# Coloring points with respect to squares

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**Abstract**

We consider the problem of 2-coloring geometric hypergraphs. Specifically, we show that there is a constant $m$ such that any finite set $\mathcal{S}$ of points in the plane can be 2-colored such that every axis-parallel square that contains at least $m$ points from $\mathcal{S}$ contains points of both colors. Our proof is constructive, that is, it provides a polynomial-time algorithm for obtaining such a 2-coloring. By affine transformations this result immediately applies also when considering homothets of a fixed parallelogram.

## 1 Introduction

In this paper we consider the problem of coloring a given set of points in the plane such that every region from a given set of regions contains a point from each color class. To state our results, known results and open problems we need the following definitions and notations.

A *hypergraph* is a pair $(\mathcal{V},\mathcal{E})$ where $\mathcal{V}$ is a set and $\mathcal{E}$ is a set of subsets of $\mathcal{V}$. The elements of $\mathcal{V}$ and $\mathcal{E}$ are called *vertices* and *hyperedges*, respectively. For a hypergraph $H=(\mathcal{V},\mathcal{E})$, let $H|_m=(\mathcal{V},\{e\in\mathcal{E}:|e|\geq m\})$. A *proper coloring* of a hypergraph is a coloring of its vertex set such that in every hyperedge not all vertices are assigned the same color. Proper colorability of a hypergraph with two colors is sometimes called *Property B* in the literature. A *polychromatic $k$-coloring* of a hypergraph is a coloring of its vertex set with $k$ colors such that every hyperedge contains at least one vertex from each of the $k$ colors.

Given a family of regions $\mathcal{F}$ in $\mathbb{R}^d$ (e.g., all disks in the plane), there is a natural way to define two types of finite hypergraphs that are dual to each other. First, for a finite set of points $\mathcal{S}$, let $H^{\mathcal{F}}(\mathcal{S})$ denote the *primal hypergraph* on the vertex set $\mathcal{S}$ whose hyperedges are all subsets of $\mathcal{S}$ that can be obtained by intersecting $\mathcal{S}$ with a member of $\mathcal{F}$. We say that a finite subfamily $\mathcal{F}_0\subseteq\mathcal{F}$ *realizes* $H^{\mathcal{F}}(\mathcal{S})$ if for every hyperedge $\mathcal{S}'\subseteq\mathcal{S}$ of $H^{\mathcal{F}}(\mathcal{S})$ there is $F'\in\mathcal{F}_0$ such that $F'\cap\mathcal{S}=\mathcal{S}'$. The *dual hypergraph* $H^*(\mathcal{F}_0)$ is defined with respect to a finite subfamily $\mathcal{F}_0\subseteq\mathcal{F}$. Its vertex set is $\mathcal{F}_0$ and for each point $p\in\mathbb{R}^d$ it has a hyperedge that consists of all the regions in $\mathcal{F}_0$ that contain $p$.

The general problems we are interested in are the following.

**Problem 1.** *For a given family of regions $\mathcal{F}$,*

*(i) Is there a constant $m$ such that for any finite set of points $\mathcal{S}$ the hypergraph $H^{\mathcal{F}}(\mathcal{S})|_m$ admits a proper 2-coloring?*

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*(ii) Is there a constant $m^*$ such that for any finite subset $\mathcal{F}_0 \subseteq \mathcal{F}$ the hypergraph $H^*(\mathcal{F}_0)|_{m^*}$ admits a proper 2-coloring?*

*(iii) Given a constant $k$, is there a constant $m_k$ such that for any finite set of points $\mathcal{S}$ the hypergraph $H^{\mathcal{F}}(\mathcal{S})|_{m_k}$ admits a polychromatic $k$-coloring? If so, is $m_k = O(k)$?*

*(iv) Given a constant $k$, is there a constant $m_k^*$ such that for any finite subset $\mathcal{F}_0 \subseteq \mathcal{F}$ the hypergraph $H^*(\mathcal{F}_0)|_{m_k^*}$ admits a polychromatic $k$-coloring? If so, is $m_k^* = O(k)$?*

Examples of families $\mathcal{F}$ for which such coloring problems are studied are translates of convex sets, homothets of convex sets, axis-parallel rectangles and half-planes. If $\mathcal{F}$ is the family of disks in the plane then these hypergraphs generalize Delaunay graphs.

The main motivation for studying proper and polychromatic colorings of such geometric hypergraphs comes from cover-decomposability problems and conflict-free coloring problems [29]. We concentrate on the first connection, as the problems we regard are in direct connection with cover-decomposability problems.

Multiple coverings and packings were first studied by Davenport and L. Fejes Tóth almost 50 years ago. Since then a wide variety of questions related to coverings and packings has been investigated. In 1986 Pach [20] published the first paper about decomposability problems of multiple coverings. It turned out that this area is rich of deep and exciting questions, and it has important practical applications as well (e.g., in the area of surveillance systems). Following Pach's papers, most of the efforts were concentrated on studying coverings by translates of some given shape. Recently, many researchers started to study also cover-decomposability of homothets of a given shape.

A family of planar sets is called an *$r$-fold covering* of a region $R$, if every point of $R$ is contained in at least $r$ members of the family. A 1-fold covering is simply called a covering. A family $\mathcal{F}$ of planar sets is called *cover-decomposable*, if there is an integer $l$ with the property that for any region $R$, any subfamily of $\mathcal{F}$ that forms an $l$-fold covering of $R$ can be decomposed into two coverings. We can generalize the problem of decomposition into more than 2 coverings, in which case we are interested in the existence of a number $l_k$ such that any subfamily of $\mathcal{F}$ that covers every point in $R$ at least $l_k$ times, can be split into $k$ subfamilies, each forming a covering. If we consider only coverings with finite subfamilies, then we call it the *finite cover-decomposition* problem. One of the first observations of Pach was that if $\mathcal{F}$ is the family of translates of an open convex set, then cover-decomposition is equivalent to finite cover-decomposition.

It is easy to see that the finite cover-decomposition problem is equivalent to Problems 1(ii) and (iv) (i.e., $l = m^*$ and $l_k = m_k^*$ in the notation above). Pach also observed that if $\mathcal{F}$ is the translates of an open set, then Problems 1(i) and (ii) are equivalent and also Problems 1(iii) and (iv) are equivalent. That is, it is enough to consider the primal hypergraph coloring problem.

Pach conjectured that translates of every open convex planar set are cover-decomposable [19]. During the years researchers acquired a good understanding of convex planar shapes whose translates are cover-decomposable. On the positive side, Pach's conjecture was verified for every open convex polygon: Pach himself proved it for every open centrally symmetric convex polygon [20], then Tardos and Tóth [30] proved the conjecture for every open triangle, and finally Pálvölgyi and Tóth [27] proved it for every open convex polygon. For open convex polygons we also know that $l_k = O(k)$ [4, 11, 24]. In [27] Pálvölgyi and Tóth also gave a complete characterization of cover-decomposable open concave polygons. Thus, the cover-decomposability problem is settled for translates of an open polygon. However, Pach's conjecture was refuted by Pálvölgyi [26] who showed that it does not hold for a disk and for convex shapes with smooth boundary.

In the three dimensional space it follows from the planar results [25, 26] that every bounded polytope is not cover-decomposable. Thus, it is not easy to come up with a cover-decomposable set in the space. An important exception is the *octant*[1], whose translates were proved to be cover-decomposable [13]. The currently best bounds are $5 \leq l \leq 9$ [16] and $l_k = O(k^{5.09})$ [8, 15, 16]. It is an interesting open problem whether $l_k = O(k)$.

For a long time no positive results were known about cover-decomposability and geometric hypergraph coloring problems concerning homothets of a given shape. For disks, the answer is negative for all parts of Problem 1 [21, 22]. As a first positive result, the cover-decomposability of octants along with a simple reduction implied that both the primal and dual hypergraphs with respect to homothets of a triangle are properly 2-colorable:

**Theorem 2** ([13, 16]). *For the family $\mathcal{F}$ of all homothets of a given triangle both Problems 1(i) and 1(iii) have a positive answer with $m = m^* \leq 9$.*

This result was later used to obtain polychromatic colorings of the primal and dual hypergraphs defined by the family of homothets of a fixed triangle. For the dual hypergraph, the best bound comes from the corresponding result about octants and so it is $m^*_k = O(k^{5.09})$. For the primal hypergraph there is a better bound $m_k = O(k^{4.09})$ [7, 14, 16]. An important tool for obtaining these results is the notion of *self-coverability* (see Section 2.2), which is also essential for proving our results. It is an interesting open problem whether $m_k = O(k)$ and $m^*_k = O(k)$ for the homothets of a given triangle.

