Curves in $\mathbb{R}^d$ intersecting every hyperplane at most $d + 1$ times

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Abstract

By a curve in $\mathbb{R}^d$ we mean a continuous map $\gamma: I \to \mathbb{R}^d$, where $I \subset \mathbb{R}$ is a closed interval. We call a curve $\gamma$ in $\mathbb{R}^d$ $(\leq k)$-crossing if it intersects every hyperplane at most $k$ times (counted with multiplicity). The $(\leq d)$-crossing curves in $\mathbb{R}^d$ are often called convex curves and they form an important class; a primary example is the moment curve $\{(t, t^2, \ldots, t^d) : t \in [0, 1]\}$. They are also closely related to Chebyshev systems, which is a notion of considerable importance, e.g., in approximation theory. Our main result is that for every $d$ there is $M = M(d)$ such that every $(\leq d + 1)$-crossing curve in $\mathbb{R}^d$ can be subdivided into at most $M$ $(\leq d)$-crossing curve segments. As a consequence, based on the work of Eliáš, Roldán, Safernová, and the second author, we obtain an essentially tight lower bound for a geometric Ramsey-type problem in $\mathbb{R}^d$ concerning order-type homogeneous sequences of points, investigated in several previous papers.

1 Introduction

The most intuitive statement of the problem investigated in this paper involves curves in $\mathbb{R}^d$. By a curve we mean an arbitrary continuous mapping $\gamma: I \to \mathbb{R}^d$, where $I \subset \mathbb{R}$ is a closed interval (we could admit an open interval as well, but this would add unnecessary technical complications). Let us say that a curve $\gamma$ in $\mathbb{R}^d$ is $(\leq k)$-crossing if it intersects every hyperplane $h$ at most $k$ times$^1$. Here the intersections are counted with multiplicity; that is, the condition of $(\leq k)$-crossing reads $|\{t \in I : \gamma(t) \in h\}| \leq k$.

It will be useful to observe that a $(\leq k)$-crossing curve is not constant on any nonempty open interval, and its image contains no segment.

$^1$For algebraic curves in the complex projective space, the number of intersections with a generic hyperplane is the degree, but we prefer using a different term, since we deal with much more general curves, which are typically not algebraic.
(≤ d)-crossing (=convex) curves. The (≤ d)-crossing curves in \( \mathbb{R}^d \) are called **convex curves** in a significant part of the literature (e.g., [Arn04, Živ04, SS00, SS05, Mus98]), and they are of considerable interest in several areas. In the plane, a convex curve in this sense is a connected piece of the boundary of a convex set. A primary example of a higher-dimensional convex curve is the **moment curve** \( \{(t, t^2, \ldots, t^d) : t \in [0, 1]\} \). The convex hull of \( n \geq d + 1 \) points on a convex curve in \( \mathbb{R}^d \) is a **cyclic polytope**, one of the most important examples in the theory of convex polytopes and in discrete geometry in general.

If we regard a convex curve \( \gamma: I \to \mathbb{R}^d \) as a \( d \)-tuple \((\gamma_0, \ldots, \gamma_d)\) of functions \( I \to \mathbb{R} \), and define \( \gamma_0 \equiv 1 \), then the \((d + 1)\)-tuple \((\gamma_0, \gamma_1, \ldots, \gamma_d)\) (or possibly \((-\gamma_0, \gamma_1, \ldots, \gamma_d)\)) forms a **Chebyshev system**[^1] which is an important notion in approximation theory, theory of finite moments, and other areas—see, e.g., [KS66, CPZ98]. Conversely, every Chebyshev system \((\gamma_0, \ldots, \gamma_d)\) on an interval \( I \) with \( \gamma_0 \equiv 1 \) (or more generally, \( \gamma_0 \) strictly monotone) gives rise to a convex curve in \( \mathbb{R}^d \).

**Subdividing (≤ d + 1)-crossing curves.** The following question is quite natural and interesting in its own right and it has been motivated by the work [EMRS13] in geometric Ramsey theory, as will be explained below. Given an integer \( d \geq 2 \), does there exist \( M = M(d) \) such that every \((≤ d + 1)\)-crossing curve \( \gamma \) in \( \mathbb{R}^d \) can be subdivided into at most \( M \) convex curves? In more detail, if \( \gamma \) is a map \( I \to \mathbb{R}^d \), we want to subdivide \( I \) into subintervals \( I_1, \ldots, I_k, k \leq M \), so that the restriction of \( \gamma \) to each \( I_i \) is convex (i.e., \((≤ d)\)-crossing). Our main result answers this question in the affirmative.

**Theorem 1.1.** For every integer \( d \geq 2 \) there exists \( M = M(d) \) such that every \((≤ d + 1)\)-crossing curve \( \gamma \) in \( \mathbb{R}^d \) can be subdivided into at most \( M \) convex curves.

We note that the value \( d + 1 \) is important, since a \((≤ d + 2)\)-crossing curve in \( \mathbb{R}^d \) in general cannot be subdivided into a bounded number of convex curves. An example for \( d = 2 \) can be obtained, e.g., by starting with a circular arc and making many very small and flat inward dents in it.

The case \( d = 2 \) is already nontrivial, but to our surprise, we haven’t found it mentioned in the literature. The following picture shows a planar curve, namely, the graph of \( x(1-x^2)^2 \) on \([-1, 1]\), which can be checked to be \((≤ 3)\)-crossing, but obviously cannot be subdivided into fewer than 4 convex arcs:

![Image of a planar curve](image)

Hence \( M(2) \geq 4 \). We can prove that \( M(2) \) actually equals 4, and that \( M(3) \leq 22 \). The proofs can be found in an earlier version of this paper [BM13] by the first two authors.

**Theorem 1.1 for polygonal paths.** For technical reasons, and also from the point of view of our motivation in geometric Ramsey theory, it is more convenient to work with polygonal paths.

