Proving the Pressing Game Conjecture on Linear Graphs *

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March 28, 2013

Abstract

The pressing game on black-and-white graphs is the following: Given a graph $G(V, E)$ with its vertices colored with black and white, any black vertex v can be pressed, which has the following effect: (a) all neighbors of v change color, i.e. white neighbors become black and *vice versa*, (b) all pairs of neighbors of v change connectivity, i.e. connected pairs become unconnected, unconnected ones become connected, (c) and finally, v becomes a separated white vertex. The aim of the game is to transform G into an all white, empty graph. It is a known result that the all white empty graph is reachable in the pressing game if each component of G contains at least one black vertex, and for a fixed graph, any successful transformation has the same number of pressed vertices.

The pressing game conjecture is that any successful pressing path can be transformed into any other successful pressing path with small alterations. Here we prove the conjecture for linear graphs. The connection to genome rearrangement and sorting signed permutations with reversals is also discussed.

Keywords: AMS MSC: 05A05 – Permutations, words, matrices, 05Cxx – Graph theory, free keywords: bioinformatics, sorting by reversals, pressing game

*This paper presents the results of the undergraduate research of E. Bixby and T. Flint in the 2012 Fall semester at the Budapest Semesters in Mathematics

1 Introduction

Sorting signed permutations by reversals (or inversions as biologists call it) is the first genome rearrangement model introduced in the scientific literature. The hypothesis that reversals change the order and orientation of genes – so-called genetic factors in that time – arose in a paper published in 1921 [6] and implicitly was verified by microscopic inferring of chromosomes a couple of decades later [7]. In the same time, geneticists realized that "The mathematical properties of series of letters subjected to the operation of successive inversions do not appear to be worked out" [8]. This computational problem has been rediscovered at the end of the XX. century, and the solution to it, now called as the Hannenhalli-Pevzner theorem has been published in 1995 and 1999 [2].

The Hannenhalli-Pevzner theorem gives a polynomial running time algorithm to find one scenario with the minimum number of reversals necessary to sort a signed permutation. However, there might be multiple solutions, and the number of solutions typically grows exponentially with the length of the permutation. Therefore, a(n almost) uniform sampler is required which gives a set of solutions from which statistical properties of the solutions can be calculated. A typical approach for sampling is the Markov chain Monte Carlo method. It starts with an arbitrary solution, and applies random perturbations on it thus exploring the solution space. In case of most parsimonious reversal sorting scenarios, two approaches are considered as perturbing the current solution:

- The first approach encodes the most parsimonious reversal sorting scenarios with the intermediate permutations visited. Then it cuts out a random window from this path, and gives a random new sorting scenario between the permutations at the beginning and end of the window
- The second approach encodes the scenarios with the series of mutations applied, and perturbs them in a sophisticated way, described in details later in this paper.

A Markov chain for sampling purposes should fulfill two conditions: **(a)** it must converge to the uniform distributions of all possibilities, and a necessary condition for it that it must be irreducible, namely, from any solution the chain must be able to get to any another solution and **(b)** the convergence must be fast.

The problem with the first approach mentioned above is that it is provenly slowly mixing [5]. This means that the necessary number of steps in the Markov chain to get sufficiently close to the uniform distribution grows exponentially with the length of the size of the permutation. Therefore this approach is not applicable in practice.

The problem with the second approach is that we even do not know if it is irreducible, nor that it is rapidly mixing. In this paper, we want to take a step towards proving that it is an irreducible Markov chain.

The paper is organized in the following way. In Section 2, we define the problem of sorting by reversals, and the combinatorics tools necessary: the graph of desire and reality and the overlap graph. Then we introduce the pressing game on the black-and-white graphs, and show that they correspond to the shortest reversal scenarios in case of a biologically important subset of permutations. We finish the section for stating the pressing game conjecture. If this was proven then this would give a proof for the irreducibility of the Markov chain applying the above mentioned second approach. In Section 3, we prove the conjecture for linear graphs. The paper is finished with a discussion and conclusions.

2 Preliminaries

Definition 1. *A* signed permutation *is a permutation of numbers from* 1 *to* n*, where each number has a* $+$*or* $-$ *sign.*

While the number of n long permutations is $n!$, the number of n long signed permutations is $2^n \times n!$.

