UNIFORM EVENTOWN PROBLEMS

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ABSTRACT. Let $n \geq k \geq l \geq 2$ be integers, and let $\mathscr{F}$ be a family of k-element subsets of an n-element set. Suppose that l divides the size of the intersection of any two (not necessarily distinct) members in $\mathscr{F}$. We prove that the size of $\mathscr{F}$ is at most $\binom{\lfloor n/l \rfloor}{k/l}$ provided n is sufficiently large for fixed k and l.

1. INTRODUCTION

Let $\mathscr{F} = \{F_1, \ldots, F_m\}$ be a family of subsets of $[n] := \{1, 2, \ldots, n\}$. Suppose that

$$|F_i \cap F_j| \text{ is even for all } i, j, \tag{1}$$

including the case $i = j$. Then the so-called Eventown Theorem claims that $m \leq 2^{\lfloor n/2 \rfloor}$, see Berlekamp[2] and Graver[12]. (See also Babai–Frankl[1] and Matoušek[14] for related problems including the oddtown theorem.) Let $A_1 \cup \cdots \cup A_{\lfloor n/2 \rfloor} \subset [n]$ be a disjoint union of 2-element sets (so $|A_i| = 2$ for all i), and consider a family

$$\Big\{ \bigcup_{i \in I} A_i : I \subset [\lfloor n/2 \rfloor] \Big\},$$

which we will call an "atomic construction." Then this family has size $2^{\lfloor n/2 \rfloor}$ and satisfies the property (1). For $n \geq l \geq 2$ let $m(n,l)$ denote the maximum size of a family $\mathscr{F} \subset 2^{[n]}$ such that $|F \cap F'| \equiv 0 \pmod{l}$ for all $F, F' \in \mathscr{F}$. Then the Eventown Theorem and the atomic construction show that

$$m(n,2) = 2^{\lfloor n/2 \rfloor}.$$

A similar atomic constructions using l-element subsets shows $m(n,l) \geq 2^{\lfloor n/l \rfloor}$, but this lower bound coincides with $m(n,l)$ only when $l = 2$. In fact, for $l \geq 3$ Frankl and Odlyzko[7] found a construction showing

$$m(n,l) \geq (8l)^{\lfloor n/(4l) \rfloor}$$

if an Hadamard matrix of order $4l$ exists, and $m(n,l) \geq 2^{8\lfloor n/(4l) \rfloor}$ in general.

In this paper we consider the corresponding problems in k-uniform families. So let $m_k(n,l)$ be the maximum size of a family $\mathscr{F} \subset \binom{[n]}{k}$ such that $|F \cap F'| \equiv k \pmod{l}$ for all $F, F' \in \mathscr{F}$. We will show that

$$m_k(n,l) = \binom{\lfloor (n-r)/l \rfloor}{(k-r)/l} \tag{2}$$

if $n > n_0(k,l)$ and $k \equiv r \pmod{l}$ where $0 \leq r < l$. Unlike the non-uniform case, the atomic constructions attain $m_k(n,l)$ for all l (if n is large enough).

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To state our main result more precisely, we need some definitions. Let $n \geq k > 0$ and let $L \subset [0,k-1] := \{0,1,\ldots,k-1\}$. We say that $\mathscr{F} \subset \binom{[n]}{k}$ is an (n,k,L)-system, or an L-system for short, if $|F \cap F'| \in L$ for all *distinct* $F, F' \in \mathscr{F}$. (Note that $k \notin L$. See [8] or [18] for more about L-systems in general.) Let $m(n,k,L)$ denote the maximum size of an (n,k,L)-system. Let $l\mathbb{N} = \{0,l,2l,\ldots\}$ denote the set of all nonnegative multiples of l, and for $i \in \mathbb{Z}$ let $L+i = \{l+i : l \in L\} \cap \mathbb{N}$ be a translation of L. Notice that negative integers are deleted in this translation, e.g., if $L = \{0,3,6,9\}$, then $L-4 = \{2,5\}$.