For polygons other than triangles, somewhat surprisingly, Kovács [18] recently provided a negative answer for Problems 1(ii) and (iv). Namely, he showed that the homothets of any given convex polygon with at least four sides are not cover-decomposable. In other words, there is no constant $m^*$ for which the dual hypergraph consisting of hyperedges of size at least $m^*$ is 2-colorable. Our main contribution is showing that this is not the case when considering 2-coloring of the primal graph. Indeed, Problem 1(i) has a positive answer for homothets of any given *parallelogram*.

**Theorem 3.** *There is an absolute constant $m_q \leq 1484$ such that the following holds. Given an (open or closed) parallelogram $Q$ and a finite set $\mathcal{S}$ of points in the plane, the points of $\mathcal{S}$ can be 2-colored in polynomial time, such that any homothet of $Q$ that contains at least $m_q$ points contains points of both colors.*

This is the first example that exhibits such different behavior for coloring the primal and dual hypergraphs with respect to the family of some geometric regions. Furthermore, combined with results about self-coverability, the proof of Theorem 3 immediately implies the following generalization to polychromatic $k$-colorings, thus partially answering also Problem 1 (iii) (it remains open whether linearly many points per hyperedge/parallelogram suffice).

**Corollary 1.1.** *Let $Q$ be a given (open or closed) parallelogram and let $\mathcal{S}$ be a set of points in the plane. Then $\mathcal{S}$ can be $k$-colored, such that any homothet of $Q$ that contains at least $m_k = \Omega(k^{11.6})$ points from $\mathcal{S}$ contains points of all $k$ colors.*

Our proof of Theorem 3 also works for homothets of a triangle, i.e., we give a new proof for the primal case of Theorem 2 (with a larger constant though):

**Theorem 4** ([13]). *There is an absolute constant $m_t$ such that the following holds. Given an (open or closed) triangle $T$ and a finite set $\mathcal{S}$ of points in the plane, the points of $\mathcal{S}$ can be 2-colored in polynomial time, such that any homothet of $T$ that contains at least $m_t$ points contains points of both colors.*

[1] An octant is the set of points $\{(x, y, z) | x < a, y < b, z < c\}$ in the space for some $a, b$ and $c$.

This paper is organized as follows. In Section 2 we introduce definitions, notations, tools and some useful lemmas. In Section 3 we describe a generalized 2-coloring algorithm and then apply it for parallelograms and for triangles. Concluding remarks and open problems appear in Section 4.

# 2 Preliminaries

Unless stated otherwise, we restrict ourselves to the two-dimensional Euclidean space $\mathbb{E}^2$. For a point $p \in \mathbb{E}^2$ let $(p)_x$ and $(p)_y$ denote the $x$- and $y$-coordinate of $p$, respectively. We denote by $\partial S$ the boundary of a subset $S \subseteq \mathbb{E}^2$ and by $Cl(S)$ the closure of $S$. A *homothet* of $S$ is a translated and scaled copy of $S$. That is, a set $S' = \alpha S + p$ for some number $\alpha > 0$ and a point $p \in \mathbb{E}^2$.

**Lemma 2.1** (e.g., [2]). *Let $C$ be a convex set and let $C_1$ and $C_2$ be homothets of $C$. Then $\partial C_1$ and $\partial C_2$ cross each other at most twice.*

## 2.1 Generalized Delaunay triangulations

For proving Theorems 3 and 4 and we will use the notion of generalized Delaunay triangulations, which are the dual of generalized Voronoi diagrams. In the generalized Delaunay triangulation of a point set $\mathcal{S}$ with respect to some convex set $C$, two points of $\mathcal{S}$ are connected by a straight-line edge if there is a homothet of $C$ that contains them and does not contain any other point of $\mathcal{S}$ in its interior. The generalized Delaunay triangulation of $\mathcal{S}$ with respect to $C$ is denoted by $\mathcal{DT}(C, \mathcal{S})$. We say that $\mathcal{S}$ is *in general position* with respect to (homothets of) $C$, if there is no homothet of $C$ whose boundary contains four points from $\mathcal{S}$. If $\mathcal{S}$ is in general position with respect to a convex polygon $P$ and no two points of $\mathcal{S}$ define a line that is parallel to a line through two vertices of $P$, then we say that $\mathcal{S}$ is in *very general position* with respect to $P$. The following properties of generalized Delaunay triangulations will be useful.

**Lemma 2.2** ([5, 17, 28]). *Let $C$ be a convex set and let $\mathcal{S}$ be a set of points in general position with respect to $C$. Then $\mathcal{DT}(C, \mathcal{S})$ is a well-defined connected plane graph whose inner faces are triangles.*

If the boundary of the unbounded face in $\mathcal{DT}(C, \mathcal{S})$ is a convex polygon, then we say that $\mathcal{S}$ is *nice* with respect to $C$.

**Lemma 2.3.** *Let $P$ be a closed convex polygon and let $\mathcal{S}$ be a set of points in the plane that is in very general position with respect to $P$. Suppose that $P'$ is a homothet of $P$ and we have a set of $i$ points ($1 \le i \le 3$) $Z = \{z_1, \ldots, z_i\} \subset \mathcal{S}$ on $\partial P'$. Then there is a homothet of $P$, denote it by $P''$, such that $P'' \cap \mathcal{S} = (P' \cap \mathcal{S}) \setminus Z$.*

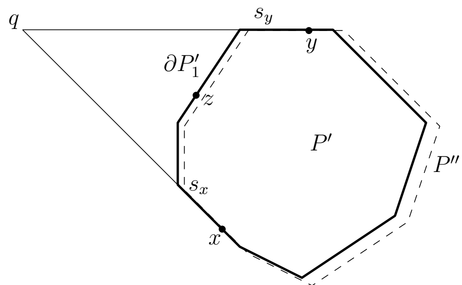
*Proof.* Since $\mathcal{S}$ is in general position, $|\partial P' \cap \mathcal{S}| \le 3$.

If $|Z| = 3$ then there are no other points on $\partial P'$, thus shrinking $P'$ from an inner point gives us the required $P''$.

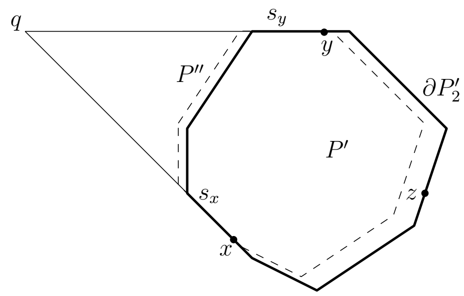
If $|Z| = 2$ then if there is no other point on $\partial P'$ then we can again shrink from an inner point slightly to get the required $P''$. Otherwise, there is exactly one point $q$ on $\partial P'$ besides the two points of $Z$. Now shrink $P'$ from $q$ slightly. As the points are in very general position, the new homothet $P''$ will contain exactly the points $(P' \cap \mathcal{S}) \setminus Z$.

Finally, if $|Z| = 1$, $Z = \{z\}$, then we consider three cases.

**Case 1:** $|\partial P' \cap \mathcal{S}| = \{z\}$. In this case we slightly shrink $P'$ with respect to some point in its interior and obtain the desired homothet $P''$.




(a) If $q$ sees $z$ then we inflate $P'$ w.r.t. $q$.




(b) If $q$ does not see $z$ then we shrink $P'$ w.r.t. $q$.

Figure 1: Illustrations for the proof of Lemma 2.3: $x$ and $y$ are not at vertices of $P'$.