Definition 2. *A* reversal *takes any consecutive part of a signed permutation and change both the order of the numbers and the sign of each number. It is also allowed that a reversal takes only one single number from the signed permutations, in that case, it changes the sign of this number.*

For example, the following reversal flips the $-3\ +6\ -5\ +4\ +7$ segment:

$$+8\ -1\ -3\ +6\ -5\ +4\ +7\ -9\ +2 \Rightarrow +8\ -1\ -7\ -4\ +5\ -6\ +3 -9\ +2$$

The sorting by reversals problem asks for the minimum number of reversals necessary to transform a signed permutation into the identity permutation, ie. the signed permutation $+1\ +2\ \dots\ +n$. This number is called the reversal distance, and the reversal distance of a signed permutation π is denoted by $d_{REV}(\pi)$. To solve this problem, we have to introduce two discrete mathematical objects, the graph of desire and reality and the overlap graph. The graph of desire and reality is a drawn graph, ie. not only the topology (which vertices are connected) but also its drawing matter. The overlap graph is a graph in terms of standard graph theory.

The graph of desire and reality of a signed permutation can be constructed in the following way. Each signed number is replaced with two unsigned number, $+i$ becomes $2i-1, 2i$, $-i$ becomes $2i, 2i-1$. The so-obtained $2n$ long permutation is framed between 0 and $2n+1$. Each number including 0 and $2n+1$ will represent one vertex in the graph of desire and reality. They are drawn in the same order as they appear in the permutation, so the graph of desire and reality is not only a graph, its drawing is also important.

We index the positions of the vertices starting with 1, and each pair of vertices in positions $2i-1$ and $2i$ are connected with an edge. We call these edges the reality edges. Each pair of vertices for numbers $2i$ and $2i+1$, $i = 0, 1, \dots n$ are connected with an arc, and they are named the desire edges. The explanation for these names is that the reality edges describes what we see in the current

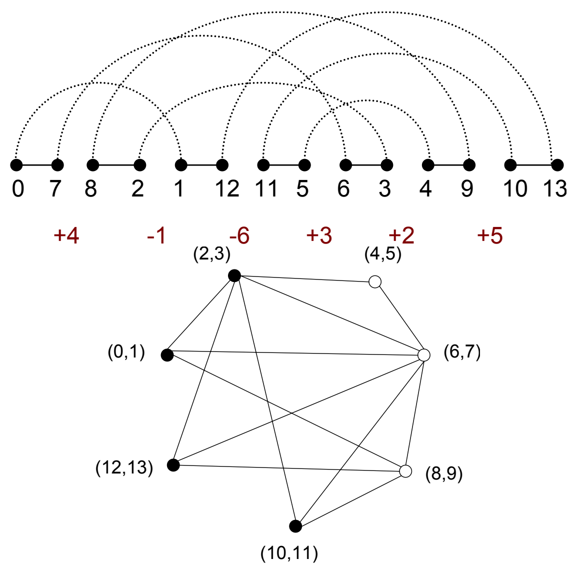


Figure 1: The graph of desire and reality and the overlap graph of the signed permutation $+4\ -1\ -6\ +3\ +2\ +5$

permutation, and the desire edges tell what neighbors we would like to see to get the $+1, +2, \ldots + n$ permutation: we would like that 1 be next to 0, 3 be next to 2, etc.

The overlap graph is constructed from the graph of desire and reality in the following way. The vertices of the overlap graph are the desire edges in the graph of desire and reality. The vertices are colored, a vertex in the overlap graph is black if the number of vertices below the desire edge it represents in the graph of desire and reality is odd. A vertex is white if the number of vertices is even. Two vertices are connected if the intervals that the corresponding desire edges span overlap but neither contain the other. On Figure 1, we give an example for the graph of desire and reality and overlap graph.

The overlap graph might fall into components. A vertex, as well as its corresponding desire edge is called oriented if the vertex is black, namely, the corresponding desire edge spans odd number of vertices. A vertex and its corresponding desire edge is unoriented if the vertex is white. A component is called oriented if it contains at least one black vertex, if the component contains only white vertices, it is called unoriented. A component is non-trivial if it contains more than one vertex. Some of the non-trivial unoriented components are hurdles. We skip the precise definition of hurdles here as we do not need it. A permutation is called fortress, if the number of its hurdles is odd, with some prescribed properties, also not detailed here.