Theorem 1. *Let $n \geq k \geq l > r \geq 0$ be integers, and let $L = l\mathbb{N} \cap [0,k-1-r]$. If $n \geq n_0(k,l)$ and $l|k$, then*

$$m(n+r,k+r,L+r) = \binom{\lfloor n/l \rfloor}{k/l}. \tag{3}$$

Moreover an $(n+r,k+r,L+r)$-system with the maximum size is uniquely determined (up to isomorphism).

Letting $n' = n+r$, $k' = k+r$, $L' = L+r$ in (3) we have $m(n',k',L') = \binom{\lfloor (n'-r)/l \rfloor}{(k'-r)/l}$. Moreover (assuming $l|k$) we also have $m_{k'}(n',l) = m(n',k',L')$. So (3) coincides with (2).

Deza, Erdős, and Frankl[6] proved that if n is sufficiently large for fixed k and L, then

$$m(n,k,L) \leq \prod_{l \in L} \frac{n-l}{k-l}. \tag{4}$$

We remark that Theorem 1 for the case $r = 0$ and $l|n$ follows from the above result.

Recently Tasaki[16, 17] observed that the problem of classifying all maximal antipodal sets in the oriented real Grassmann manifold consisting of oriented real vector subspaces of dimension k in $\mathbb{R}^n$ can be reduced to the problem of classifying all maximal $(n,k,2)$-system, and, among other results, he showed $m_4(n,2) = \binom{\lfloor n/2 \rfloor}{2}$ for $n \geq 12$, and $m_5(n,2) = \binom{\lfloor (n-1)/2 \rfloor}{2}$ for $n \geq 87$. In this paper we also consider $m_6(n,2)$ using the linear programming bound, and we will show the following.

Theorem 2. *If $n \geq 26$ and $L = \{0,2,4\}$, then*

$$m_6(n,2) = m(n,6,L) \leq \frac{n(n-2)(n-4)}{6 \cdot 4 \cdot 2}. \tag{5}$$

We remark that Theorem 2 verifies (4) for this case. If $n = 2a$ is even, then (5) reads $m_6(2a,L) \leq \binom{a}{3}$, which is a special case of Theorem 1. The point is that the lower bound for n in Theorem 2 is much smaller than that of Theorem 1 or (4). We will comment on this at the end of the next section.

Finally we mention that some lower bound for n is necessary in Theorem 1.

Theorem 3. *For every $a \geq 2$ one has $m_{2a}(4a,2) > \binom{2a}{a}$.*

Proof. Let $\mathscr{H} = \{\emptyset, [8]\} \cup \mathscr{F}_0 \subset 2^n$ be the set of codewords in the $[8,4]$ binary Hamming code. Then $\mathscr{F}_0$ is an $(8,4,2\mathbb{N})$-system of size 14. Let $\mathscr{G} \subset 2^{[9,4a]}$ be the atomic construction with $|\mathscr{G}| = 2^{2a-4}$. Now we construct a $(4a,2a,2\mathbb{N})$-system

$$\mathscr{F} := \{G \cup H \in \binom{[4a]}{2a} : G \in \mathscr{G}, H \in \mathscr{G}\}.$$

Then it follows

$$|\mathscr{F}| = \binom{2a-4}{a} + 14\binom{2a-4}{a-2} + \binom{2a-4}{a-4}.$$

A simple computation shows that

$$|\mathscr{F}|/\binom{2a}{a} = \frac{4a^2-6a+3}{4a^2-8a+3} > 1$$

for all $a \geq 2$. □

It is also readily seen that $m_{2a+1}(4a+1,2) > \binom{2a}{a}$ for all $a \geq 2$.

The authors do not know any general lower bound for $m(n,k,L)$, but there is a conjecture due to Füredi[10] which would give a strong lower bound for $m(n,k,L)$ (if true) in terms of the so-called *rank* of (n,k,L)-systems. See [8, 15, 18] for more details on this subject.

2. Proof of Theorem 1

For the proof we will need the following lemma.