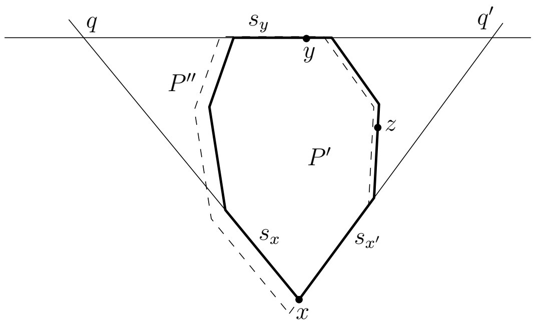
**Case 2:** $|\partial P' \cap \mathcal{S}| = 2$. Let $\{x, z\} = \partial P' \cap \mathcal{S}$. Since $\mathcal{S}$ is in very general position, $x$ and $z$ are on different sides of $\partial P'$. Therefore if we slightly shrink $P'$ such that $x$ remains on its boundary the resulting homothet of $P$ does not contain $z$ and contains any other point in $P' \cap \mathcal{S}$.
**Case 3:** $|\partial P' \cap \mathcal{S}| = 3$. Let $\{x, y, z\} = \partial P' \cap \mathcal{S}$. Since $\mathcal{S}$ is in very general position, at least two of these three points are not at a vertex of $P'$. Suppose first that $x$ and $y$ are not at vertices of $P'$. Since $\mathcal{S}$ is in very general position, $x$ and $y$ are on different sides of $\partial P'$, denote them by $s_x$ and $s_y$, respectively. Let $q$ be the intersection point of the lines through $s_x$ and $s_y$ (if these sides are parallel, then $q$ is a point at infinity). Denote by $\partial P'_1$ (resp, $\partial P'_2$) the segment of $\partial P \setminus (s_x \cup s_y)$ that is (resp., not) 'visible' from $q$ (see Figure 1). Note that $z \in \partial P'_1 \cup \partial P'_2$ since $\mathcal{S}$ is in very general position. If $z \in \partial P'_1$, then by inflating $P'$ a little with respect to $q$ we obtain a homothet of $P$ that contains every point in $P' \cap \mathcal{S}$ but $z$ (see Figure 1a). Otherwise, if $z \in \partial P'_2$, then by shrinking $P'$ a little with respect to $q$ we obtain a homothet of $P$ that contains every point in $P' \cap \mathcal{S}$ but $z$ (see Figure 1b).

It remains to consider the case that exactly one of $x$ and $y$ is at a vertex of $P'$. Assume without loss of generality that it is $x$. Let $s_x$ and $s_{x'}$ be the sides of $P'$ that have a common endpoint at $x$ and let $s_y$ be the side of $P'$ that contains (in its interior) $y$. Let $q$ (resp., $q'$) be the intersection point of the lines through $s_x$ (resp., $s_{x'}$) and $s_y$ (if these sides are parallel, then it is a point at infinity). There are two subcases to consider. First, suppose that $q$ and $q'$ are on different sides of $y$ on the line through $s_y$ (see Figure 2a). Then, as before, we inflate $P$ with respect to the point among $q$ and $q'$ that 'sees' $z$ and obtain the desired homothet $P''$. Suppose now that $q$ and $q'$ are on the same side of $y$ on the line through $s_y$, such that $q'$ is between $q$ and $y$. Then, as before, if $q'$ and $q$ 'see' $z$ we then inflate $P'$ with respect to $q'$ and get the desired homothet $P''$ (see Figure 2b). Otherwise, if $q'$ and $q$ do not 'see' $z$, then we shrink $P'$ with respect to $q$ and get the desired homothet $P''$ (see Figure 2c). □

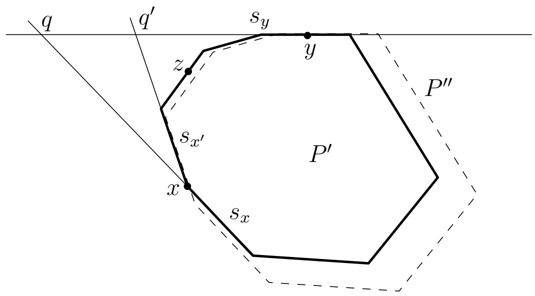
For a homothet $C'$ of a convex set $C$ we denote by $\mathcal{DT}(C, \mathcal{S})[C']$ the subgraph of $\mathcal{DT}(C, \mathcal{S})$ that is induced by the points of $\mathcal{S} \cap C'$. Note that it is not the same as $\mathcal{DT}(C, \mathcal{S} \cap C')$, however the following is true.

**Lemma 2.4.** *Let $P$ be a closed convex polygon, let $\mathcal{S}$ be a set of points in very general position with respect to $P$, and let $P'$ be a homothet of $P$. Then $\mathcal{DT}(P, \mathcal{S})[P']$ is a connected graph that is contained in $P'$.*

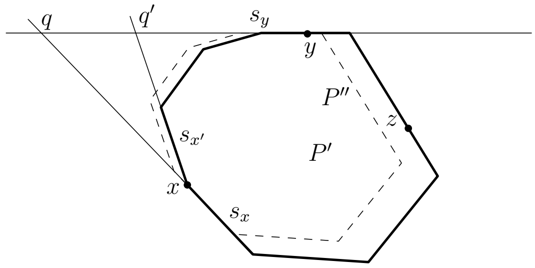
*Proof.* Denote $\mathcal{DT} := \mathcal{DT}(P, \mathcal{S})$. Obviously all the points in $\mathcal{DT}[P']$ are in $P'$ by definition. An edge in $\mathcal{DT}[P']$ must also be in $P'$ since $P'$ is convex. We prove that $\mathcal{DT}[P']$ is connected by induction on $|\mathcal{S} \cap P'|$. This is trivially true when there is only one point from $\mathcal{S}$ in $P'$, so suppose it also holds whenever a homothet of $P$ contains $k-1$ points from $\mathcal{S}$ and let $P'$




(a) If $q$ and $q'$ are on different sides of $y$, then we inflate w.r.t. the point among them that sees $z$.




(b) If $q'$ is between $y$ and $q$ and both $q$ and $q'$ see $z$, then we inflate w.r.t. $q'$.




(c) If $q'$ is between $y$ and $q$ and both $q$ and $q'$ do not see $z$, then we shrink w.r.t. $q$.

Figure 2: Illustrations for the proof of Lemma 2.3: $x$ is at a vertex of $P'$.

be a homothet of $P$ that contains exactly $k$ points from $\mathcal{S}$. We may assume without loss of generality that $P'$ contains a point $z$ on its boundary, for otherwise we can continuously shrink $P'$ until such a point exists. It follows from Lemma 2.3 that there is a homothet of $P$, denote it by $P''$, such that $\mathcal{S} \cap (P' \setminus P'') = \{z\}$. By the induction hypothesis $\mathcal{DT}[P'']$ is connected and therefore there is a path in $\mathcal{DT}[P']$ between every two points that are different from $z$.

Therefore, it is enough to show that there is a path between $z$ and some other point in $\mathcal{S} \cap P''$. Indeed, we can continue shrinking $P'$ such that $z$ remains on its boundary, until the boundary contains at least one more point $x \in \mathcal{S}$. If $P_x \cap \mathcal{S} = \{x, z\}$, then $xz$ is an edge in $\mathcal{DT}$ and we are done. Otherwise, it follows from Lemma 2.3 that there is a homothet of $P$, denote it by $P'_x$, such that $\mathcal{S} \cap (P_x \setminus P'_x) = \{x\}$ and $P'_x$ contains at least one point $y \in \mathcal{S}$, $y \neq z$. Thus, by applying the induction hypothesis to $P'_x$ we conclude that there is a path in $\mathcal{DT}[P'_x]$ between $z$ and $y$. Since this path also exists in $\mathcal{DT}[P']$, it follows that $\mathcal{DT}[P']$ is connected. □

**Corollary 2.5.** *Let $P$ be a closed convex polygon and let $\mathcal{S}$ be a set of points in very general position with respect to $P$. Suppose that $P'$ is a homothet of $P$ and $e$ is an edge of $\mathcal{DT}(P, \mathcal{S})$ that crosses $\partial P'$ twice and thus splits $P'$ into two parts. Then one of these parts does not contain a point from $\mathcal{S}$.*

A *rotation* of a vertex $v$ in a plane graph $G$ is the clockwise order of its neighbors. For three straight-line edges $vx, vy, vz$ we say that $vy$ is *between* $vx$ and $vz$ if $x, y, z$ appear in this order in the rotation of $v$ and $\angle xvz < \pi$ ($\angle xvz$ is the angle by which one has to rotate the vector $\vec{vx}$ around $v$ clockwise until its direction coincides with that of $\vec{vz}$). The following will be useful later on.