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[^1]: Let \( A \) be a linearly ordered set of at least \( k + 1 \) elements. A (real) **Chebyshev system** on \( A \) is a system of continuous real functions \( f_0, f_1, \ldots, f_k: A \to \mathbb{R} \) such that for every choice of elements \( t_0 < t_1 < \cdots < t_k \) in \( A \), the matrix \( (f_i(t_j))_{i,j=0}^k \) has a (strictly) positive determinant.
paths. A polygonal path is a curve made of finitely many straight segments; we call these segments the edges of the polygonal path, and their endpoints are the vertices. For a point sequence \((p_1, p_2, \ldots, p_n)\), we write \(p_1 p_2 \cdots p_n\) for the polygonal path consisting of the segments \(p_1 p_2, \ldots, p_{n-1} p_n\).

The definition of \((≤k)\)-crossing needs to be modified: we call a polygonal path \(\pi\) \((≤k)\)-crossing if it intersects every hyperplane in at most \(k\) points, with the exception of the hyperplanes that contain an edge of \(\pi\). Moreover, we will also consider only polygonal paths in general position, meaning that every \(k \leq d + 1\) vertices of the polygonal path are affinely independent. The polygonal path version of Theorem 1.1 says the following.

**Theorem 1.2.** For every integer \(d \geq 2\) there exists \(M = M(d)\) such that every \((≤d + 1)\)-crossing polygonal path \(\pi\) in \(\mathbb{R}^d\) can be subdivided into at most \(M\) convex (i.e., \((≤d)\)-crossing) polygonal paths.

In Section 6 we prove by a limit argument that Theorem 1.2 implies Theorem 1.1.

**Order-type homogeneous subsequences.** Now we come to the geometric Ramsey-type problem motivating our work.

Let \(T = (p_1, \ldots, p_{d+1})\) be an ordered \((d + 1)\)-tuple of points in \(\mathbb{R}^d\). We recall that the sign (or orientation) of \(T\) is defined as \(\text{sgn}\ det X\), where the \(j\)th column of the \((d + 1) \times (d + 1)\) matrix \(X = (1, p_{i1}, p_{i2}, \ldots, p_{jd})\), with \(p_{ij}\) denoting the \(i\)th coordinate of \(p_j\). Geometrically, the sign is +1 if the \(d\)-tuple of vectors \(p_1 - p_{d+1}, \ldots, p_d - p_{d+1}\) forms a positively oriented basis of \(\mathbb{R}^d\), it is −1 if it forms a negatively oriented basis, and it is 0 if these vectors are linearly dependent.

We call a sequence \((p_1, p_2, \ldots, p_n)\) of points in \(\mathbb{R}^d\) in general position order-type homogeneous if all \((d + 1)\)-tuples \((p_{i1}, \ldots, p_{id+1})\), \(i_1 < \cdots < i_{d+1}\), have the same sign (which is nonzero, by the general position assumption).

Let \(\text{OT}_d(n)\) be the smallest \(N\) such that every sequence of \(N\) points in general position in \(\mathbb{R}^d\) contains an order-type homogeneous subsequence of length \(n\). The existence of \(\text{OT}_d(n)\) for all \(d\) and \(n\) follows immediately from Ramsey’s theorem, but several recent papers \(\text{EM13, CFP}^{+13, \text{Suk}13, \text{EMRS}13}\) considered the order of magnitude of \(\text{OT}_d(n)\), for \(d\) fixed and \(n\) large.

For \(d = 2\), the classical paper of Erdős and Szekeres \(\text{ES}35\) implies that \(\text{OT}_2(n) = 2^{\Theta(n)}\). \(\text{Suk}13, \text{Suk}13\), improving on a somewhat weaker bound by Conlon et al. \(\text{CFP}^{+13}, \text{proved the upper bound } \text{OT}_d(n) \leq \text{twr}_d(O(n))\) for every fixed \(d\), where the tower function \(\text{twr}_d(x)\) is defined by \(\text{twr}_1(x) = x\) and \(\text{twr}_{i+1}(x) = 2^{\text{twr}_i(x)}\). He conjectured this to be optimal, but so far matching lower bounds were known only for \(d = 2\) (by \(\text{ES}35\)) and \(d = 3\) \(\text{EM13}\).

By combining the results of \(\text{EMRS}13\) with Theorem 1.2, we obtain a matching lower bound for all \(d \geq 2\):

**Theorem 1.3.** We have \(\text{OT}_d(n) \geq \text{twr}_d(\Omega(n))\).

The argument is given in Section 7.

\(^3\text{We employ the usual asymptotic notation for comparing functions: } f(n) = O(g(n)) \text{ means that } |f(n)| \leq C|g(n)| \text{ for some } C \text{ and all } n, \text{ where } C \text{ may depend on parameters declared as constants (in our case on } d); f(n) = \Omega(g(n)) \text{ is equivalent to } g(n) = O(f(n)); \text{ and } f(n) = \Theta(g(n)) \text{ means that both } f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n)). \)
2 Order-type homogeneity and path convexity

We need the following fact.

**Lemma 2.1.** A sequence $P = (p_1, p_2, \ldots, p_n)$ in general position in $\mathbb{R}^d$ is order-type homogeneous iff the polygonal path $\pi = p_1p_2 \cdots p_n$ is convex.

**Proof.** First we assume that $P$ is not order-type homogeneous. Then it has two $(d+1)$-tuples, of the form $Q = (q_1, \ldots, q_{d+1})$ and $R = (r_1, \ldots, r_{d+1})$, with opposite signs (both $Q$ and $R$ are subsequences of $P$, i.e., the $q_i$ and the $r_j$ appear in $P$ in this order).

It is easy to check that we can also find $Q$ and $R$ with opposite signs that differ in a single point; more precisely, there is an index $k$ such that $q_i = r_i$ for all $i \neq k$. Indeed, given arbitrary $Q$ and $R$ with opposite signs, we can convert $Q$ into $R$ by a sequence of moves, each of them changing a single element: we always move the first element in which the current $Q$ differs from $R$ to the correct position. Then at least one of the moves involves two $(d+1)$-tuples with opposite signs.