Any reversal changes the topology of the graph of desire and reality on two reality vertices. Any desire edge is a neighbor of two reality edges, and we say that the reversal acts on this desire edge if it changes the topology on the two

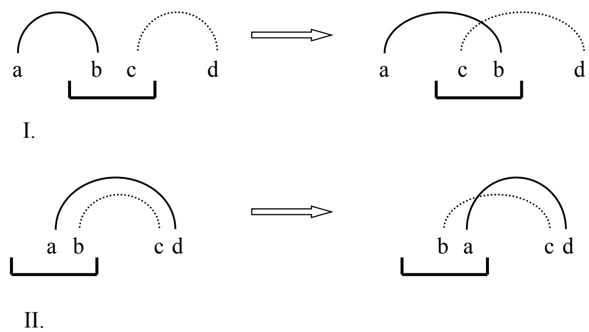


Figure 2: This picture show how a reversal can change the overlap of two desire edges. The reverted fragment is indicated with a thick black line.

neighbor reality edges.

How do such reversals change the graph of desire and reality and thus the overlap graph? We set up a Lemma below explaining this.

Lemma 1. *Let v be an orientd desire edge on which the reversal acts. Then the reversal*

- *change the orientation of any desire edge crossing v*
- *change the overlap of any pair of desire edges crossing v*
- *the desire edge itself become an unoriented edge without any overlap with any other edges.*

Proof. The reversal flips one of the 'legs' of each overlapping desire edge, namely, the reality edge connected to the desire edge. Therefore it changes the parity of the vertices below the desire edge and thus the orientation of it.

Two edges which are both overlap with v but not with each other, can overlap only from the two ends of v, see also Fig. 2, case **I**. A reversal acting on v will flip one-one of their endpoints, so they will indeed overlap. If two edges overlap with v, but by definition not with each other since the interval of one of them contains the interval of the other, then they come from one end of v. It is easy to see that after the reversal they will overlap by definition, see Fig. 2, case **II**. It is also easy to see that overlapping pair of edges which are also overlap with each other are the cases on the right hand side of Fig. 2, so after the reversal, they will not overlap.

Finally, the oriented edge on which the reversal acts becomes an unoriented edge forming a small cycle with a reality edge, and thus it cannot overlap with any other desire edge.

□

This lemma also shows the connection between sorting by reversals and the pressing game on black and white graphs: a pressing of a black vertex is equivalent with a reversal acting on the corresponding desire edge. Below we define the pressing game on black-and-white graphs:

Definition 3. *Given a graph $G(V, E)$ with its vertices colored with black and white. Any black vertex v can be pressed, which has the following effect: (a) all neighbors of v change color, white neighbors become black and* vice versa*, (b) all pair of neighbors of v change connectivity, connected pairs become unconnected, unconnected ones become connected, (c) and finally, v becomes a separated white vertex. The aim of the game is to transform G into an all white, empty graph.*

If each component of G contains at least one black vertex, then the pressing game always has at least one solution, as it turns out from the Hannenhalli-Pevzner theorem.

Theorem 2. (Hannenhalli-Pevzner), [2]

$$d_{REV}(\pi) = n + 1 - c(\pi) + h(\pi) + f(\pi)$$

where n is the length of the permutation π, $c(\pi)$ is the number of cycles in the graph of desire and reality, $h(\pi)$ is the number of hurdles in the permutation and $f(\pi)$ is the fortress indicator, it is 1 if the permutation is a fortress, otherwise 0.

It is easy to see that any reversal can increase the number of cycles in the graph of desire and reality at most by 1, hence the Hannenhalli-Pevzner theorem also says if a permutation does not contain any hurdle (and thus it is not a fortress) then any optimal reversal sorting path increases the number of cycles to $n + 1$ without creating any hurdle. Below we state this theorem.

Theorem 3. *Let π be a permutation which is not the identical permutation and whose overlap graph does not contain any non-trivial unoriented component. Then a reversal exists that acts on an oriented desire edge, thus increases $c(\pi)$ by 1 and does not create any non-trivial unoriented component.*

Furthermore, if G is an arbitrary black-and-white graph such that each component contains at least one black vertex, then at least one black vertex can be pressed without making a non-trivial unoriented component.

The proof can be found in [1], and we skip it here. The proof consider only the overlap graph, and in fact, it indeed works for every black-and-white graph. A clear consequence is the following theorem.

Theorem 4. *Let G be a black-and-white graph such that each component on it contains at least one black vertex. Then G can be transformed into the all-white empty graph in the pressing game.*

Proof. It is sufficient to use iteratively Theorem 3. Indeed, according to Theorem 3, we can find a black vertex v, such that pressing it does not create a non-trivial all-white component, on the other hand, v become a separated white vertex, and it will remain a separated white vertex afterward. Hence, the number of vertices in non-trivial components decreases at least by one, and in a finite number of steps, G is transformed into the all-white, empty graph. □

Consider the set of vertices as an alphabet, any sequence over this alphabet is called a pressing path. It is a valid pressing path when each vertex is black when it is pressed, and it is successful, if it is valid and leads to the all-white, empty graph. The length of the pressing path is the number of vertices pressed in it. The following theorem is also true.