Lemma 4 (Deza–Erdős–Frankl[6]). *Let $\mathscr{F}$ be an (n,k,L)-system with $b = |L|$ and $r = \min L$. If $n > n_1(k,L)$ and $|\mathscr{F}| \geq \Omega(n^{b-1})$, then there exists an r-element set A such that $A \subset F$ holds for all $F \in \mathscr{F}$.*

Notation. *For $\mathscr{F} \subset 2^{[n]}$ and $A \subset [n]$ let $\mathscr{F}(A) := \{F \setminus A : A \subset F \in \mathscr{F}\}$. For $x \in [n]$ we write $\mathscr{F}(x)$ for $\mathscr{F}(\{x\})$.*

Proof of Theorem 1. Let $n = la+q$, $k = lb$ $(0 \leq q < l)$. First we verify the lower bound for the case $r = 0$ by constructing an (n,k,L)-system $\mathscr{F}_{n,k}$ with size $\binom{a}{b} = \binom{\lfloor n/l \rfloor}{k/l}$. For this, let $[n] = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_a$ be a partition, where $A_i = [(i-1)l+1, il]$, and define

$$\mathscr{F}_{n,k} := \Big\{ \bigsqcup_{i \in I} A_i : I \subset \tbinom{[a]}{b} \Big\}.$$

This example shows that $m(n,k,L) \geq |\mathscr{F}_{n,k}| = \binom{\lfloor n/l \rfloor}{k/l} = \Theta(n^b)$. For the cases $r > 0$ let

$$\mathscr{F}_{n+r,k+r} := \{F \cup [n+1, n+r] : F \in \mathscr{F}_{n,k}\},$$

which is an $(n+r, k+r, L+r)$-system with $|\mathscr{F}_{n+r,k+r}| = |\mathscr{F}_{n,k}| = \Theta(n^b)$.

Now suppose that $\mathscr{F}$ is an $(n+r,k+r,L+r)$-system of size $|\mathscr{F}| = m(n+r,k+r,L+r)$, where $|L+r| = b$. By (4) it follows $|\mathscr{F}| = O(n^b)$. Since $|\mathscr{F}| \geq |\mathscr{F}_{n+r,k+r}|$ we have $|\mathscr{F}| = \Theta(n^b)$. Then, by Lemma 4, there is an r-element set A which is contained in all $F \in \mathscr{F}$. In this case $\mathscr{F}(A)$ is an (n,k,L)-system. Consequently we have

$$m(n,k,L) \geq |\mathscr{F}(A)| = |\mathscr{F}| = m(n+r,k+r,L+r) \geq |\mathscr{F}_{n+r,k+r}| = |\mathscr{F}_{n,k}|.$$

Thus to conclude $m(n,k,L) = m(n+r,k+r,L+r) = |\mathscr{F}_{n,k}|$ it suffices to show $|\mathscr{F}_{n,k}| \geq m(n,k,L)$. Namely, the cases $r > 0$ are reduced to the case $r = 0$.

So let $\mathscr{F}$ be one of the largest (n,k,L)-system (thus $|\mathscr{F}| \geq |\mathscr{F}_{n,k}|$), and we are going to show $|\mathscr{F}| \leq |\mathscr{F}_{n,k}|$. We say that $A \in \bigcup_{i \geq l} \binom{[n]}{i}$ is an *atom* of $\mathscr{F}$ if

$$\text{either } A \subset F \text{ or } A \cap F = \emptyset \text{ for any } F \in \mathscr{F}, \tag{6}$$