Having $Q$ and $R$ as above with $q_i = r_i$ for all $i \neq k$, we consider the hyperplane $h$ spanned by the points of $Q' := \{q_i : i \neq k\}$. Then $q_k$ and $r_k$ lie on opposite sides of $h$, and hence $\pi$ intersects $h$ between $q_k$ and $r_k$. Together with the $d$ points $Q'$, we have $d+1$ intersections of $\pi$ with $h$.

This $h$ may still contain edges of $\pi$, so we may need to move it slightly. For simpler description, we think of $h$ as horizontal, and say that $q_k$ is below $h$, $r_k$ is above $h$, and $q_k$ precedes $r_k$ in $P$. Then, since $Q'$ is affinely independent, we can move $h$ by an arbitrarily small amount to a new position $h'$ so that the points in the sequence $(q_1, q_2, \ldots, q_{k-1}, q_k, r_k, q_{k+1}, \ldots, q_{d+1})$ are alternatingly above and below $h'$. This implies that $\pi$ intersects $h'$ at least $d+1$ times, and since the move of $h$ was generic, we may assume that $h'$ contains no edges of $\pi$.

For the reverse implication, we need the following claim: If $P = (p_1, p_2, \ldots, p_n)$ is an order-type homogeneous sequence and $q$ is an interior point of the segment $p_ip_{i+1}$, then the sequence $P' = (p_1, p_2, \ldots, p_i, q, p_{i+2}, \ldots, p_n)$ ($p_{i+1}$ replaced with $q$) is order-type homogeneous as well.

To verify this claim, we suppose w.l.o.g. that all $(d+1)$-tuples of $P$ are positive, and we consider an arbitrary $(d+1)$-tuple in $P'$ involving $q$, of the form

$$T = (p_{j_1}, \ldots, p_{j_{k-1}}, q, p_{j_{k+1}}, \ldots, p_{j_{d+1}}), \quad 1 \leq j_1 < \cdots < j_{k-1} < i + 1 < j_{k+1} < \cdots < j_{d+1} \leq n.$$  

We think of $q$ moving from $p_i$ to $p_{i+1}$ along the segment $p_ip_{i+1}$. The determinant whose sign defines the sign of $T$ is an affine function of $q$ (considering the remaining points of $T$ fixed). For $q = p_i$ it is either 0 (if $j_{k-1} = i$) or strictly positive, and for $q = p_{i+1}$ it is strictly positive. Therefore, for $q$ in between, it is strictly positive too, which proves the claim.

Now we assume for contradiction that the sequence $P = (p_1, \ldots, p_n)$ is order-type homogeneous, but the corresponding polygonal path $\pi$ is not convex, and so it has at least $d+1$ intersections with some hyperplane $h$ not containing an edge of $\pi$. Let us fix intersections $q_1, q_2, \ldots, q_{d+1}$; at least one of them, call it $q_\ell$, is an interior point of an edge $p_jp_{j+1}$ of $\pi$ (since the $p_i$ are in general position).

Using the claim above, we now want to replace $\pi$ by another polygonal path $\pi'$, whose vertex sequence is still order-type homogeneous and includes all $q_\ell$ with $i \neq \ell$, as well as $p_j$ and $p_{j+1}$. To this end, we first observe that no two $q_i$ share a segment of $\pi$ (since $h$ contains no such segment).
When producing \( \pi' \), first, if there is a \( q_i \) with \( i > \ell \) that is not a vertex of the current polygonal path, we take the last such \( q_i \). We replace the vertex of the current polygonal path immediately following \( q_i \) with \( q_i \). By the claim, the new vertex sequence is still order-type homogeneous. We repeat this step until all \( q_i \) with \( i > \ell \) become vertices.

Then we proceed analogously with the \( q_i \), \( i < \ell \), that are not vertices. This time we start with the smallest \( i \), and \( q_i \) always replaces the vertex immediately preceding it (and we apply the claim to the reversal of the considered sequences). Here is an illustration:

![Polygonal Path Illustration]

In this way, we obtain the polygonal path \( \pi' \) with order-type homogeneous vertex sequence that is intersected by the hyperplane \( h \) in the \( d \) vertices \( q_i \), \( i \neq \ell \), and in \( q_\ell \), which is an interior point of the segment \( p_jp_{j+1} \) (neither \( p_j \) nor \( p_{j+1} \) have been replaced). But then the \((d+1)\)-tuples \((q_1, \ldots, q_{\ell-1}, p_j, q_{\ell+1}, \ldots, q_{d+1})\) and \((q_1, \ldots, q_{\ell-1}, p_{j+1}, q_{\ell+1}, \ldots, q_{d+1})\) have opposite signs—a contradiction.

\[ \square \]

### 3 A combinatorial property of \((\leq d+1)\)-crossing paths

Here we prove a combinatorial property of point sequences in \( \mathbb{R}^d \) for which the corresponding polygonal path is \((\leq d+1)\)-crossing. In the two subsequent sections we will derive Theorem 1.2 from this property in a purely combinatorial way.

Let \( P = (p_1, \ldots, p_n) \) be a sequence in general position in \( \mathbb{R}^d \) and let \( \pi = p_1 \cdots p_n \) be the corresponding polygonal path. For notational convenience, for \( Q \subset P \) with \(|Q| = d+1\), we define \( \text{sgn} \ Q \) as the sign of the sequence \((p_{i_1}, \ldots, p_{i_{d+1}})\), where \( Q = \{p_{i_1}, \ldots, p_{i_{d+1}}\} \) with \( i_1 < i_2 < \ldots < i_{d+1} \). For a fixed subset \( R \subset P \) with \(|R| = d \), we consider the following sequence, which we call the sign sequence of \( R \):

\[
\left( \text{sgn}(\{p_i\} \cup R) : i = 1, 2, \ldots, n, \ p_i \not\in R \right) \in \{-1, +1\}^{n-d}.
\]

**Lemma 3.1.** If \( \pi \) is \((\leq d+1)\)-crossing, then for every \( R \) as above, the sign sequence \((3.1)\) of \( R \) has at most one sign change.