Theorem 5. *Let G be a black-and-white graph such that each component on it contains at least one black vertex. Then each successful pressing path of G has the same length.*

The proof can be found in [3]. We are ready to state the pressing path conjecture.

Conjecture 6. *Let G be a black-and-white graph such that each component on it contains at least one black vertex. Construct a metagraph, M whose vertices are the successful pressing paths on G. Connect two vertices if the length of the longest common subsequence of the pressing paths they represent is at most* 4 *less than the common length of the pressing paths. The conjecture is that M is connected.*

The conjecture means that with small alterations, we can transform any pressing path into any other pressing path, whatever G is. The small alteration means that we remove at most 4, not necessary consecutive vertices from a pressing path, and add at most 4 vertices, not necessarily to the same places where the old vertices were removed from, and not necessarily to consecutive places. Although it is generally not true that only the pressing paths are the reversal sorting paths of a signed permutation, as there might be cycle-increasing reversals not acting on a desire edge, for a class of permutations, it is true. Specially, if a signed permutation is such that in its graph of desire and reality each cycle contains only one or two desire edges, then all cycle-increasing reversals act on desire edges. These signed permutations are the permutations that can be considered in the so-called infinite site model [4].

In this paper, we prove the pressing game conjecture for linear graphs. Actually, we can prove more, the metagraph will be already connected if we require that neighbor vertices have longest common subsequence at most 2 less than the common length of their pressing paths.

3 Proof of the Conjecture on Linear Graphs

The proof of our main theorem is recursive, and for this, we need the following notations. Let G be a black-and-white graph, and v a black vertex in it. Then Gv denotes the graph we get by pressing vertex v. Similarly, if P is a valid pressing path of G (namely, each vertex is black when we want to press it, but P does not necessary yield the all-white, empty graph), then GP denotes the graph we get after pressing all vertices in P in the indicated order. Finally, let P^k denote the suffix of P starting in position $k + 1$.

The simplicity of the linear graphs is that they have a simple structure and furthermore, the pressing game on linear graphs is self-reducible as the following observation states.

Observation 1. *Let G be a linear black-and-white graph and v a black vertex in it. Then Gv is also a linear graph and the separated white vertex v.*

Since any separated white vertex does not have to be pressed again, it is sufficient to consider $Gv \setminus \{v\}$, which is a linear graph. We are ready to state and prove our main theorem.

Theorem 7. *Let G be an arbitrary, finite, linear black-and-white graph, and let M be the following graph. The vertices of M are the successful pressing paths on G, and two vertices are connected if the length of the longest common subsequence of the pressing paths they represent is at most 2 less than the common length of the pressing paths. Then M is connected.*

Proof. It is sufficient to show that for any successful pressing paths X and $Y = v_1 v_2 \ldots v_k$ there is a series $X_1, X_2, \ldots X_m$ such that for any $i = 1, 2, \ldots m-1$, the length of the longest common subsequence of X_i and X_{i+1} is at most 2 less than the common length of the paths, and X_m starts with v_1. Indeed, then both X_m and Y starts with v_1, and both X_m^1 and Y^1 are successful pressing paths on $Gv_1 \setminus \{v_1\}$. We can use the induction to transform X_m into a pressing path which starts v_2, then we consider its suffix which is a successful pressing path on $Gv_1v_2 \setminus \{v_1, v_2\}$, etc.

Furthermore, to show that v_1 can be moved to the first position to the current pressing path, it is sufficient to show that it can be moved towards the first position with some series of allowed alterations of the path.

The first question is if v_1 is in X. X is a successful pressing path of G and v_1 is a black vertex in G (since it is the first vertex in the valid pressing path Y). Then either v_1 is pressed or it become a separated white vertex by pressing a neighbor of v_1. Since G is a linear graph, the only possibility for the later case is that the remaining linear part of G contains two vertices: v_1 and some u, both of them are black and connected, and u is pressed in the pressing path. But then pressing v_1 instead of u has the same effect. Replacing u to v_1 in the pressing path means that the length of the longest common subsequence is one less than the common length of the paths.