and moreover A is inclusion maximal with this property (6). Notice that atoms are pairwise disjoint. In fact if both A and A' satisfy (6) with $A\cap A'\neq\emptyset$, then $A\cup A'$ also satisfies (6). Let $\mathscr{A}\subset\binom{[n]}{l}$ be the set of atoms of size l, and let $X_1\subset[n]$ be the union of all atoms in $\mathscr{A}$. Then we have a partition $X_1=A_1\sqcup A_2\sqcup\cdots\sqcup A_t$, where $A_i\in\mathscr{A}$ and $t=|\mathscr{A}|$. Let $X_0=[n]\setminus X_1$, $\mathscr{F}_{X_1}=\{F\in\mathscr{F}:F\subset X_1\}$, and $\mathscr{F}_{X_0}=\mathscr{F}\setminus\mathscr{F}_{X_1}$. Then $|X_1|=lt$ and $|X_0|=n-lt$. By definition if $F\in\mathscr{F}_{X_1}$ then it can be uniquely partitioned into b atoms as $F=A_{i_1}\sqcup\cdots\sqcup A_{i_b}$. Recall that $n=la+q$ and $q<l$. So $t\le a$ and $|\mathscr{F}_{X_1}|\le\binom{t}{b}\le\binom{a}{b}=|\mathscr{F}_{n,k}|$. If $X_0=\emptyset$ then $|\mathscr{F}|=|\mathscr{F}_{X_1}|$, and we are done. Now assume that $X_0\neq\emptyset$. For each $x\in X_0$ we will examine the size of $\mathscr{F}(x)$.

Claim 5. *It follows* $|\mathscr{F}(x)|=O(n^{b-2})$ *for* $x\in X_0$.

Proof. Let $x\in X_0$. First suppose that there is an atom $A\subset X_0$ with $x\in A$. Then $c:=|A|>l$ (otherwise $A\in\mathscr{A}$ and $A\subset X_1$) and $|\mathscr{F}(x)|=|\mathscr{F}(A)|$. Since $\mathscr{F}(A)$ is an $(n-c,k-c,L-c)$-system with $|L-c|\le b-2$ we have $|\mathscr{F}(A)|=O(n^{b-2})$ by (4). Next suppose that there is no atoms containing x. Recall that $\mathscr{F}(x)$ is an $(n-1,k-1,L')$-system where $L'=L-1$ has size $b-1$ and $\min L'=l-1$. Let $\bigcap\mathscr{F}(x)=\bigcap_{G\in\mathscr{F}(x)}G$. If $|\bigcap\mathscr{F}(x)|<l-1$, then we have $|\mathscr{F}(x)|=o(n^{b-2})$ by Lemma 4. So suppose $|\bigcap\mathscr{F}(x)|\ge l-1$. Choose $Y\subset\bigcap\mathscr{F}(x)$ with $|Y|=l-1$. Then, for all $F\in\mathscr{F}$ with $x\in F$, it follows $Y\subset F$. But $\{x\}\cup Y$ is not contained in atoms, and there is $F_1\in\mathscr{F}$ such that $x\notin F_1$ and $F_1\cap Y\neq\emptyset$. In this case if $x\in F\in\mathscr{F}$ then $|F\cap F_1|\ge l$, and if $G\in\mathscr{F}(x)$ then $G\cap(F_1\setminus Y)\neq\emptyset$. For $z\in F_1\setminus Y$ let $W_z=\{x\}\cup Y\cup\{z\}$. Then $\mathscr{F}(W_z)$ is an $(n-l-1,k-l-1,L')$-system where $L'=L-l-1$ has size $b-2$, and (4) yields $|\mathscr{F}(W_z)|=O(n^{b-2})$. Thus we get $|\mathscr{F}(x)|\le\sum_{z\in F_1\setminus Y}|\mathscr{F}(W_z)|=|F_1\setminus Y|O(n^{b-2})=O(n^{b-2})$, which completes the proof of Claim 5. □

Let $m:=|X_0|=n-lt\ge q$. It follows from Claim 5 that

$$|\mathscr{F}_{X_0}|\le\sum_{x\in X_0}|\mathscr{F}(x)|=|X_0|O(n^{b-2})=O(mn^{b-2}).$$

On the other hand it follows $|\mathscr{F}_{X_1}|\le\binom{t}{b}$, where $t=|\mathscr{A}|=(n-m)/l$. Therefore we have

$$\begin{aligned}\binom{(n-q)/l}{b}=\binom{a}{b}&=|\mathscr{F}_{n,k}|\le|\mathscr{F}|=|\mathscr{F}_{X_1}|+|\mathscr{F}_{X_0}|\\&\le\binom{(n-m)/l}{b}+O(mn^{b-2}).\end{aligned}\tag{7}$$