**A simple case.** For proving the lemma, we first consider a simple special case. Letting \( H \) be the hyperplane spanned by \( R \), we assume that \( R \) contains no consecutive elements from \( P \), and moreover, that \( H \) separates \( p_{i-1} \) from \( p_{i+1} \) whenever \( p_i \in R \).

Because of the \((\leq d+1)\)-crossing condition, \((\pi \cap H) \setminus R \) is either the empty set or a single point, which we call \( q \). Then for \( x \in \pi \), we have \( \text{sgn}(\{x\} \cup R) = 0 \) iff \( x \in R \) or \( x = q \).

Let us think of \( x \) moving along \( \pi \). When it passes through a point \( p \in R \), \( \text{sgn}(\{x\} \cup R) \) does not change because \( x \) moves from one side of \( H \) to the other, while \( x \) changes places with \( p \) in the order on \( \pi \). The same argument shows that \( \text{sgn}(\{x\} \cup R) \) changes only if \( x \) passes through \( q \).

**Auxiliary claims.** Next, we make preparations for proving the lemma in general.
The set $P \setminus R$ is non-empty, so we fix one of its elements and call it $p_\alpha$. We define $\mathcal{R}_\delta$ as the set of all sequences $(q_i \in \pi : p_i \in R)$ such that $|q_i - p_i| < \delta$, and for $i > \alpha$, $q_i$ lies on the open segment $(p_{i-1}, p_i)$, while for $i < \alpha$ it lies on $(p_i, p_{i+1})$. Here is a schematic illustration:

$$
\begin{array}{c}
\text{H} \\
p_1 & p_2 \\
p_3 \\
p_\alpha \\
\end{array}
\begin{array}{c}
\text{Q} \\
p_\pi \\
\end{array}
$$

Since $R$ spans the hyperplane $H$, every set $Q \in \mathcal{R}_\delta$ for sufficiently small $\delta$ spans a hyperplane as well. By general position, we have $\varepsilon_0 := \text{dist}(P \setminus R, H) > 0$. By continuity, we also get the next claim:

**Claim 3.2.** There is $\delta_1 > 0$ such that $\text{dist}(P \setminus R, \text{aff } Q) > \frac{1}{2}\varepsilon_0$ for all $Q \in \mathcal{R}_{\delta_1}$.

This has the following consequence:

**Corollary 3.3.** If $p_{h}, p_{h+1} \notin R$ and $H \cap p_{h} p_{h+1} \neq \emptyset$, then $\text{aff } Q \cap p_{h} p_{h+1} \neq \emptyset$ for all $Q \in \mathcal{R}_{\delta_1}$.

**Claim 3.4.** There is a $\delta_2 \in (0, \delta_1)$ such that $P \cap \text{aff } Q = \emptyset$ for all $Q \in \mathcal{R}_{\delta_2}$.

**Proof.** If not, then there is a sequence $\delta_m \rightarrow 0$ and $Q_m \in \mathcal{R}_{\delta_m}$ with $P \cap \text{aff } Q_m \neq \emptyset$. Then, for a suitable subsequence, $P \cap \text{aff } Q_m$ contains a fixed element $p_h \in P$. We have $p_h \in R$ because the $Q_m$ have distance at least $\varepsilon_0/2$ to $P \setminus R$.

Let $(p_i, p_{i+1}, \ldots, p_j)$ be the string of $R$ containing $p_h$, i.e., a maximal contiguous subsequence of $P$ whose points all lie in $R$ (i.e., $p_{i-1}, p_{j+1} \notin R$; we also admit $i = 1$ and $j = n$, as well as $i = j$). Thus $i \leq h \leq j$ and the polygonal path $p_i \ldots p_j$ is contained in $H$.

Let us assume $h > \alpha$; then $i > \alpha$ as well. Since $p_h \in \text{aff } Q_m$ and $q_h \in Q_m$, the whole line aff $\{p_h, q_h\}$ is contained in aff $Q_m$. Since $p_{h-1}$ is on this line, it is in aff $Q_m$ as well. This shows (by induction) that $p_h, p_{h-1}, \ldots, p_i, p_{i-1} \in \text{aff } Q_m$. Thus $p_{i-1} \in \text{aff } Q_m$, which contradicts Claim 3.2. The argument for $h < \alpha$ is symmetric. \[\square\]

**Proof of Lemma 3.1.** We fix some $\delta \in (0, \delta_2)$ and $Q \in \mathcal{R}_\delta$, and set $H^* = \text{aff } Q$. We observe that $H$ and $H^*$ separate the points of $P \setminus R$ the same way. Moreover, if $(p_i, \ldots, p_j)$ is a string of $R$ and $i > \alpha$, then the points $p_{i-1}, p_i, \ldots, p_j$ lie alternately on the two sides of $H^*$. This follows from the fact that the path $p_{i-1} p_i \ldots p_j$ intersects $H^*$ in the points $q_i, \ldots, q_j$. Similarly, for $i < \alpha$, the points $p_i, \ldots, p_j, p_{j+1}$ lie alternately on the two sides of $H^*$.

We again let $x$ move along $\pi$. With $R = (p_{i_1}, \ldots, p_{i_{d+1}})$, we have

$$\text{sgn}\{x\} = \text{sgn det} \begin{pmatrix}
1 & \cdots & 1 & 1 & \cdots & 1 \\
p_{i_1} & \cdots & p_{i_{j-1}} & x & p_{i_j} & \cdots & p_{i_{d+1}}
\end{pmatrix}$$

where the position of the column with $x$ is determined by $x$ lying between $p_{i_{j-1}}$ and $p_{i_j}$. Then

$$\text{sgn}\{x\} = \text{sgn det} \begin{pmatrix}
1 & \cdots & 1 & 1 & \cdots & 1 \\
q_{i_1} & \cdots & q_{i_{j-1}} & x & q_{i_j} & \cdots & q_{i_{d+1}}
\end{pmatrix}$$

where $Q = (q_{i_1}, \ldots, q_{i_{d+1}})$ and the same remark applies to the position of the $x$ column.