Case 1. So from now we assume that v_1 is part of the current pressing path, which we denote by $P_1 w_1 v_1 P_2$, both P_1 and P_2 might be empty. If w_1 and v_1 are not neighbors in GP_1, then $P_1 v_1 w_1 P_2$ is also a valid pressing path, and one of the longest common subsequences of $P_1 w_1 v_1 P_2$ and $P_1 v_1 w_1 P_2$ is $P_1 w_1 P_2$, one vertex less then the original pressing paths. In this way, we can move v_1 to a smaller index position in the pressing path, and this is what we want to prove.

Case 2. If w_1 and v_1 are neighbors, then v_1 is white in GP_1, and then w_1 makes it black again. However, v_1 is black in G, since it is the first vertex in the valid pressing path Y. Then there have to be at least one vertex in P_1 that made v_1 white. Let w_2 be the last such vertex in P_1, and let we denote

$P_1 = P_{1a}w_2P_{1b}$. We claim that none of the vertices in P_{1b} are neighbors of w_2 in GP_{1a}. Indeed, if there were a neighbor of w_2 in P_{1b}, denote it by w_3, then w_3 would become a neighbor of v_1 after pressing w_2, and then pressing w_3 would make v_1 black, and then either v_1 was black before pressing w_1, a contradiction, or there were further vertices in P_{1b} making v white, contradicting that w_2 is the last such vertex. Since P_{1b} does not contain a vertex which is a neighbor of w_2 in GP_{1a}, we can recursively bubble down w_2 next to w_1. We get that the pressing path is now $P_1w_2w_1v_1P_2$, where P_1 is now a different pressing path, and possibly empty, and P_2 might also be empty. The topology and the colors of w_2, w_1 and v_1 in GP_1 is one of the following:

Case 2a. Assume that P_2 is not empty, then the $\{w_1, w_2, v_1\}$ triplet has at least one neighbor, call it u, and u either is pressed in P_2, or we can replace a vertex in P_2 with u such that it is still a successful pressing path on $GP_1w_2w_1v_1$. So we can assume that at least one neighbor of the $\{w_1, w_2, v_1\}$ triplet is pressed in P_2. It is easy to see that the neighbors of the $\{w_1, w_2, v_1\}$ triplet changes their color in the same way by pressing only v_1 and pressing $w_2w_1v_1$, see Figure 3. Therefore we can press v_1 instead of $w_2w_1v_1$, and the pressing path P_2 will be still valid up to the point when u_1 or u_2 is pressed. Assume that u_1 is pressed before u_2 in P_2, and $P_2 = P_{2a}u_1P_{2b}$ Figure 4 shows that the color of u_2 and a possible second neighbor of u_1 denoted by u_3 will be the same in $GP_1w_2w_1v_1P_{2a}u_1$ and $GP_1v_1P_{2a}u_1w_1w_2$. Therefore $P_1v_1P_{2a}u_1w_1w_2P_{2b}$ will be also a successful pressing path on G, since no more vertices are affected by the given alteration of the pressing path. One of the longest common subsequences of $P_1w_2w_1v_1P_{2a}u_1P_{2b}$ and $P_1v_1P_{2a}u_1w_1w_2P_{2b}$ is $P_1v_1P_{2a}u_1P_{2b}$, 2 vertices less than the entire pressing paths. v_1 is in a smaller index position of the pressing path, and this is what we wanted to prove. The case when u_2 is pressed first in P_2 is similar to the discussed case.

Case 2b Finally, assume that P_2 is empty. This means that the w_1, w_2, v_1 triplet might have at most one more vertex that becomes a separated white vertex when v_1 is pressed. This additional vertex is white if a neighbor of w_1 or w_2 and black if it is a neighbor of v_1 (it can be only when w_2 is a neighbor of w_1.

Then P_1 cannot be empty, otherwise $w_2w_1v_1$ would be the only successful pressing path, contradicting that a successful pressing path exists that starts with v_1.

If the last vertex in P_1 is a neighbor of v_1 when it is pressed, then it makes v_1 black, namely, before pressing the last vertex in P_1, v_1 is white. However, v_1 is black in G, so there has to be further vertices in P_1 changing the color of v_1. The last vertex in P_1 making v_1 white can be bubbled down to the last but one position of P_1 just as we did with w_2. Let P' be the path obtained from path P_1 in this way, excluding the last two vertices. Then the graph GP' contains the black vertex v_1, all of its neighbors are black, and all further vertices are