**Proposition 2.6.** *Let $C$ be a convex set and let $\mathcal{S}$ be a nice set of points with respect to $C$. Let $C'$ be a homothet of $C$ and let $v$ be a vertex in $\mathcal{DT}(C, \mathcal{S})[C']$. Suppose that $x$ and $z$ are two vertices such that $z$ immediately follows $x$ in the rotation of $v$ in $\mathcal{DT}(C, \mathcal{S})[C']$,*

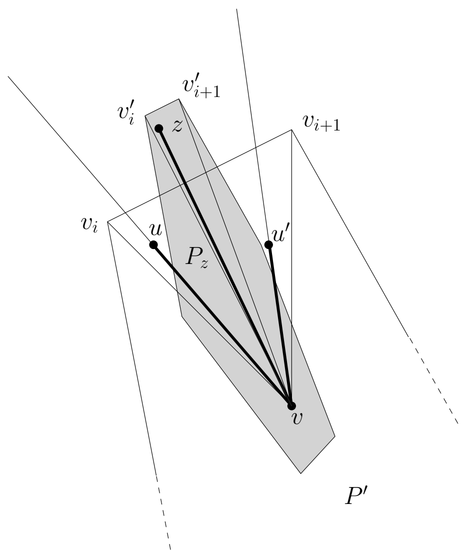



Figure 3: An illustration for the proof of Proposition 2.9.

*$\angle xvz < \pi$ and $xz \notin \mathcal{DT}(C, \mathcal{S})$. Then there exists an edge $vy \in \mathcal{DT}(C, \mathcal{S})$ between $vx$ and $vz$ (which implies that $y \notin C'$).*

*Proof.* Suppose that $x$ and $z$ are also consecutive in the rotation of $v$ in $\mathcal{DT}(C, \mathcal{S})$. Then the face that is incident to $vx$ and $vz$ and is to the right of $\vec{vx}$ and to the left of $\vec{vz}$ cannot be the unbounded face since $\angle xvz < \pi$ and $\mathcal{S}$ is nice. However, since this face is an inner face, then by Lemma 2.2 it must be a triangle and so $xz \in \mathcal{DT}(C, \mathcal{S})$. □

**Lemma 2.7.** *For every closed convex polygon $P$ there is a constant $\Delta = \Delta(P)$ such that the following holds. Let $\mathcal{S}$ be a set of points in very general position with respect to $P$ and let $P'$ be a homothet of $P$ such that $G' := \mathcal{DT}(P, \mathcal{S})[P']$ is a tree. Then for every $v \in \mathcal{S} \cap P'$ we have $\deg_{G'}(v) \leq \Delta$.*

*Proof.* Let $n$ be the number of vertices of $P$ and let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of $P'$ listed in their clockwise order around $P'$. Let $v \in \mathcal{S} \cap P'$ be a point and let $N_i := \{u \in \mathcal{S} \cap P' \mid u \in \triangle v v_i v_{i+1}, vu \in G'\}$ be the neighbors of $v$ in $G'$ that are also in the triangle $v v_i v_{i+1}$, for every $i = 0, \ldots, n-1$ (addition is modulo $n$). Let $\alpha = \alpha(P)$ be the smallest angle formed by three vertices of $P$ (hence, also of $P'$).

**Observation 2.8.** *For every point $p'$ in the interior of $P'$ and every $0 \leq i < j \leq n-1$ we have $\angle v_i p' v_j \geq \alpha$.*

**Proposition 2.9.** *For every $i = 0, \ldots, n-1$ and every $u, u' \in N_i$ we have $\angle uvu' \geq \alpha$.*

*Proof.* It follows from Proposition 2.6 that there is a point $z \notin P'$ such that $z$ is a neighbor of $v$ in $\mathcal{DT}(P, \mathcal{S})$ and is between $u$ and $u'$ in the rotation of $v$. Thus, there is a homothet of $P$, denote it by $P_z$, such that $P_z \cap \mathcal{S} = \{v, z\}$. Let $v'_i v'_{i+1}$ be the side of $P_z$ that corresponds and is parallel to the side $v_i v_{i+1}$ of $P'$ (see Figure 3). Note that $v'_i v'_{i+1}$ is outside of $P'$, since $z \notin P'$. Furthermore, since $u, u' \notin P_z$, the side $v'_i v'_{i+1}$ lies inside the wedge whose apex is $v$ and whose boundary consists of the two rays that emanate from $v$ and go through $u$ and $u'$, respectively. Therefore, $\angle v'_i v v'_{i+1} < \angle uvu'$. It follows from Observation 2.8 that $\angle v'_i v v'_{i+1} \geq \alpha$, thus we have $\angle uvu' \geq \alpha$. □

To complete the proof Lemma 2.7 consider the neighbors of $v$ in $G'$ in their clockwise order around $v$, and for every set $N_i$ remove the extreme neighbor in this order. It follows

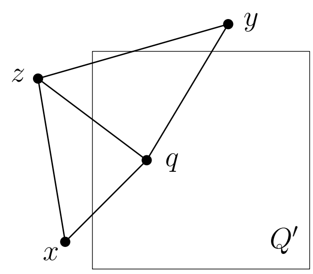



Figure 4: Considering homothets of an axis-parallel square, $x$-$q$-$y$ is a good 2-path whereas $x$-$q$-$z$ is not since the square $Q'$ separates it and both $qx$ and $qz$ cross the left side of $Q'$.

from Proposition 2.9 that the angle between any two remaining neighbors of $u$ is at least $\alpha$. Therefore, $\deg_{G'}(v) \leq n + \frac{2\pi}{\alpha}$. □

### 2.2 Self-coverability of convex polygons and polychromatic $k$-coloring

Keszegh and Pálvölgyi introduced in [14] the notion of *self-coverability* and its connection to polychromatic $k$-coloring. In this section we list the definition and results from their work that we use.

**Definition 2.10** ([14])**.** *A collection of closed sets $\mathcal{F}$ in a topological space is* self-coverable *if there exists a* self-coverability function *$f$ such that for any set $F \in \mathcal{F}$ and for any finite point set $\mathcal{S} \subset F$, with $|\mathcal{S}| = l$ there exists a subcollection $\mathcal{F}' \subset \mathcal{F}$, $|\mathcal{F}'| \leq f(l)$ such that $\bigcup_{F' \in \mathcal{F}'} F' = F$ and no point of $\mathcal{S}$ is in the interior of some $F' \in \mathcal{F}'$.*

**Theorem 5** ([14])**.** *For every convex polygon $P$ there is a constant $c_f = c_f(P)$ such that the family of all homothets of $P$ is self-coverable with $f(l) \leq c_f l$.*

**Theorem 6** ([14])**.** *The family of all homothets of a square is self-coverable with $f(l) := 2l+2$ and this is sharp.*

**Theorem 7** ([14])**.** *The family of all homothets of a given triangle is self-coverable with $f(l) := 2l + 1$ and this is sharp.*

**Theorem 8** ([14, Theorem 2])**.** *If $\mathcal{F}$ is self-coverable with a monotone self-coverability function $f(l) > l$ and any finite set of points can be colored with two colors such that any member of $\mathcal{F}$ with at least $m$ points contains both colors, then any finite set of points can be colored with $k$ colors such that any member of $\mathcal{F}$ with at least $m_k = m(f(m-1))^{\lceil \log k \rceil - 1} \leq k^d$ points contains all $k$ colors (where $d$ is a constant that depends only on $\mathcal{F}$).*

Theorems 3 and 8 immediately imply Corollary 1.1.