Clearly $\text{sgn}\{x\} = \text{sgn}\{x\}$ when $x \in P \setminus R$. Thus, it suffices to check how $\text{sgn}\{x\}$ changes when $x$ moves through $q_1, \ldots, q_j$ for the string $p_i, \ldots, p_j$. Note that $\text{sgn}\{x\}$ changes only when $x$ passes some point in $Q \cap \pi$. 

6
Just like in the basic case, $\text{sgn}\{(x) \cup Q\}$ does not change when $x$ passes $q_h$ because then $x$ moves from one side of $H^*$ to the other and it also changes places with $q_h$. Thus, $\text{sgn}\{(p_{i-1}) \cup Q\} = \text{sgn}\{(x) \cup Q\}$ when $x$ just passed $q_j$.

Now we assume that $\alpha < i$; the other option $\alpha > i$ is symmetric and follows the same way. There are two cases.

**Case 1:** when $p_j$ and $p_{j+1}$ are on the same side of $H^*$. Then $\text{sgn}\{(p_j) \cup Q\} = \text{sgn}\{(p_{j+1}) \cup Q\}$, and so $\text{sgn}\{(p_{i-1}) \cup Q\} = \text{sgn}\{(p_{j+1}) \cup Q\}$, implying $\text{sgn}\{(p_{i-1}) \cup R\} = \text{sgn}\{(p_{j+1}) \cup R\}$. So there is no sign change between $p_{i-1}$ and $p_{j+1}$ in the sign sequence of $R$.

**Case 2:** when $p_j$ and $p_{j+1}$ are on opposite sides of $H^*$. Then $H^* \cap p_jp_{j+1}$ is a point $q$, and $\text{sgn}\{(x) \cup Q\}$ changes sign when $x$ moves through $q$. Consequently, $\text{sgn}\{(p_{j+1}) \cup R\} = -\text{sgn}\{(p_{i-1}) \cup R\}$, and there is a sign change in the sign sequence of $R$ here.

But since $H^* \cap \pi$ contains already $d+1$ points, Case 2 cannot occur anywhere else. Also, the case in Claim 3.3 cannot come up either, since that would mean $H^* \cap \pi$ contains $d+2$ points. Thus, the only sign change in the sign sequence of $R$ occurs between $p_{i-1}$ and $p_{j+1}$. □

## 4 k-sequences and flip k-sequences

Now we will define a combinatorial abstraction of point sequences in $\mathbb{R}^k$. A **k-sequence** is a sequence $S = (a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are distinct (abstract) elements, together with a mapping $\text{sgn}$ that assigns either $+1$ or $-1$ to every $(k+1)$-element subset $A \subseteq \{a_1, \ldots, a_n\}$ (sometimes we will regard $A$ as a subsequence, with the elements in the same order as in $S$). We will also say that $A$ is **positive** or **negative** if $\text{sgn}A = 1$ or $\text{sgn}A = -1$, respectively.

We subdivide the sequence $S$ into contiguous blocks with one-point overlaps: The first block is $B_1 = (a_1, \ldots, a_{i_1})$ with $i_1$ maximal such that all $(k+1)$-point subsequences in $B_1$ have the same sign $\sigma_1$. The next one is $B_2 = (a_{i_1}, \ldots, a_{i_2})$ with $i_2$ maximal such that all $(k+1)$-point subsequences in $B_2$ have the same sign $\sigma_2$, and so on, up until some $B_m = (a_{i_{m-1}}, \ldots, a_n)$, where $B_m$ either has at most $k$ elements, or it has more than $k$ elements and every $(k+1)$-tuple in it has the same sign $\sigma_m$.

We call this partition the **greedy partition** of $S$; here both $m = m(S)$ and the blocks $B_j$ are uniquely determined. Note that each $B_j$, $j < m$, contains a subset $D_j$ of size $k$ such that $\text{sgn}\{(a_{i_{j+1}}) \cup D_j\} \neq \sigma_j$.

The following lemma shows that $S$ has a short subsequence $S^*$ whose greedy partition is similar to that of $S$.

**Lemma 4.1.** There is a subsequence $S^*$ of $S$, which we call the reduced version of $S$, such that $m(S^*) = m(S)$, every block of the greedy partition of $S^*$ contains at most $k+3$ elements, and the last one exactly 2. Moreover, every string of $2k+5$ consecutive elements of $S^*$ contains both a positive $(k+1)$-tuple and a negative one.

**Proof.** Let $B_j = (a_{i_{j-1}}, \ldots, a_{i_j})$ be a block of the greedy partition of $S$ with $j < m$. Let us fix a $d$-element subset $D_j$ of $B_j$ as above, i.e., with $\text{sgn}\{(a_{i_{j+1}}) \cup D_j\} \neq \sigma_j$.

The subsequence $S^*$ contains the following elements of $B_j$: $a_{i_{j-1}}, a_{i_{j-1}+1}, a_{i_j}$, the elements of $D_j$, and one more (arbitrarily chosen) element if the first three are all contained in $D_j$. All the other elements are discarded. From the last block we keep the first two elements.

Let us consider the greedy partition of $S^*$. By induction on $j$, it is easy to see that for $j < m$, the $j$th block $B_j^*$ starts with $a_{i_{j-1}}$, ends with $a_{i_j}$, and the sign of $D_j \cup \{a_{i_{j+1}}\}$ is different from $\sigma_j$, which is the sign of (all) $(k+1)$-tuples in $B_j^*$. 