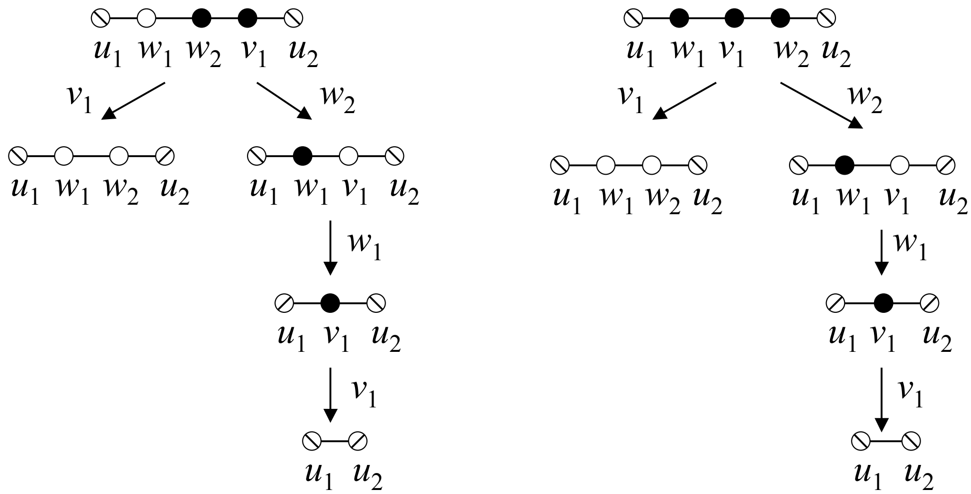


Figure 3: On the indicated two configurations, the neighbors of the w_1, w_2, v_1 triplet, u_1 and u_2 changes color in the same way by pressing only v_1 and pressing $w_2w_1v_1$. The color change on u_1 and u_2 is indicated with the flipping of their crossing line.

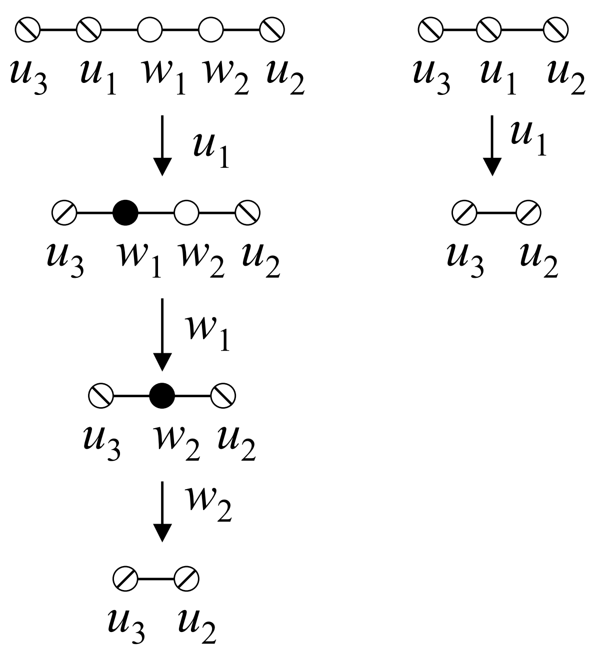
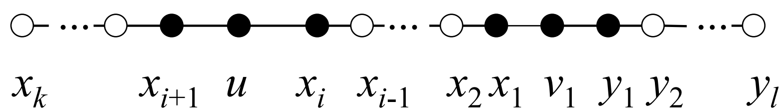


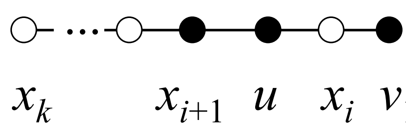
Figure 4: The color of u_2 and u_3 changes in the same way on the two indicated configurations. See text for details.

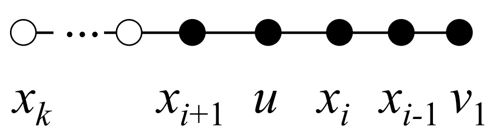
white. In this graph, v_1 cannot be the first vertex of a successful pressing path, since pressing it would create an all-white non-trivial component. Then further vertices must be in P'. If the last vertex of P' is a neighbor of v_1, we can do the same thing, creating a path P'' such that GP'' contains the black vertex v_1, all of its neighbors are black, and all other vertices are white.