## 3 A 2-coloring algorithm

In this section we prove Theorems 3 and 4. In fact, we prove a more general result, for which we need the following definitions. Let $P$ be an (open or closed) convex polygon, let $\mathcal{S}$ be a finite set of points, and let $x$-$y$-$z$ be a 2-path in $\mathcal{DT}(P, \mathcal{S})$ (i.e., a simple path of length two). If $P'$ is a homothet of $P$ that contains $y$ and does not contain $x$ and $z$, then we say that $P'$ *separates* the 2-path $x$-$y$-$z$. Call $x$-$y$-$z$ a *good* 2-path, if there is no homothet of $P$ that separates it such that the edges $yx$ and $yz$ cross the same side of this homothet of $P$ (see Figure 4 for an example).

**Definition 3.1** (Property H)**.** *We say that an (open or closed) polygon $P$ satisfies* Property H *if there is a constant $c_H = c_H(P)$ such that for any finite set of points $\mathcal{S}$ that is in very general position and is nice with respect to $P$ the following holds: if $P'$ is a homothet of $P$ that contains at least $c_H$ points from $\mathcal{S}$ and $\mathcal{DT}(P,\mathcal{S})[P']$ is acyclic, then $P'$ contains two good 2-paths $x$-$y$-$z$ and $x'$-$y'$-$z'$ such that $yy'$ is an edge in $\mathcal{DT}(P,\mathcal{S})$ (note that this implies that $y \neq y'$, however other points may coincide).*

**Theorem 9.** *Let $P$ be an (open or closed) convex polygon with $n$ vertices that satisfies Property H with a constant $c_H = c_H(P)$, and let $f(l) \leq c_f l$ be a self-coverability function of the family of homothets of $Cl(P)$ (where $c_f$ is a constant). Then there is a constant $m = m(P) \leq c_H f(n) + n$ such that it is possible to 2-color in polynomial-time the points of any given finite set of points $\mathcal{S}$ such that every homothet of $P$ that contains at least $m$ points from $\mathcal{S}$ contains points of both colors.*

Theorems 3 and 4 immediately follow from Theorems 6, 7, 9, and the following.

**Lemma 3.2.** *Every triangle satisfies Property H with a constant $c_H \leq 7382$.*

**Lemma 3.3.** *Every parallelogram satisfies Property H with a constant $c_H \leq 148$.*

We remark that in both cases the two good 2-paths that find are actually a path of length three $x_1$-$x_2$-$x_3$-$x_4$ such that $x_1$-$x_2$-$x_3$ and $x_2$-$x_3$-$x_4$ are two good 2-paths.

### 3.1 Proof of Theorem 9

Let $P$ be an (open or closed) convex polygon and let $\mathcal{S}$ be a set of points in the plane. We first argue that it is enough to prove Theorem 9 when $P$ is a closed polygon. Indeed, suppose that $P$ is open, let $\mathcal{P}$ be the family of homothets of $P$ and let $\mathcal{P}_0 \subseteq \mathcal{P}$ be a finite subfamily that realizes $H^{\mathcal{P}}(\mathcal{S})$. By slightly shrinking every homothet of $P$ in $\mathcal{P}_0$ with respect to an interior point, we get a subfamily $\mathcal{P}'_0 \subseteq \mathcal{P}$ that realizes $H^{\mathcal{P}}(\mathcal{S})$ such that there is no $p \in \mathcal{S}$ and $P' \in \mathcal{P}'_0$ with $p \in \partial P'$. Let $\bar{P} = Cl(P)$ be the closed polygon that is the closure of $P$, let $\bar{\mathcal{P}}$ be the family of homothets of $\bar{P}$, and let $\bar{\mathcal{P}}'_0 \subseteq \bar{\mathcal{P}} = \{Cl(P') | P' \in \mathcal{P}'_0\}$. Since there is no homothet of $P$ in $\mathcal{P}'_0$ that contains a point of $\mathcal{S}$ on its boundary, every hyperedge of $H^{\mathcal{P}}(\mathcal{S})$ appears also in $H^{\bar{\mathcal{P}}}(\mathcal{S})$. Thus, it is enough to show that $\bar{P}$ satisfies Theorem 9.

Suppose therefore that $P$ is a closed convex polygon that satisfies Property H with a constant $c_H = c_H(P)$. Let $\mathcal{P}$ be the family of homothets of $P$ and let $\mathcal{P}_0 \subseteq \mathcal{P}$ be the smallest subfamily that realizes $H^{\mathcal{P}}(\mathcal{S})$. It would convenient to pick $\mathcal{P}_0$ such that every $P' \in \mathcal{P}$ is minimal in the sense that it does not contain any other homothet of $P$ that contains the same set of points from $\mathcal{S}$. We may also assume that $\mathcal{S}$ is in very general position with respect to $P$. Indeed, otherwise note that a small perturbation of the points will achieve that and observe also that for every homothet $P' \in \mathcal{P}_0$ a slightly inflated $P'$ will contain the (perturbed) points that correspond to the points in $\mathcal{S} \cap P'$ and no other points. After a perturbation every homothet in $\mathcal{P}_0$ is 'almost' minimal, which is fine for our purposes. It will also be convenient to assume that after the perturbation, the boundaries of every two polygons in $\mathcal{P}_0$ do not overlap, and no edge in $\mathcal{DT} = \mathcal{DT}(P,\mathcal{S})$ crosses the boundary of a polygon in $\mathcal{P}_0$ at one of its vertices. Since $\mathcal{P}$ is a family of *pseudo-disks*[2] by Lemma 2.1, it follows from [6] that $|\mathcal{P}_0| = O(|\mathcal{S}|^3)$.

We can also assume that $\mathcal{S}$ is nice with respect to $P$: Set $-P := \{(-x,-y) \mid (x,y) \in P\}$ and let $-P'$ be a homothet of $-P$ that contains in its interior all the polygons in $\mathcal{P}_0$. By adding the vertices of $-P'$ to $\mathcal{S}$ (and perturbing again if needed) we obtain a nice set of points such that $-P'$ is the boundary of the unbounded face in its generalized Delaunay

[2] In a family of pseudo-disks the boundaries of every two regions cross at most twice.

triangulation with respect to $P$. Moreover, we have only extended the set of point subsets $\mathcal{S}'$ that contain at least $m$ points and for which there is a homothet $P' \in \mathcal{P}$ such that $\mathcal{S}' \cap P' = \mathcal{S}'$. Therefore a valid 2-coloring of the new set of points induces a valid 2-coloring of the original set of points.

Let $f(l)$ be a self-coverability function of $\mathcal{P}$ and set $m := (c_H + 3)f(n)$, where $n$ is the number of vertices of $P$. Recall that $\mathcal{DT}$ is a plane graph, and therefore, by the Four Color Theorem, we can color the points in $\mathcal{S}$ with four colors, say $1, 2, 3, 4$, such that there are no adjacent vertices in $\mathcal{DT}$ with the same color. In order to obtain two color classes, we recolor all the vertices of colors 1 or 2 with the color *light red* and all the vertices of colors 3 or 4 with the color *light blue*.

Call a homothet $P' \in \mathcal{P}_0$ *bad* if it contains exactly $c_H$ points from $\mathcal{S}$ and all of them are of the same color. Obviously, if there are no bad homothets, then we are done since $m > c_H$. Suppose that $P'$ is a bad homothet of $P$. Observe that $\mathcal{DT}[P']$ is a tree, for otherwise it would contain a cycle which in turn would contain a triangle by Lemma 2.2. That triangle must be 3-colored in the initial 4-coloring, so not all of its points can be light red or light blue, contradicting the monochromaticity of the points in $P'$.

Since $P$ has Property H and $P'$ contains $c_H$ points, $P'$ contains two good 2-paths $x$-$y$-$z$ and $x'$-$y'$-$z'$ such that $yy' \in \mathcal{DT}$. We associate them with $P'$. Suppose that $P'$ is a bad light red homothet of $P$. Then one of $y$ and $y'$ was originally colored 1 and the other was originally colored 2. Recolor the one whose original color was 1 with the color *dark blue*. Similarly, if $P'$ is a bad light blue homothet of $P$, then one of $y$ and $y'$ was originally colored 3 and the other was originally colored 4. In this case we recolor the one whose original color was 3 with the color *dark red*. Repeat this for every bad homothet, and, finally, in order to obtain a 2-coloring, merge the color classes light red and dark red into one color class — red, and merge the color classes light blue and dark blue into one color class — blue.