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7
It follows that every string of $2k + 5$ consecutive elements of $S^*$ contains a full block $B_j^*$ plus the next element $a_{i_j+1}$. The sign of the first $k + 1$ elements of $B_j^*$ is different from $\sgn(D_j \cup \{a_{i_j+1}\})$. \hfill $\Box$

A $k$-sequence $S = (a_1, \ldots, a_n)$ is called a flip $k$-sequence if it has the property as in Lemma 3.1 that is, for every $k$-element $A \subset \{a_1, \ldots, a_n\}$, the sign sequence of $A$

$$\left(\sgn(\{a_i\} \cup A) : i = 1, 2, \ldots, n, \ a_i \notin A\right)$$

has at most one sign change. The following result of combinatorial nature is the key step in the proof of Theorem 1.2.

**Theorem 4.2.** For every $k \geq 1$ there is $c(k)$ such that the greedy partition of every flip $k$-sequence has at most $c(k)$ blocks.

We prove this result in the next section. Now we show how it implies Theorem 1.2.

**Proof of Theorem 4.2** We assume that $P = (p_1, \ldots, p_n) \subset \mathbb{R}^d$ is in general position. Let $\pi = p_1 \cdots p_n$ be the corresponding polygonal path. Lemma 3.1 shows that $(p_1, \ldots, p_n)$ with the sign of $(d + 1)$-tuples given by their orientation is a flip $d$-sequence. Theorem 4.2 says that its greedy partition has at most $c(d)$ blocks. All $(d + 1)$-tuples in $B_j$ have the same sign, so $B_j = (p_{i_j-1}, \ldots, p_i)$ is order-type homogeneous, and thus the polygonal path $p_{i_j-1} \cdots p_i$ is convex. It follows that $M(d) \leq c(d).$ \hfill $\Box$

**5 Proof of Theorem 4.2**

**Proof.** We proceed by induction on $k$.

**The case $k = 1$.** We will show that $c(1) = 3$ (instead of reading this part, the reader may perhaps prefer to find a simple proof of $c(1) \leq 5$, say).

Let $S = (a_1, \ldots, a_n)$ be a flip 1-sequence, and let $B_1, \ldots, B_m$ be the blocks of its greedy partition. Each $B_i$ has the form $(b_i, x_i, \ldots, c_i)$ where $b_{i+1} = c_i$, and $B_i$ contains an element $d_i$ such that $\sgn(d_i, x_{i+1}) \neq \sigma_i$. Note that $x_1$ and $d_m$ are undefined.

**Observation.** If $B_i$ and $B_{i+1}$ are two consecutive blocks, both positive, then $d_i$, $c_i = b_{i+1}$, and $x_{i+1}$ are three distinct elements of $S$. Moreover, for every $a \in S$ preceding $d_i$ we have $(a, x_{i+1})$ negative, and similarly, for every $a$ following $x_{i+1}$ we have $(d_i, a)$ negative.

Only the last two statements need an explanation. Since $(c_i, x_{i+1})$ is positive and $(d_i, x_{i+1})$ is negative, $(a, x_{i+1})$ must be negative for $a$ preceding $d_i$, for otherwise, there are two sign changes in the sign sequence of $\{x_{i+1}\}$. The statement about $(d_i, a)$ is proved in the same way.

The proof of $c(1) \leq 3$ comes in fives steps. We assume w.l.o.g. that $(a_1, a_2)$ is positive.

**Step 1.** If all $(a_i, a_{i+1})$ are positive, then $m < 4$. Indeed, supposing $B_4$ exists, all blocks are positive, $(d_1, x_2)$ is negative, and $(d_1, d_3)$ is negative by the observation. Also, $(d_3, x_4)$ is negative and there are two sign changes in the sign sequence of $\{d_3\}$. Since $(b_3, d_3)$ or $(d_3, c_3)$ (or both) are positive. 

8
Step 2. If $j$ is the smallest index with $(a_j, a_{j+1})$ negative, then $a_j = c_i = b_{i+1}$, $B_i$ is a positive block, and $B_{i+1}$ is a negative one. Assume $B_{i-1}$ exists. Then it is positive, $(d_{i-1}, x_i)$ is negative, and thus $(d_{i-1}, a_j)$ is negative by the observation. But then there are two sign changes in the sign sequence of $\{a_j\}$: $(d_{i-1}, a_j)$ and $(a_j, x_{i+1})$ are negative and $(b_i, a_j)$ is positive. Thus $B_{i-1}$ cannot exist, $i = 1$, and there is a single block before $a_j$.

Step 3. Thus $B_1$ is positive and $B_2$ negative. Assume $B_3$ negative; then $(d_2, x_3)$ is positive and so is $(b_2, x_3)$ by the observation. Consequently, there are two sign changes in the sign sequence of $\{b_2\}$: $(b_1, b_2)$ and $(b_2, x_3)$ are positive but $(b_2, x_2)$ is negative. We conclude that $B_3$ is a positive block.

Step 4. Assume $B_4$ exists and is positive. Then $(d_3, x_4)$ is negative and so is $(b_3, x_4)$ by the observation. Then there are two sign changes in the sign sequence of $\{b_3\}$: $(b_2, b_3)$ and $(b_3, x_4)$ are negative and $(b_3, c_3)$ is positive.

Step 5. We are left with the case when $B_1, B_3$ are positive and $B_2, B_4$ negative. If $(b_2, b_4)$ is positive, then there are two sign changes in the sign sequence of $\{b_2\}$: $(b_1, b_2)$ and $(b_2, b_4)$ are positive and $(b_2, c_2)$ negative. Similarly, if $(b_2, b_4)$ is negative, then there are two sign changes in the sign sequence of $\{b_4\}$.

Consequently, $B_4$ does not exist: $m < 4$ and so $c(1) \leq 3$.

The example $S = (a_1, a_2, a_3, a_4)$ with $a_1, a_2$ and $a_3, a_4$ positive and all other pairs negative shows that $c(1) = 3$.