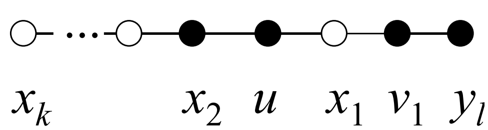
Since there is a successful pressing path which starts with v_1 after separating down a few – possibly 0 – couples of vertices from P_1, we have to find a vertex, call it u, which is not a neighbor of v_1. Let the so-emerging pressing path be $P_{1a}uP_{1b}v_1$. Note that we also incorporate w_1 and w_2 into P_{1b}. The vertices in P_{1b} are all neighbors of v_1 when pressed, and at least one of them are neighbor of u. Let the left neighbors of v_1 be denoted by $x_1, x_2 \ldots x_k$ and the let the right neighbors be denoted by $y_1, y_2, \ldots y_l$. Without loss of generality we can assume that u is in the left neighbors (swap left and right if this was not the case). Obviously, any x is not a neighbor of y, so we can rearrange them in P_{1b} such that first the y vertices are pressed then the x vertices. After a finite number of allowed alterations, $P_{1b} = y_1y_2 \ldots y_lx_1x_2 \ldots x_k$ and GP_{1a} is



Similarly, we move down vertex u before x_i in the pressing path. We consider the graph $GP_{1a}y_1 \ldots y_lx_1 \ldots x_{i-1}$ if v_1 is black in it (the runs of x vertices might be empty if $i = 1$), and otherwise the graph $GP_{1a}y_1 \ldots y_lx_1 \ldots x_{i-2}$ (also the runs of x vertices might be empty if $i = 2$) or $GP_{1a}y_1 \ldots y_{l-1}$ if $i = 1$. We have one of the following graphs







on which $ux_i \ldots x_kv_1$, $ux_{i-1} \ldots x_kv_1$, $y_lux_1 \ldots x_kv_1$ is the current successful pressing path, respectively.

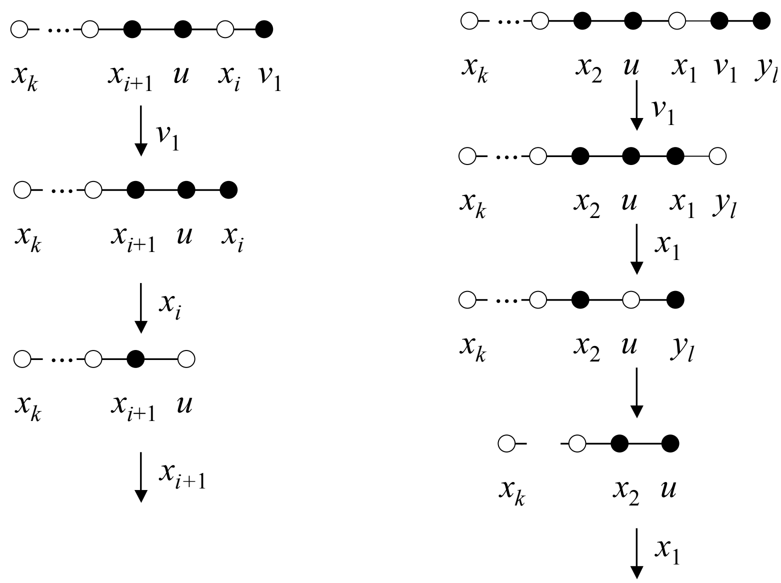


Figure 5: Alternative pressing paths for two cases. See text for details.

A successful pressing path replacing $ux_i \ldots x_k v_1$ is $v_1 x_i \ldots x_k u$, as can be seen on the left hand side of Figure 5. The length of the longest common subsequence of the two pressing paths is 2 less than their common length, as required. The pressing path $y_l u x_1 \ldots x_k v_1$ can be replaced to $u x_1 y_l x_2 \ldots x_k v_1$ since y_l is a neighbor neither u nor x_1. Then this pressing path can be replaced to $v_1 x_1 y_l x_2 \ldots x_k u$, as can be seen on the right hand side of Figure 5. The length of the longest common subsequence of $u x_1 y_l x_2 \ldots x_k v_1$ and $v_1 x_1 y_l x_2 \ldots x_k u$ is again 2 less than their common length.

Finally, the pressing path $u x_{i-1} \ldots x_k v_1$ can be replaced in two steps, first it is changed to $x_i x_{i+1} u x_{i-1} x_{i+2} \ldots x_k v_1$, then to $x_i x_{i+1} v_1 x_{i-1} x_{i+2} \ldots x_k u$, as can be checked on Figure 6. In both setps, the length of the longest common subsequences of two consecutive pressing paths is 2 less than their common length as required.

We proved that in any case, v_1 can be moved into a smaller index position with a finite series of allowed perturbations. Iterating this, we can move v_1 to the first position. Then we can do the same thing with v_2 on the graph $Gv_1 \setminus \{v_1\}$, and eventually transform X into Y with allowed perturbations.

