \mathcal{G}_5(n, 8)$, then $s = 3$ and a_1 is adjacent to each component in $G \setminus A$. So the edge $a_2 a_3$ is contained in a unique triangle, namely $a_1 a_2 a_3$.

If $G \in \mathcal{G}_6(n, 8) \cup \mathcal{G}_7(n, 8)$, then $s = 4$ and the endpoints of all J_3 -bridges are $\{a_1, a_2\}$ while one of the neighbors of some isolated vertex c of $G \setminus A$ is a_1 in case of $\mathcal{G}_6(n, 8)$ and $N(c) = \{a_3, a_4\}$ for all c in case of $\mathcal{G}_7(n, 8)$.

If $G \in \mathcal{G}_8(n, 8)$, then $s = 5$ and $N(S) = \{a_1, a_2\}$ for each component S of $G - A$.

Theorem 5.1. *Let $4 \leq k \leq 8$ and $n \geq k$. Let G be an n -vertex 2-connected graph with no cycle of length at least k . Then either $7 \leq k \leq 8$ and $e(G) \leq h(n, k, t-1)$ edges or G is a subgraph of a graph in $\mathcal{G}(n, k)$, where*

- (1) $\mathcal{G}(n, 4) = \emptyset$,
- (2) $\mathcal{G}(n, 5) := \mathcal{G}_1(n, 5)$,
- (3) $\mathcal{G}(n, 6) := \mathcal{G}_1(n, 6) \cup \mathcal{G}_2(n, 6)$,
- (4) $\mathcal{G}(n, 7) := \{H_{n,7,3}\} \cup \mathcal{G}_1(n, 6) \cup \mathcal{G}_2(n, 6) \cup \mathcal{G}_3(n, 6)$,
- (5) $\mathcal{G}(n, 8) := \bigcup_{1 \leq i \leq 8, i \neq 4} \mathcal{G}_i(n, 8)$.

The proof scheme is that we consider a graph G satisfying the conditions of the theorem and take a longest cycle C with vertex set, say $X := \{x_0, x_1, x_2, \dots, x_r\}$. Moreover, we will assume that C has the maximum sum of the degrees of its vertices among the longest cycles in G . Analyzing possibilities, we will derive that $G \in \mathcal{G}(n, k)$.

A *bridge* of C is the vertex set of a component of $G - X$.

We start from a sequence of simple claims on the structure of bridges and the edges between X and the bridges. For brevity we denote by $d_C(i, j)$ the distance on C between x_j and x_i , i.e. $\min\{|j - i|, r + 1 - |j - i|\}$. For a bridge S and neighbors x, x' of S on C , an (x, x', S) -path is an x, x' -path whose all internal vertices are in S .

The maximality of $|C|$ implies our first claim:

Claim 5.2. *For every bridge S and any $x_i, x_j \in N(S) \cap X$, the length of any (x_i, x_j, S) -path is at most $d_C(i, j)$. In particular, if S contains distinct c_1, c_2 such that $x_i c_1, x_j c_2 \in E(G)$, then $d_C(i, j) \geq 3$.*

If $|S| \geq 2$, then by the 2-connectedness of G , there are two vertex-disjoint S, X -paths. Thus if $G[S]$ contains a cycle, then for some $x_i, x_j \in N(S) \cap X$, the length of the longest (x_i, x_j, S) -path is at least 4. Hence, since $|C| \leq k - 1 \leq 7$, by Claim 5.2, we get the next claim:

Claim 5.3. *For every bridge S of X and any distinct $x_i, x_j \in N(S) \cap X$, the length of any (x_i, x_j, S) -path is at most 3. In particular, $G[S]$ is acyclic (a tree).*

Suppose that for some bridge S , and two leaves c_1, c_2 of the tree $G[S]$, there is a c_1, c_2 -path P in $G[S]$ of length at least 3. Then by Claim 5.3, each of c_1 and c_2 has exactly one neighbor in X , and this is the same vertex, say x_i . Again by the 2-connectedness of G , there is $x_j \in X \cap N(S) \setminus \{x_i\}$. Then there is an (x_j, x_i, S) -path of length at least 4 through either c_1 or c_2 , which contradicts Claim 5.3. Thus we get:

Claim 5.4. *For every bridge S of X , $G[S]$ is a star. Moreover, if $|S| \geq 3$, then all leaves of $G[S]$ have degree 2 in G and the same neighbor, $x(S)$, in X .*

Suppose $|S| \geq 2$ and $|N(S) \cap X| \geq 3$, say $\{x, x', x''\} \subseteq N(S) \cap X$. Let c_1 be a leaf of $G[S]$. If $|S| \geq 3$, then by Claim 5.3 it has a unique neighbor in X , say x . It follows that there are an (x, x', S) -path and an (x, x'', S) -path of length at least 3. Also there is an (x', x'', S) -path of length at least 2. Then by Claim 5.2, the distance on C from x to x' and to x'' is at least 3 and between x' and x'' is at least 2. Thus $|X| \geq 3 + 3 + 2 = 8$, a contradiction. Similarly, if $S = \{c_1, c_2\}$, then by symmetry we may assume that $x \in N(c_1) \cap X$ and $\{x', x''\} \subseteq N(c_2) \cap X$. In this case again by Claim 5.2, $|X| \geq 3 + 3 + 2 = 8$, a contradiction. Thus summarizing this with the previous claims, we have proved the following.

Claim 5.5. *For every bridge S of X with $|S| \geq 2$, $|N(S) \cap X| = 2$. Moreover, if $|S| \geq 3$, then $G[S]$ is a star and all leaves of $G[S]$ have degree 2 in G and the same neighbor, $x(S)$, in X . In other words, each bridge S with $|S| \geq 2$ is a J_3 -bridge of X .*

From Claims 5.2 and 5.5 we deduce:

Claim 5.6. *For every J_3 -bridge S of X with endpoints x_i and x_j , $d_C(i, j) \geq 3$.*

If there are $i_1 < i_2 < i_3 < i_4 \leq r$ and bridges S_1 and S_2 such that G contains an (x_{i_1}, x_{i_3}, S_1) -path P_1 and an (x_{i_2}, x_{i_4}, S_2) -path P_2 , then we can construct two new cycles C_1 and C_2 such that each of them contains the edges of P_1 and P_2 and each edge of C belongs to exactly one of C_1 and C_2 . Then the total length of C_1 and C_2 is at least $|E(C)| + 2(|E(P_1)| + |E(P_2)|) \geq (k - 1) + 8 \geq 2k - 1$. Thus at least one of them is longer than C , a contradiction. Thus we have:

Claim 5.7. *There are no $i_1 < i_2 < i_3 < i_4 \leq r$ and bridges S_1 and S_2 of X such that G contains an (x_{i_1}, x_{i_3}, S_1) -path and an (x_{i_2}, x_{i_4}, S_2) -path. In particular, since $k - 1 \leq 7$, any two J_3 -bridges share an endpoint.*

We now can prove Theorem 5.1. Indeed, by Claim 5.2, $|X| \geq 4$. This proves $\mathcal{G}(n, 4) = \emptyset$, i.e., Part 1 of the theorem.

We will consider 3 cases according to the value of $|X|$. As mentioned above, $|X| \geq 4$.

Case 1: $4 \leq |X| \leq 5$. Then by Claims 5.5 and 5.6, each bridge is a singleton. Furthermore, by Claim 5.2 each such singleton has exactly two (necessarily nonconsecutive) neighbors in X . If $|X| = 4$, Claim 5.7 yields that this pair of neighbors is the same for all bridges, say it is $\{x_0, x_2\}$. Then G is contained in $H_{n,5,2}$ with $A = \{x_0, x_2\}$, as claimed. This proves Part 2.