**Lemma 3.4.** *There is no homothet $P' \in \mathcal{P}_0$ that contains $m$ points from $\mathcal{S}$ all of which of the same color.*

*Proof.* Suppose for contradiction that $P'$ is a homothet of $P$ that contains $m$ points from $\mathcal{S}$ all of which of the same color. We may assume without loss of generality that all the points in $P'$ are colored red, therefore, before the final recoloring each point in $P'$ was either light red or dark red. Recall that $n$ is the number of vertices of $P$. We consider two cases based on the number of dark red points in $P'$.

**Case 1: There are at most $n$ dark red points in $P'$.** By Theorem 5 there is a set $\mathcal{P}'$ of at most $f(n)$ homothets of $P$ whose union is $P'$ such that no dark red point in $P'$ is in the interior of one of these homothets. Using Lemma 2.3 we can change these homothets slightly such that none of them contains a dark red point yet all light red points are still covered by these homothets. Thus the at least $m - n = c_H f(n)$ light red points are covered by these at most $f(n)$ homothets. By the pigeonhole principle one of these homothets, denote it by $P''$, contains at least $\frac{c_H f(n)}{f(n)} = c_H$ light red points and no other points. However, in this case it follows from Lemma 2.3 that there is a bad (light red) homothet in $\mathcal{P}_0$ that contains exactly $c_H$ points from $\mathcal{S} \cap P''$. Therefore, the coloring algorithm should have found within this bad homothet two good 2-paths $x$-$y$-$z$ and $x'$-$y'$-$z'$ such that $yy' \in \mathcal{DT}$ and color one of $y$ and $y'$ with dark blue and then blue. This contradicts the assumption that all the points in $P'$ are red.

**Case 2: There are more than $n$ dark red points in $P'$.** Let $y$ be one of these dark red points. Then there is a good 2-path $x$-$y$-$z$ within a bad light blue homothet $P_y \in \mathcal{P}_0$ with whom this 2-path is associated. Furthermore, the original color of $y$ is 3 and therefore the

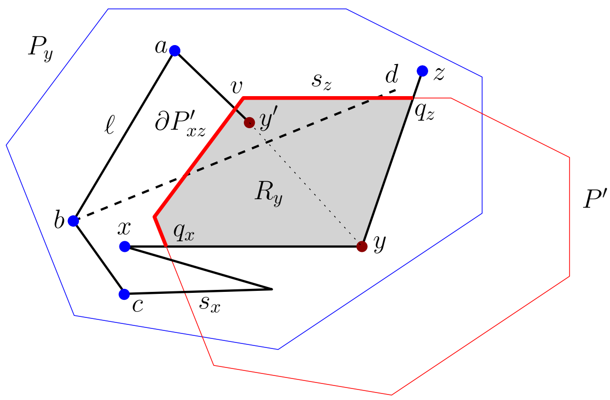



Figure 5: An illustration for the proof of Lemma 3.4.

original color of $x$ and $z$ is 4, and thus their final color is blue. It follows that $P'$ separates $x$-$y$-$z$, moreover, since $x$-$y$-$z$ is a good 2-path the edges $yx$ and $yz$ cross different sides of $P'$. Let $s_x$ (resp., $s_z$) be the side of $P'$ that is crossed by $yx$ (resp., $yz$), and let $q_x$ (resp., $q_z$) be the crossing point of $yx$ and $s_x$ (resp., $yz$ and $s_z$). See Figure 5. Note that $\partial P'$ and $\partial P_y$ cross each other exactly twice. Indeed, this follows from Lemma 2.1 and the fact that there are points from $\mathcal{S}$ in each of $P' \cap P_y$ (e.g., $y$), $P_y \setminus P'$ (e.g., $x$ and $z$) and $P' \setminus P_y$ (since $|P' \cap \mathcal{S}| \geq c > c_H = |P_y \cap \mathcal{S}|$). The points $q_x$ and $q_z$ partition $\partial P'$ into two parts $\partial P'_1$ and $\partial P'_2$. Note that since $q_x, q_z \in P' \cap P_y$ the two crossing points between $\partial P'$ and $\partial P_y$ must lie either in $\partial P'_1$ or in $\partial P'_2$. Assume without loss of generality that both of them lie in $\partial P'_1$. Thus $\partial P'_2 \subset P_y$. Let $v$ be a vertex of $P'$ in $\partial P'_2$ (note that since $s_x \neq s_z$ each of $\partial P'_1$ and $\partial P'_2$ contains a vertex of $P'$). We associate the vertex $v$ with the dark red point $y$. We also define $R_y$ to be the region whose boundary consists of segment $yq_x$ of $yx$, the segment $yq_z$ of $yz$, and the part of $\partial P'_2$ whose endpoints are $q_x$ and $q_z$ (call this part $\partial P'_{xz}$). Observe that $R_y \subseteq P' \cap P_y$.

**Claim 3.5.** *There is no other dark red point but $y$ in $R_y$.*

*Proof.* Suppose that the claim is false and let $y' \neq y$ be a dark red point in $R_y$. Since $x$ and $y'$ both lie in (the light blue) homothet $P_y$, they are connected by a path in $\mathcal{DT}[P_y]$ that alternates between points of colors 3 and 4 (considering the initial 4-coloring). We may assume without loss of generality that $y'$ is the first point in $R_y$ along this path from $x$ to $y'$: indeed, there are no points of color 4 in $R_y$, and if there is point of color 3 before $y'$, then we can name it $y'$. Denote by $\ell$ the path (in $\mathcal{DT}$) from $y$ to $y'$ that consists of the edge $yx$ and the above-mentioned path from $x$ to $y'$. Consider the polygon $\hat{P}$ whose boundary consists of $\ell$ and a straight-line segment $yy'$ ($\hat{P}$ is not a homothet of $P$). Since $y'$ and $y$ are the only vertices of $\hat{P}$ in $R_y$, there is no edge of $\ell$ that crosses $yy'$. Indeed, if there was such an edge then it would split $P'$ into two parts such that one contains $y$ and the other contains $y'$. This would contradict Corollary 2.5. Hence $\hat{P}$ is a simple polygon.

Since every simple polygon has at least three convex vertices, $\hat{P}$ has a convex vertex different from $y$ and $y'$ (thus this vertex is not in $R_y$). Denote this vertex by $b$ and let $a$ and $c$ be its neighbors along $\ell$ such that $\angle abc < \pi$. Observe that since $\angle abc < \pi$ it is impossible that the unbounded face of $\mathcal{DT}$ is incident to $a, b, c$ and lies inside $\hat{P}$. Therefore, since the initial color of $a$ and $c$ is the same (thus $ac \notin \mathcal{DT}$), it follows from Proposition 2.6 that there is a neighbor of $b$ in $\mathcal{DT}$ in between $a$ and $c$. Furthermore, it is not hard to see that there is such a neighbor $d$ whose initial color is 1 or 2. Note that $\hat{P} \subseteq P_y$ since all of its edges are inside $P_y$. Thus $d \notin \hat{P}$ and also $d \notin P_y$ since $P_y$ does not contain vertices of color 1 or 2. Now consider the directed edge $bd$: it starts inside $\hat{P}$ (since $d$ is in between $a$ and $c$)

and so it must cross $yy'$. Before doing so $bd$ must cross $\partial R_y$ and so it crosses $\partial P'_{xz}$, since it cannot cross $yq_z$ or $yq_x$. After crossing $yy'$, the directed edge $bd$ must cross $\partial P'_{xz}$ again, since $d \notin R_y$. But then $bd$ splits $P'$ into two parts such that one contains $y$ and the other contains $y'$, which is impossible by Corollary 2.5. □

In a similar way to the one described above, we associate a vertex of $P'$ with every dark red point in $P'$. Since there are more than $n$ dark red points in $P'$, there are two of them, denote them by $y$ and $y'$, that are associated with the same vertex of $P'$, denote it by $v$. Let $x$-$y$-$z$ (resp., $x'$-$y'$-$z'$) be the good 2-path that corresponds to $y$ (resp., $y'$) as above. Let $yq_x$ and $yq_z$ (resp., $yq_{x'}$ and $yq_{z'}$) be the edge-segments of $yx$ and $yz$ (resp., $y'x'$ and $y'z'$) as above, and let $R_y$ (resp., $R_{y'}$) be the region as defined above.