The inductive step from $k = 1$ to $k$. Assuming that the greedy partition of each flip $(k - 1)$-sequence has at most $c(k - 1)$ blocks, we will show that the greedy partition of an arbitrary flip $k$-sequence $S = (a_1, \ldots, a_n)$ has at most $c(k) := 1 + (4k + 10)c(k - 1)/k$ blocks.

So we suppose the contrary that $S$ as above has $m > c(k)$ blocks. We can further assume that $S$ is reduced in the sense of Lemma 4.1 for otherwise, we can replace $S$ by $S^*$. Since each $B_i$, $i < m$, has at least $k + 1$ elements, and $|B_m| = 2$, the length of $S$ is at least

$$n \geq (m - 1)k + 2 > (4k + 10)c(k - 1) + 2.$$  

We consider the sequence $T = (a_1, \ldots, a_{n-1})$ and regard it as a $(k - 1)$-sequence by defining, for a $k$-element $A \subset \{a_1, \ldots, a_{n-1}\}$, the sign $\text{sgn} A := \text{sgn}(A \cup \{a_n\})$. It is clear that $T$ is a flip $(k - 1)$-sequence, and so its greedy partition has at most $c(k - 1)$ blocks. One of the blocks, which we call $B$, has at least $(n - 1)/c(k - 1) \geq 4k + 10$ elements. We may assume w.l.o.g. that $\text{sgn} A = +1$ for every $k$-element subset of $B$.

Since $S$ is reduced, there is a positive $(k + 1)$-tuple $(b_1, \ldots, b_{k+1})$ among the first $2k + 5$ elements of $B$, and a negative $(k + 1)$-tuple $(b_{k+2}, \ldots, b_{2k+2})$ among the last $2k + 5$ elements of $B$. The sign of the $(k + 1)$-tuple $(b_i, \ldots, b_{i+k})$ changes from $+1$ to $-1$ as $i$ moves through $1, 2, \ldots, k + 2$, and so there is some $j$ with $\text{sgn}(b_j, \ldots, b_{j+k+1}) = +1$ and $\text{sgn}(b_{j+1}, \ldots, b_{j+k+2}) = -1$.

We set $A = \{b_{j+1}, \ldots, b_{j+k+1}\}$. Then we have $\text{sgn}(\{b_j\} \cup A) = +1$ and $\text{sgn}(A \cup \{b_{j+k+2}\}) = -1$, while $\text{sgn}(A \cup \{a_n\}) = +1$ by the choice of the block $B$. Hence the sign sequence of $A$ has at least two sign changes, contradicting the assumption that $S$ is a flip $k$-sequence. This contradiction finishes the proof of Theorem 1.2.

Remark. This argument gives $c(k) = \exp(O(k))$. We note that $c(1) = 3$ and $M(1) = 3$. The above proof gives $c(2) \leq 22$ while $M(2) = 4$.  

9
6 From polygonal paths to curves: proof of Theorem 1.1

Here we show how Theorem 1.2 implies Theorem 1.1.

We assume \( \gamma: I \to \mathbb{R}^d \) is a \((\leq d + 1)\)-crossing curve.

Let us say that an \( n \)-tuple \( T = (t_1, \ldots, t_n) \), \( t_1, \ldots, t_n \in I \), \( t_1 < \cdots < t_n \), is an \( \epsilon \)-sample if every subinterval of \( I \) of length \( \epsilon \) contains some \( t_i \). Let \( \pi = \pi(\gamma, T) = \gamma(t_0)\gamma(t_1) \cdots \gamma(t_n) \) be the polygonal line determined by \( T \).

First we observe that for every \( \epsilon > 0 \), there is an \( \epsilon \)-sample \( T \) with \( \pi(\gamma, T) \) in general position. Indeed, having already placed \( k \) points of \( T \), so that their \( \gamma \)-images are in general position, we consider the finitely many hyperplanes spanned by \( d \)-tuples of these \( \gamma \)-images. Since \( \gamma \) is \((\leq d + 1)\)-crossing, each of these hyperplanes contains at most one extra point of \( \gamma \), and so at every step of the construction, we have only finitely many excluded points of \( I \). Thus, we can construct an \( \epsilon \)-sample as desired.

Next, for every \( \epsilon > 0 \), we fix an \( \epsilon \)-sample \( T = T(\epsilon) \) with \( \pi(\gamma, T(\epsilon)) \) in general position. Let \( M = M(d) \) be as in Theorem 1.2; by that theorem, we can also fix a subdivision of \( I \) into \( M \) subintervals such that the restriction of \( \pi(T(\epsilon), \gamma) \) on each of them is convex. By compactness, these subdivisions have a cluster point for \( \epsilon \to 0 \); we denote its intervals by \( I_1, \ldots, I_M \).

It remains to show that \( \gamma \) restricted to each \( I_j \) is convex. This follows from the next lemma, applied with \( I = I_j \) and \( \gamma = \gamma_j \).

**Lemma 6.1.** Let \( \gamma: I \to \mathbb{R}^d \) be a \((\leq d + 1)\)-crossing curve, and let us suppose that for every \( \epsilon > 0 \) there is an \( \epsilon \)-sample \( T(\epsilon) \) such that the corresponding polygonal path \( \pi(\gamma, T(\epsilon)) \) is in general position and convex. Then \( \gamma \) is convex as well.

**Proof.** For contradiction, we suppose that there is a hyperplane \( h \) intersecting \( \gamma \) in at least \( d + 1 \) points.

First we observe that these points can be assumed to span \( h \): if their affine hull \( F \) had dimension smaller than \( d - 1 \), then since \( \gamma \not\subset F \), we could rotate \( h \) around \( F \) and thus get more than \( d + 1 \) intersections.

Let us say that a point \( \gamma(t) \in h, t \in I \), is a generic intersection with \( h \) if for an arbitrarily small neighborhood \( U \) of \( t \), \( \gamma(U) \) intersects both of the open halfspaces bounded by \( h \) (as usual, we count generic intersections with multiplicity, so the generic intersection is actually determined by \( t \)). We claim that there is a hyperplane \( h' \) with at least \( d + 1 \) generic intersections.