We compare the coefficients of n^b and n^{b-1} on both sides of (7). Recall that $q\le m\le n$. If $m\ge\Omega(n)$, then the coefficient of n^b in the RHS is clearly smaller than that in the LHS, and (7) fails. If $m=o(n)$, then the $O(mn^{b-2})$ term in the RHS does not affect n^b and n^{b-1} terms. In this case, by comparing the n^{b-1} term on both sides, we see from (7) that $m=q$, namely, $t=a$. Thus we have $|X_0|=q<l$ and $|X_1|=la$. Since $|F\cap X_1|$ is a multiple of l for every $F\in\mathscr{F}$ it follows that $|F\cap X_0|$ is also a multiple of l, namely, $|F\cap X_0|=0$. Consequently we have $\mathscr{F}=\mathscr{F}_{X_1}$, then $|\mathscr{F}|=|\mathscr{F}_{n,k}|$ follows from (7). This completes the proof for the inequality $|\mathscr{F}|\le|\mathscr{F}_{n,k}|$. Moreover, our proof also shows that if equality holds, then $\mathscr{F}$ is isomorphic to $\mathscr{F}_{n,k}$ if $r=0$, and thus isomorphic to $\mathscr{F}_{n,k,r}$ if $r\ge 0$. This completes the proof of Theorem 1. □

We did not attempt to reduce the lower bound $n_0(k,l)$ for n in the above proof. We used (4) and Lemma 4, which already require $n \geq n_1(k,L) \geq 2^k k^3$. But we believe the true lower bound $n_0(k,l)$ is much smaller, perhaps polynomial in k.

3. Proof of Theorem 2

Let $k = 2t$ be even, and let $L = \{0,2,\dots,k-2\}$. First we recall a general method (see Delsarte[5], Wilson[19]) to obtain an upper bound for $m(n,k,L)$. Let $G = (V,E)$ be a Kneser graph corresponding to (n,k,L)-system, namely, let $V = \binom{[n]}{k}$ and $xy \in E$ iff $|x \cap y| \notin L$. Let us define a pseudo-adjacency matrix M for G. So let $M = (m_{xy})$ be an $\binom{n}{k} \times \binom{n}{k}$ matrix indexed by V whose (x,y)-entry is defined by

$$m_{xy} := \begin{cases} a_j & \text{if } |x \cap y| = j \in \{1,3,\dots,k-1\}, \\ 0 & \text{if } |x \cap y| \in L \cup \{k\} = \{0,2,\dots,k\}, \end{cases} \tag{8}$$

where $a_1, a_3, \dots, a_{k-1} = a_{2t-1}$ are t variables. If we substitute some real values into these t variables, then M becomes a real symmetric matrix. Let $\lambda_{\max}$ (resp. $\lambda_{\min}$) be the largest (resp. least) eigenvalue of the resulting matrix. Suppose moreover that the all-ones vector is contained in the $\lambda_{\max}$-eigenspace of M. Then the independence number $\alpha(G)$ of G is bounded in terms of $\lambda_{\max}$ and $\lambda_{\min}$:

$$\alpha(G) \leq \frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}} \binom{n}{k}. \tag{9}$$

This is a generalization of Hoffman's ratio bound due (among others) to Delsarte[4] (see also [13], §3.5 of [3], §9.6 of [11]). On the other hand, every independent set in G is an (n,k,L)-system, and it follows

$$m(n,k,L) \leq \alpha(G).$$

Thus to get a better upper bound for $m(n,k,L)$ we need to find specific values for a_i's so that the ratio bound, the RHS of (9), is minimized. In particular, if

$$\frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}} \binom{n}{k} = \frac{n!!}{\big((n-k)!!\big)(k!!)} \tag{10}$$

holds, then we would get $m(2a, 2t, L) = \binom{a}{t}$.