It follows from Claim 3.5 that $y \notin R_{y'}$ and $y' \notin R_y$. However, $\partial R_y$ and $\partial R'_y$ both contain $v$. This implies that one of the segments $yq_x$ and $yq_z$ crosses one of the segments $y'q_{x'}$ and $y'q_{z'}$, which is impossible since these are segments of edges of a plane graph. Lemma 3.4 is proved. □

To complete the proof of Theorem 9, we need to argue that the described algorithm runs in polynomial-time. Indeed, constructing the generalized Delaunay triangulation and 4-coloring it can be done in polynomial-time. Recall that there are at most $O(|\mathcal{S}|^3)$ combinatorially different homothets of $P$. Among them, we need to consider those that contain exactly $c_H$ points, and for each such monochromatic (bad) homothet $P'$ we need to find two good 2-paths in $\mathcal{DT}[P']$ whose midpoints are also neighbors, for the final recoloring step. This takes a constant time for every bad homothet, since $c_H$ is a constant. Therefore, the overall running time is polynomial with respect to the size of $\mathcal{S}$.

### 3.2 Triangles: proof of Lemma 3.2

Let $T$ be a triangle, let $\mathcal{S}$ be a set of points which is nice and in very general position with respect to $T$, and let $\mathcal{DT} = \mathcal{DT}(T, \mathcal{S})$ be the generalized Delaunay triangulation of $\mathcal{S}$ with respect to $T$. By applying an affine transformation, if needed, we may assume without loss of generality that $T$ is an equilateral triangle. Suppose that $T'$ is a homothet of $T$ that contains at least 7382 points from $\mathcal{S}$ and that $\mathcal{DT}[T']$ is a tree. We will show that $T'$ contains two good 2-paths $x$-$y$-$z$ and $x'$-$y'$-$z'$ such that $yy'$ is an edge in $\mathcal{DT}$.

It follows from the proof of Lemma 2.7 that for every point $v \in T' \cap \mathcal{S}$ we have $\deg_{\mathcal{DT}[T']} \leq 9$. Since $\mathcal{DT}[T']$ is a tree with at least $7382 = 1 + 9 + 9^2 + 9^3 + 9^4 + 1$ vertices of maximum degree 9, it contains a simple path of length 9. Let $Z = v_1$-$v_2$-$\ldots$-$v_{10}$ be such a path. We will prove that there is $2 \leq i \leq 8$ such that $v_{i-1}$-$v_i$-$v_{i+1}$ and $v_i$-$v_{i+1}$-$v_{i+2}$ are good 2-paths, and therefore $T$ satisfies Property H. Call a 2-path $v_{i-1}$-$v_i$-$v_{i+1}$ $(2 \leq i \leq 9)$ *bad* if it is not good, that is, there is a homothet of $T$, $T_i$, such that $T_i$ contains $v_i$, does not contain $v_{i-1}$ and $v_{i+1}$, and the edges $v_iv_{i-1}$ and $v_iv_{i+1}$ cross the same side of $T_i$.

Denote the sides of $T$ by $s_1, s_2, s_3$. For $j = 1, 2, 3$, let $B_j$ be the set of bad 2-paths $v_{i-1}$-$v_i$-$v_{i+1}$ such that there is a homothet $T_i$ that contains $v_i$ and does not contain $v_{i-1}$ and $v_{i+1}$, and the edges $v_iv_{i-1}$ and $v_iv_{i+1}$ both cross the side of $T_i$ that is homothetic to $s_j$. Suppose for contradiction that $Z$ does not contain two consecutive good 2-paths. Then, at least one of the sets $B_j$ contains two bad 2-paths. Assume without loss of generality that $B_1$ contains two bad 2-paths $v_{i-1}$-$v_i$-$v_{i+1}$ and $v_{k-1}$-$v_k$-$v_{k+1}$ such that $i < k$. We may further assume that $s_1$ is horizontal and that $T$ lies above it.

There is a homothet of $T$ that separates $v_{i-1}$-$v_i$-$v_{i+1}$ such that $v_{i-1}v_i$ and $v_iv_{i+1}$ both cross its side that is homothetic to $s_1$, therefore both $v_{i-1}$ and $v_{i+1}$ lie below $v_i$. Similarly, both $v_{k-1}$ and $v_{k+1}$ lie below $v_k$. Let $v_r$ be the lowest point among $v_i, \ldots, v_k$. Since $v_{i+1}$ is lower than $v_i$ and $v_{k-1}$ is lower than $v_k$ it follows that $r \neq i, k$ and so $v_r$ is lower than $v_{r-1}$

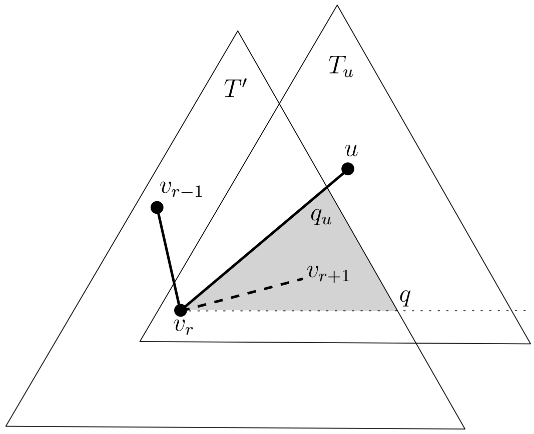



Figure 6: An illustration for the proof of Lemma 3.2.

and $v_{r+1}$. Suppose without loss of generality that $v_{r+1}$ is to the right of the line through $v_r$ and $v_{r-1}$. It follows from Proposition 2.6 that there is at least one neighbor of $v_r$ between $v_{r-1}$ and $v_{r+1}$ that lies outside of $T'$. Let $u$ be such a neighbor of $v_r$ and let $T_u$ be a homothet of $T$ that contains $v_r$ and $u$ and no other point from $\mathcal{S}$. Note that $u$ is higher than $v_r$, thus $v_r u$ crosses either the right or the left side of $T'$. Suppose without loss of generality that $v_r u$ crosses the right side of $T'$ at a point $q_u$ (refer to Figure 6). It follows that the right side of $T_u$ is to the right of the right side of $T'$. Thus, a horizontal ray that begins at $v_r$ and goes to the right will first cross the right side of $T'$ (denote this crossing point by $q$) and then cross the right side of $T_u$ (note that this ray does not cross the left sides of $T_u$ and $T'$ since $v_r \in T_u \cap T'$). Now consider the triangle $\triangle q_u q v_r$. All of its vertices are in $T_u \cap T'$, therefore $\triangle q_u q v_r \in T_u \cap T'$. However, since $v_{r+1}$ follows $u$ in the rotation of $v_r$, it follows that the edge $v_r v_{r+1}$ lies in $\triangle q_u q v_r$ since it cannot cross none of its sides. This is impossible since $v_{r+1}$ should be outside of $T_u$ and hence outside of $\triangle q_u q v_r$.

### 3.3 Parallelograms: proof of Lemma 3.3

Let $Q$ be a parallelogram, let $\mathcal{S}$ be a set of points which is nice and in very general position with respect to $Q$, and let $\mathcal{DT} = \mathcal{DT}(Q, \mathcal{S})$ be the generalized Delaunay triangulation of $\mathcal{S}$ with respect to $Q$. By applying an affine transformation, we may assume without loss of generality that $Q$ is an axis-parallel square. Since $\mathcal{S}$ is in very general position, no two points in $\mathcal{S}$ share the same $x$- or $y$-coordinate.

Suppose that $Q'$ is a homothet of $Q$ that contains at least 148 points from $\mathcal{S}$ and that $\mathcal{DT}[Q']$ is a tree. We will show that $Q'$ contains two good 2-paths $x$-$y$-$z$ and $x'$-$y'$-$z'$ such that $yy'$ is an edge in $\mathcal{DT}$.

Let $q \in \mathcal{S}$ be a point. We partition the points of the plane into four open quadrants according to their position with respect to $q$: NE($q$) (North-East), NW($q$) (North-West), SE($q$) (South-East), and SW($q$) (South-West).