Let $|X| = 5$. If also each bridge has the same pair of neighbors in X , say $\{x_0, x_2\}$, then since $n \geq |X| + 1 = 6$, x_1 is not adjacent to $\{x_3, x_4\}$ to avoid a 6-cycle. Thus in this case, G is contained in $H_{n,6,2}$ with $A = \{x_0, x_2\}$, and so $e(G) \leq h(n, 6, 2)$. Otherwise by Claim 5.7, there are exactly two distinct pairs of neighbors of the bridges, and they share a vertex. Suppose these pairs are $\{x_0, x_2\}$ and $\{x_0, x_3\}$ and for $j \in \{2, 3\}$, Y_j is the set of vertices adjacent to x_0 and x_j . Then to avoid a 6-cycle, edges x_1x_4, x_1x_3 and x_2x_4 are not present in G . Then $G \in \mathcal{G}_2(n, 6)$ with $A = \{x_0, x_2\}$, $B = Y_2 \cup \{x_3\}$ and $J = Y_3 \cup \{x_4\}$. Since $H_{n,6,2}$ contains $H_{n,5,2}$, this together with the previous paragraph proves Part 3 of the theorem.

Case 2: $|X| = 6$. By Claims 5.5–5.7, it is enough to consider the following three subcases.

Case 2.1: X has a bridge S with $|N(S) \cap X| \geq 3$. By Claim 5.5, S is a single vertex, say z , and by Claim 5.2, z has exactly 3 (nonconsecutive) neighbors on C , say x_0, x_2 and x_4 . In view of the cycle $x_0zx_2x_3x_4x_5$ and the maximality of the degree sum of C , $d(x_1) \geq d(z) \geq 3$. By Claim 5.7, x_1 has no neighbors outside of C . In order to avoid a 7-cycle in G , $x_1x_3, x_1x_5 \notin E(G)$. So $x_1x_4 \in E(G)$. Similarly, $x_2x_5, x_0x_3 \in E(G)$, so G contains $K_{3,4}$ with parts $A = \{x_0, x_2, x_4\}$ and $B = \{x_1, x_3, x_5, z\}$. Moreover, B is independent. Let C be the vertex set of any component of $G - A - B$. If C has a neighbor in B or is not a singleton, then $G[A \cup B \cup C]$ has a cycle of length at least 7. Thus each component of $G - A - B$ is a singleton and has no neighbors in B . This means A meets all edges and so G is a subgraph of $H_{n,7,3}$.

Case 2.2: X has a J_3 -bridge S . Then by Claim 5.2 and symmetry, we may assume $N(S) = \{x_0, x_3\}$. In this case, G has 3 internally disjoint x_0, x_3 -paths of length 3. Thus to have $c(G) \leq 6$, $\{x_0, x_3\}$ separates internal vertices of distinct paths. It follows that $G - \{x_0, x_3\}$ is a collection of J_3 -bridges of $\{x_0, x_3\}$ and isolated vertices each having only x_0 and x_3 as endpoints. Thus G is a subgraph of a graph in $\mathcal{G}_3(n, 6)$.

Case 2.3: $V \setminus X$ is independent and each $z \in V \setminus X$ has degree 2. By Theorem 1.3, for each $z \in V \setminus X$, graph $G[X \cup \{z\}]$ has at most $h(7, 7, 2) = 14$ edges, which yields $e(G) \leq 2n = h(n, 7, 2)$. This proves Part 4 of Theorem 5.1.

Case 3: $|X| = 7$. By Claims 5.5–5.7, it is enough to consider the following four subcases.

Case 3.1: X has a bridge S with $|N(S) \cap X| \geq 3$. As in Case 2.1, S is a single vertex, say z , and we may assume $N(S) \cap X = \{x_0, x_2, x_4\}$. Again, similarly to Case 2.1, in view of the 7-cycle $x_0zx_2x_3x_4x_5x_6$, we obtain that $d(x_1) \geq d(z) \geq 3$, and that (to avoid a long cycle in G) the third neighbor of x_1 is x_4 . Similarly, $x_0x_3 \in E(G)$. Thus, G has a subgraph consisting of $K_{3,3}$ with parts $A := \{x_0, x_2, x_4\}$ and $B := \{x_1, x_3, z\}$ and an attached 3-path $x_4x_5x_6x_0$. Moreover, $d(x_1) = d(x_3) = d(z) = 3$ and these are isolated vertices in $G \setminus A$. Let Y be the vertex set of the component of $G - A$ containing $\{x_5, x_6\}$. If there is another component Y' of $G - A$ with $|Y'| \geq 2$, then to avoid a ≥ 8 -cycle, G must be a subgraph of a graph in $\mathcal{G}_3(n, 8)$. If all the bridges of A apart from A are singletons, then G is a subgraph of a graph in either $\mathcal{G}_1(n, 8)$ (if $|Y| = 2$) or $\mathcal{G}_2(n, 8)$ (if $|Y| \geq 3$).