To determine the values of a_i's we do some "reverse engineering," c.f., Friedgut[9]. For $0 \leq i \leq k$ let B_i be an $\binom{n}{k} \times \binom{n}{k}$ matrix whose (x,y)-entry is given by $\binom{|y \setminus x|}{i}$. It is known that $B_0, \dots, B_k$ form a basis of the Bose–Mesner algebra of Johnson scheme, in particular, these matrices are simultaneously diagonalizable, and the eigenvalues of B_f are given by

$$\mu_f(e) = (-1)^e \binom{k-e}{f-e} \binom{n-f-e}{k-e} \text{ for } e = 0,1,\dots,k,$$

see e.g., [19] for the proof of this fact.

We will choose b_i for $i = 0,\dots,k$ so that

$$M = \sum_{i=0}^{k} b_i B_i.$$

To this end, by (8), we consider $k+1$ equations for $j=0,\dots,k$:

$$\sum_{i=0}^{k-j} b_i \binom{k-j}{i} = \begin{cases} a_j & \text{if } j \in \{1,3,\dots,k-1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

We have $t+(k+1)$ variables: $a_1, a_3, \dots, a_{2t-1}$, and $b_0, b_1, \dots, b_k$. So we need t more equations. For this we normalize M so that

$$a_1 = 1. \tag{12}$$

Also we require that there is only one negative value in the eigenvalues of M:

$$\lambda_2 = \lambda_4 = \cdots = \lambda_{2t}, \tag{13}$$

where

$$\lambda_e = \sum_{i=0}^{k} b_i \mu_i(e). \tag{14}$$

Consequently we have $(k+1)+t$ equations with the same number of unknowns. By solving this system of equations (11), (12), and (13), we determine all values of unknowns and eigenvalues. Then the question is whether these values satisfy (10) or not. We will shortly see that they do satisfy (10) if $k=6$ and $n \geq 26$. Some numerical experiments suggest that most likely this is the case for all even k if n is relatively large (but the lower bound for n seems much smaller than $n_0(k,l)$ in Theorem 1).

Proof of Theorem 2. Let $n \geq 26$, $k=6$, and $L=\{0,2,4\}$. We follow the method explained above. By solving the system of equations, we get all $b_0, b_1, \dots, b_6$, and

$$a_1 = 1, \quad a_3 = \frac{(n-12)(n-22)}{8(n-18)}, \quad a_5 = \frac{(n-10)(n-16)}{8}.$$

Then the eigenvalues of M are given by (14):

$$\begin{aligned}
\lambda_0 &= \frac{(n-12)(n-6)\left(n^2-3n+5\right)\left(3n^2-86n+536\right)}{60(n-18)}, \\
\lambda_1 &= \frac{n^6-63n^5+1465n^4-16110n^3+89944n^2-262752n+385920}{120(n-18)}, \\
\lambda_2 = \lambda_4 = \lambda_6 &= -\frac{(n-12)\left(3n^2-86n+536\right)}{4(n-18)}, \\
\lambda_3 &= \frac{n^5-49n^4+872n^3-6380n^2+12720n+29952}{48(n-18)}, \\
\lambda_5 &= \frac{n^4-50n^3+912n^2-7376n+22464}{8(n-18)}.
\end{aligned}$$

Finally it follows from a direct computation that if $n=6,7,8$, or $n \geq 26$, then $\lambda_{\max} = \lambda_0$ and $\lambda_{\min} = \lambda_2$. This gives us (10), which completes the proof. $\square$

In the same way one can verify (10) in the following cases:

$$\begin{aligned}
&k=4 \text{ and } n\geq 12,\\
&k=6 \text{ and } n=6,7,8, \text{ or } n\geq 26,\\
&k=8 \text{ and } n=12,31,32,33, \text{ or } n\geq 47,\\
&k=10 \text{ and } n=10,11,12, \text{ or } 57\leq n\leq 66, \text{ or } n\geq 78.
\end{aligned}$$

Namely, in the above cases we have

$$m(n,k,2\mathbb{N}\cap[k-1])\leq \frac{n!!}{\big((n-k)!!\big)(k!!)}.$$

In particular,

$$m(2a,k,\{0,2,4,\ldots\})=\binom{a}{k/2}$$

follows if $k=8$ and $a\geq 24$, or $k=10$ and $a\geq 39$.

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