For easier description, let us imagine \( h \) horizontal. An intersection that is not generic is either an endpoint of \( \gamma \), or it is a point \( p \) where \( \gamma \) touches \( h \), with a sufficiently small open neighborhood of \( p \) on \( \gamma \) lying all strictly above \( h \) or all strictly below it; let us call such intersections top-touching or bottom-touching.

Let \( q_1, q_2, \ldots, q_k \) be the non-generic intersections of \( \gamma \) with \( h \). At least \( k - 1 \) of these are affinely independent, say \( q_1, \ldots, q_{k-1} \), and thus we can make an arbitrarily small movement of \( h \) so that a prescribed subset of \( \{ q_1, \ldots, q_{k-1} \} \) ends up below \( h \) and its complement above \( h \). The previously generic intersections remain generic, provided that the movement was sufficiently small.

Now if \( q_i \) was bottom-touching and it lies above \( h \) after the move, then it yields (at least) two generic intersections with \( h \), and similarly for top-touching. If \( q_i \) is an endpoint, then it yields at least one generic intersection, provided that \( h \) was moved in the right direction.
Hence by an appropriate move we can always get at least \(d + 1 - k + 2(k - 3) + 2 = d + k - 3\) generic intersections, which is at least \(d + 1\) for \(k \geq 4\). So it remains to discuss the cases \(1 \leq k \leq 3\).

For \(k \leq 2\), the non-generic intersections are distinct and thus affinely independent, and so we can get \(k\) new generic intersections by a suitable move. For \(k = 3\), there are two affinely independent non-generic intersections, at least one of them top-touching or bottom-touching, and hence we can also get 3 new generic intersections by a suitable move. Thus, we have obtained a hyperplane \(h\) with at least \(d + 1\) generic intersections as required.

Let \(t_1, \ldots, t_{d+1} \in I, t_1 < \cdots < t_{d+1}\), be the parameter values corresponding to these generic intersections with \(h\). To finish the proof of the lemma, we fix a sufficiently small \(\varepsilon > 0\) and intervals \(J_1^+, J_1^-, \ldots, J_{d+1}^+, J_{d+1}^- \subset I\), each of length at least \(\varepsilon\), such that \(J_i^+\) and \(J_i^-\) are in a small neighborhood of \(t_i\) (and thus they lie left of \(J_{i+1}^+ \cup J_{i+1}^-\)), and \(\gamma(J_i^+)\) lies above \(h\) and \(\gamma(J_i^-)\) below it.

Suppose that \(J_i^+\) precedes \(J_i^-\), for example. Then we choose points \(u_0, u_1, \ldots, u_{d+2} \in T(\varepsilon)\) with \(u_0 \in J_1^+, u_1 \in J_i^-, u_2 \in J_2^+, u_3 \in J_i^-, u_4 \in J_4^+, \) etc. Then the polygonal line \(\pi(\gamma, T(\varepsilon))\) changes sides of \(h\) at least \(d + 1\) times, and thus it has at least \(d + 1\) intersections with \(h\).

Since the position of \(h\) is generic, this shows that \(\pi(\gamma, T(\varepsilon))\) is not convex—a contradiction proving the lemma, and also concluding the proof of Theorem 1.1.

\[\square\]

7 The lower bound for order-type homogeneous subsequences

Super-order type homogeneity. The following strengthening of order-type homogeneity was considered in \[EMRST\]: a point sequence \(P = (p_1, p_2, \ldots, p_n)\) in \(\mathbb{R}^d\) is super-order type homogeneous if, for every \(k = 1, 2, \ldots, d\), the projection of \(P\) to the first \(k\) coordinates is order-type homogeneous (this includes the assumption that all of these projections are in general position—let us abbreviate this by saying that \(P\) is in super-general position).

It is easily seen, e.g., by Ramsey’s theorem, that for every \(d\) and \(n\) there is \(N\) such that every \(N\)-point sequence in super-general position in \(\mathbb{R}^d\) contains a super-order type homogeneous subsequence of length \(n\). Let us denote the corresponding Ramsey function by \(\text{OT}^*_d(n)\).

It was shown in \[EMRST\] that \(\text{OT}^*_d(n) \geq \text{twr}_d(n - d)\). Thus, to prove Theorem 1.3, the lower bound for \(\text{OT}^*_d(n)\), and having Theorem 1.2 at our disposal, it suffices to verify the following.

**Lemma 7.1.** For all \(d \geq 2\), \(\text{OT}^*_d(n) \geq \text{OT}^*_d(\Omega(n))\).

**Proof.** Given \(n\), let us set \(N = \text{OT}^*_d(n)\), and consider an \(N\)-point sequence in super-general position in \(\mathbb{R}^d\). By definition, it contains an \(n\)-point order-type homogeneous subsequence \(P_1\).

By Lemma 2.1 the polygonal path given by \(P_1\) is convex, i.e., \((\leq d)\)-crossing, and hence its projection onto the first \(d - 1\) coordinates is \((\leq d)\)-crossing as well. So by the assumption, it can be subdivided into at most \(M(d - 1)\) polygonal paths that are \((\leq d - 1)\)-crossing. One of them corresponds, by Lemma 2.1 again, to a subsequence \(P_2\) of \(P_1\) of length at least \(n/M(d - 1)\) whose projection to the first \(d - 1\) coordinates is order-type homogeneous.

Analogously we construct \(P_3, \ldots, P_d\), where \(|P_i| \geq |P_{i-1}|/M(d - i + 1)\) and the projections of \(P_i\) to the first \(k\) coordinates, for \(k = d - i + 1, d - i + 2, \ldots, d\), are order-type homogeneous. In particular, \(P_d\) is the desired super-order type homogeneous subsequence of length \(\Omega(n)\). \[\square\]
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