Case 3.2: G has J_3 -bridges S_1 and S_2 of X with $N(S_1) \neq N(S_2)$. By Claims 5.7 and 5.6, we may assume $N(S_1) = \{x_0, x_3\}$ and $N(S_2) = \{x_0, x_4\}$. By the 2-connectivity of G , we may assume that there is an (x_0, x_3, S_1) -path $x_0y_1y_2x_3$ and an (x_4, x_0, S_2) -path $x_4y_5y_6x_0$. Let $A = \{x_0, x_3, x_4\}$. Then the edges $y_1y_2, y_5y_6, x_1x_2, x_5x_6$ belong to distinct components of $G \setminus A$. Thus to avoid long cycles in G , no bridge of A is adjacent to both, x_3 and x_4 and none of the bridges S of A contains an (x_0, x_3, S) -path or an (x_0, x_4, S) -path of length at least 4. It follows that G is a subgraph of a graph in $\mathcal{G}_5(n, 8)$.

Case 3.3: G has a J_3 -bridge S of X , and every other J_3 -bridge of X (if exists) has the same neighbors as S in X . We may assume that $N(S) \cap X = \{x_0, x_4\}$ and G contains an (x_0, x_4, S) -path $x_0y_6y_5x_4$. Then the edges y_5y_6, x_1x_2, x_5x_6 belong to three distinct components of $G \setminus \{x_0, x_4\}$. Let Y be the component of $G \setminus \{x_0, x_4\}$ containing $\{x_1, x_2, x_3\}$. By the case, all other components are either isolated vertices or J_3 -bridges of $\{x_0, x_4\}$. Also, every vertex $y \in (Y \setminus \{x_1, x_2, x_3\})$ has only neighbors in X (i.e., $N(y) \subset \{x_0, x_1, \dots, x_4\}$).

If $|Y| = 3$ we obtain that G is a subgraph of a member of $\mathcal{G}_8(n, 8)$ with $A = \{x_0, x_1, x_2, x_3, x_4\}$. Suppose $|Y| \geq 4$. If there is $y \in Y \setminus \{x_1, x_2\}$ with $N_G(y) = \{x_0, x_3\}$, then to avoid an 8- or 9-cycle, $x_1x_4 \notin E(G)$ and no $y' \in Y \setminus \{x_2, x_3\}$ has $N_G(y') = \{x_1, x_4\}$. So, either $\{x_0, x_3\}$ is a cut set in G or $x_2x_4 \in E(G)$. In the former case, G is a subgraph of a graph in $\mathcal{G}_5(n, 8)$ with $A = \{x_0, x_3, x_4\}$ and $a_1 = x_0$. In the latter case, in order to avoid an (x_0, x_4, Y) -path of length ≥ 5 , graph $G[\{x_1, x_2, x_3, x_4, y\}]$ has only the 5 edges we already know and no vertex $y' \in Y - X - y$ has $N(y') \subseteq \{x_1, x_2, x_3, x_4, y\}$. This means G is a subgraph of a graph in $\mathcal{G}_6(n, 8)$ with $A = \{x_0, x_4, x_2, x_3\}$, where $a_1 = x_0$ and $a_2 = x_4$. The case of $y \in Y \setminus \{x_1, x_2\}$ with $N_G(y) = \{x_1, x_4\}$ is symmetrical. If there is $y \in Y \setminus \{x_1\}$ with $N(y) = \{x_0, x_2\}$, then in order to avoid an (x_0, x_4, Y) -path of length ≥ 5 , $x_1x_3 \notin E(G)$ and every $y' \in Y - X$ is adjacent to x_2 . This means G is a subgraph of a graph in $\mathcal{G}_2(n, 8) \cup \mathcal{G}_3(n, 8)$ with $A = \{x_2, x_4, x_0\}$. The last possibility is that $N(y) = \{x_1, x_3\}$ for every $y \in Y - X$. Since $|Y| \geq 4$, this yields $x_2x_0, x_2x_4 \notin E(G)$. Thus G is a subgraph of a member of $\mathcal{G}_7(n, 8)$ with $\{a_1, a_2\} := \{x_0, x_4\}$ and $\{a_3, a_4\} := \{x_1, x_3\}$.