□

4 Discussion and Conslusions

In this paper, we proved the pressing game conjecture for linear graphs. Although the linear graphs are very simple, the proving technique shows a direction how to prove the general case. Indeed, it is generally true that if a vertex v

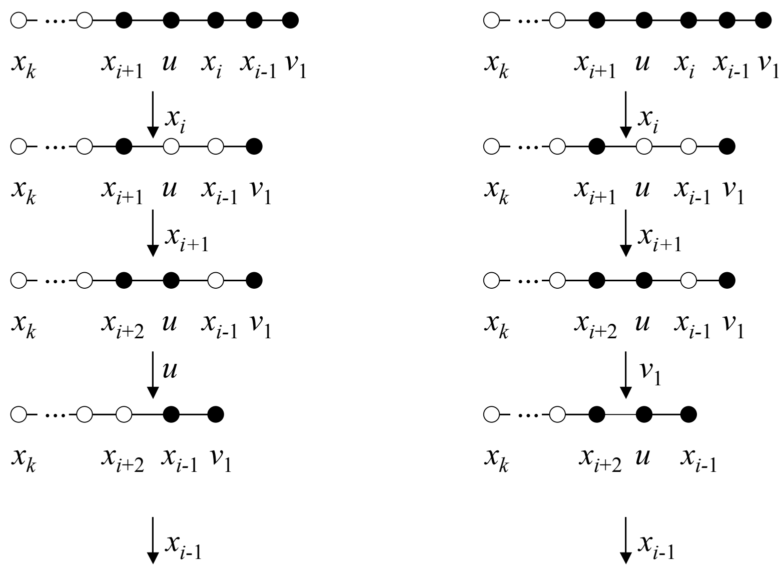


Figure 6: Changing the pressing path $ux_{i-1}\dots x_k v_1$ in two steps such that v_1 is in a smaller index position. See text for details.

is not in a successful pressing path P, then a successful pressing path P' exists which contains v and the length of the longest common subsequence of P and P' is only 1 less than their common length. Case 1 in the proof of Theorem 7 holds for arbitrary graphs, and in a working manuscript, we were able to prove that the conjecture is true for Case 2a using a linear algebraic techniques similar to that one used in [3]. The only missing part is Case 2b, which seems to be very complicated for general graphs.

A stronger theorem holds for the linear case that is conjectured for the general case. One possible direction above proving the general conjecture is to study the emerging Markov chain on the solution space of the pressing game on linear graphs. We proved that a Markov chain that randomly removes two vertices from the current pressing path, adds two random vertices to it, and accepts it if the result is a successful pressing path is irreducible. It is easy to set the jumping probabilities of the Markov chain such that it converges to the uniform distribution of the solutions. The remaing question is the speed of convergence of this Markov chain.

Acknowledgements

I.M. was supported by OTKA grant PD84297. Alexey Medvedev is thanked for fruitful discussions.

References

[1] Bergeron, A.: A Very Elementary Presentation of the Hannenhalli-Pevzner Theory. *Proceedings of the CPM* 106–117, 2001.

[2] Hannenhalli, S., Pevzner, P.: Transforming Cabbage into Turnip: Polynomial Algorithm for Sorting Signed Permutations by Reversals *Journal of the ACM*, vol. 46, no. 1, pp. 1–27, 1999.

[3] Hartman, T., Verbin, E.: Matrix Tightness: A Linear-Algebraic Framework for Sorting by Transpositions *Proceedings of SPIRE!06*, pp. 279–290, 2006.

[4] Ma, J., Ratan, A., Raney, B.J., Suh, B.B., Miller, W., Haussler, D.: The infinite sites model of genome evolution *PNAS*, vol. 105, no. 38, pp. 14254–14261, 2008

[5] Miklós, I., Mélykúti, B., Swenson, K.: The Metropolized Partial Importance Sampling MCMC mixes slowly on minimum reversal rearrangement paths *ACM/IEEE Transactions on Computational Biology and Bioinformatics*, vol. 4, no. 7, 763–767, 2010.

[6] Sturtevant, A.H.: A case of rearrangement of genes in Drosophila *Proc Natl Acad Sci U S A.*, vol. 7 no. 8, pp 235–237, 1921.

[7] Sturtevant, A.H., Novitski, E.: The homologies of chromosome elements in the genus Drosophila. *Genetics*, vol. 26, pp. 517–541, 1941.

[8] Sturtevant, A.H., Tan, C.C.: The comparative genetics of Drosophila pseudoobscura and D. melanogaster. *J.Genet.*, vol. 34, pp. 415–432, 1937.