Case 3.4: $G \setminus X$ consists of isolated vertices only, each having degree 2 in G . By Theorem 1.3, for each $z \in V \setminus X$, graph $G[X \cup \{z\}]$ has at most $h(8, 8, 2) = 19$ edges, which yields $e(G) \leq 2n + 3 = h(n, 8, 2)$. \square

Theorem 5.1 yields the following analog of Theorem 4.1(1) for a smaller range of $e(G)$.

Corollary 5.8. *Suppose that G is a 2-connected, n -vertex graph with $c(G) < 7$, $n \geq 8$. If $e(G) \geq \lfloor (5n - 6)/2 \rfloor$ then G is a subgraph of $H_{n,7,3}$, and this bound is best possible.* \square

6 Concluding remarks

It could be that for $k \geq 11$, Theorem 1.4 holds already for $n \geq 5k/4$. Note that by Theorem 1.3, it does not hold for $n < 5k/4$. It may also be possible, albeit complicated, to describe the structure of 2-connected n -vertex graphs with no cycles of length at least $k = 2t + 1$ and at least $h(n, k, t - 2)$ edges. We leave these as avenues for further research.

Acknowledgment. We thank both referees and R. Luo for very helpful comments.

References

- [1] J. A. Bondy and V. Chvátal, A method in graph theory. *Discrete Math.* **15** (1976), 111–135.
- [2] J. A. Bondy and B. Jackson, Long paths between specified vertices of a block, *Ann. Discrete Math.* **27** (1985), 195–200.
- [3] V. Chvátal, On Hamilton’s ideals. *J. Combinatorial Theory Ser. B* **12** (1972), 163–168.
- [4] G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc. (3)* **2**, (1952). 69–81.
- [5] H. Enomoto, Long paths and large cycles in finite graphs, *J. Graph Theory* **8** (1984), 287–301.
- [6] P. Erdős, Remarks on a paper of Pósa, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962), 227–229.
- [7] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.* **10** (1959), 337–356.
- [8] R. J. Faudree and R. H. Schelp, Ramsey type results, *Infinite and Finite Sets, Colloq. Math. J. Bolyai* **10**, (ed. A. Hajnal et al.), North-Holland, Amsterdam, 1975, pp. 657–665.
- [9] R. J. Faudree and R. H. Schelp, Path Ramsey numbers in multicolorings, *J. Combin. Theory Ser. B* **19** (1975), 150–160.
- [10] Z. Füredi and M. Simonovits, The history of degenerate (bipartite) extremal graph problems, *Bolyai Math. Studies* **25** pp. 169–264, *Erdős Centennial* (L. Lovász, I. Ruzsa, and V. T. Sós, Eds.) Springer, 2013. Also see: [arXiv:1306.5167](https://arxiv.org/abs/1306.5167).
- [11] G. N. Kopylov, Maximal paths and cycles in a graph, *Dokl. Akad. Nauk SSSR* **234** (1977), 19–21. (English translation: *Soviet Math. Dokl.* **18** (1977), no. 3, 593–596.)
- [12] M. Lewin, On maximal circuits in directed graphs. *J. Combinatorial Theory Ser. B* **18** (1975), 175–179.

- [13] L. Lovász, Combinatorial Problems and Exercises, *2nd* Ed., North-Holland, Amsterdam, 1993.
- [14] L. Pósa, A theorem concerning Hamilton lines, Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 225–226.
- [15] L. Pósa, On the circuits of finite graphs, Magyar. Tud. Akad. Mat. Kutató Int. Közl. **8** (1963/1964), 355–361.
- [16] D. R. Woodall, Maximal circuits of graphs I, Acta Math. Acad. Sci. Hungar. **28** (1976), 77–80.
- [17] R. Zamani and D. B. West, Spanning cycles through specified edges in bipartite graphs, J. Graph Theory **71** (2012), 1–17.