**Proposition 3.6.** *Let $x, y, z$ be three points in $\mathcal{S}$ such that $xy$ and $xz$ are edges in $\mathcal{DT}$. Then for every quadrant* Qd $\in$ {NW, NE, SW, SE} *if $y \in$ Qd($x$) then $z \notin$ Qd($y$).*

*Proof.* Suppose for contradiction and without loss of generality that $y \in$ NE($x$) and $z \in$ NE($y$). Then the smallest rectangle that contains $x$ and $z$ has $x$ at its bottom-left corner, $z$ at its top-right corner and $y$ in its interior. Therefore, there is no square that contains $x$ and $z$ and does not contain $y$ and so $xz$ cannot be an edge in $\mathcal{DT}$. □

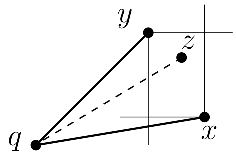



Figure 7: An illustration for the proof of Proposition 3.7.

**Proposition 3.7.** *For every point $q \in \mathcal{S} \cap Q'$ there are no two neighbors of $q$ in $\mathcal{DT}[Q']$ that lie in the same quadrant of $q$.*

*Proof.* Suppose for contradiction that $q$ has two neighbors, $x$ and $y$, that lie in the same quadrant. Assume without loss of generality that $x, y \in \text{NE}(q)$, such that $qx$ forms a smaller angle with the $x$-axis than $qy$ (refer to Figure 7). It follows from Proposition 2.6 that there is a point $z \notin Q'$ such that $z$ is a neighbor of $q$ in $\mathcal{DT}$ and is in between $x$ and $y$ in the rotation of $q$. By Proposition 3.6 we have $y \notin \text{NE}(x)$. Since $qx$ forms a smaller angle with the $x$-axis than $qy$ we have $y \notin \text{SE}(x)$. If $y \in \text{SW}(x)$ then $x \in \text{NE}(y)$ which is impossible by Proposition 3.6. Thus, $y \in \text{NW}(x)$. Using the same arguments we get that $x \in \text{SE}(y) \cap \text{SE}(z)$, $y \in \text{NW}(x) \cap \text{NW}(z)$, and $z \in \text{NW}(x) \cap \text{SE}(y)$. However, the latter implies that $z$ is contained in any axis-parallel rectangle that contains $x$ and $y$ and thus $z \in Q'$, a contradiction. □

The following will also be useful.

**Proposition 3.8.** *Let $x$ and $y$ be two neighbors of $q$ in $\mathcal{DT}[Q']$. Let $z \notin Q'$ be a neighbor of $q$ in $\mathcal{DT}$ that lies between $x$ and $y$ in the rotation of $q$ and let $Q_z$ be a square that contains $q$ and $z$ and no other point from $\mathcal{S}$. Then:*

- *if $x \in \text{NW}(q)$ and $y \in \text{NE}(q)$, then $qz$ crosses the top side of $Q'$, $x$ is to the left of $Q_z$ and $y$ is to the right of $Q_z$;*
- *if $x \in \text{NE}(q)$ and $y \in \text{SE}(q)$, then $qz$ crosses the right side of $Q'$, $x$ is above $Q_z$ and $y$ is below $Q_z$;*
- *if $x \in \text{SE}(q)$ and $y \in \text{SW}(q)$, then $qz$ crosses the bottom side of $Q'$, $x$ is to the right of $Q_z$ and $y$ is to the left of $Q_z$; and*
- *if $x \in \text{SW}(q)$ and $y \in \text{NW}(q)$, then $qz$ crosses the left side of $Q'$, $x$ is below $Q_z$ and $y$ is above $Q_z$.*

*Proof.* By symmetry it is enough to consider the first case, that is, $x \in \text{NW}(q)$ and $y \in \text{NE}(q)$. Since $z$ is between $x$ and $y$ in the rotation of $q$ we have $z \notin \text{SW}(x)$ and $z \notin \text{SE}(y)$. By Proposition 3.6 $z \notin \text{NW}(x)$ and $z \notin \text{NE}(y)$. Thus $z$ is to the right of $x$ and to the left of $y$. It follows that $z$ is above $Q'$ and $qz$ crosses the top side of $Q'$. Therefore, the top side of $Q_z$ is above $Q'$. Thus, $qx$ (resp., $qy$) cannot cross the top side of $Q_z$ so it must cross its left (resp., right) side. This implies that $x$ (resp., $y$) lies to the left (resp., right) of $Q_z$. □

Call a (simple) path in $\mathcal{DT}$ *$x$-monotone* (resp., $y$-monotone) if there is no vertical (resp., horizontal) line that intersects the path in more than one point.

**Proposition 3.9.** *Every path in $\mathcal{DT}[Q']$ is $x$-monotone or $y$-monotone.*

*Proof.* Suppose for contradiction that there is a path $p := q_1$-$q_2$-…-$q_k$ which is neither $x$-monotone nor $y$-monotone. Since $p$ is a polygonal path, it follows that there are two points, $q_i$ and $q_j$, that are “witnesses” to the non-$x$- and non-$y$-monotonicity of $p$, respectively. That is, both $q_{i-i}$ and $q_{i+1}$ are to the left of $q_i$ or both of them are to its right, and both $q_{j-1}$ and $q_{j+1}$ are above $q_j$ or both of them are below $q_j$. We choose $i$ and $j$ such that $|i - j|$

is minimized, and assume without loss of generality that $i < j$ (note that it follows from Proposition 3.7 that $i \neq j$). Thus, the sub-path $p' := q_i\text{-}q_{i+1}\text{-}\ldots, q_{j-1}\text{-}q_j$ is both $x$-monotone and $y$-monotone.

By reflecting about the $x$- and/or $y$-axis if needed, we may assume that $p'$ is ascending, that is, for every $l = i, \ldots, j-1$ we have $q_{l+1} \in \mathrm{NE}(q_l)$. Then it follows from Proposition 3.7 that $q_{i-1} \in \mathrm{SE}(q_i)$ and $q_{j+1} \in \mathrm{SE}(q_j)$. By Proposition 2.6 there is a point $x \notin Q'$ which is a neighbor of $q_i$ and is between $q_{i-1}$ and $q_{i+1}$ in the rotation of $q_i$, and it follows from Proposition 3.8 that $q_i x$ crosses the right side of $Q'$. The same argument implies that there is a point $y \notin Q'$ which is a neighbor of $q_j$ and is between $q_{j+1}$ and $q_{j-1}$ in the rotation of $q_j$, such that $q_j y$ crosses the bottom side of $Q'$. However, since $q_j$ is to the right of $q_i$ and above it, the edges $q_i x$ and $q_j y$ must cross, which is impossible. □

Call a 2-path $w$-$q$-$z$ *bad* if it is not good, that is, there is an axis-parallel square $Q''$ that contains $q$, does not contain $w$ and $z$, and $qw$ and $qz$ are edges in $\mathcal{DT}$ that cross the same side of $Q''$. We say that $w$-$q$-$z$ is a bad *left* 2-path if $qw$ and $qz$ cross the left side of $Q''$, and define *right*, *top*, and *bottom* bad 2-paths analogously.

**Proposition 3.10.** *Let $w$-$q$-$z$ be a 2-path. Then:*

- *$w$-$q$-$z$ is a bad left 2-path iff $w \in \mathrm{SW}(q)$ and $z \in \mathrm{NW}(q)$, or vice versa;*
- *$w$-$q$-$z$ is a bad right 2-path iff $w \in \mathrm{SE}(q)$ and $z \in \mathrm{NE}(q)$, or vice versa;*
- *$w$-$q$-$z$ is a bad top 2-path iff $w \in \mathrm{NW}(q)$ and $z \in \mathrm{NE}(q)$, or vice versa; and*
- *$w$-$q$-$z$ is a bad bottom 2-path iff $w \in \mathrm{SW}(q)$ and $z \in \mathrm{SE}(q)$, or vice versa.*

*Proof.* By symmetry it is enough to consider the first claim. If $w$-$q$-$z$ is a bad left 2-path, then there is a square $Q''$ that separates it such that the edges $qw$ and $qz$ cross the left side of $Q''$. Therefore, these edges goes leftwards from $q$ and so $w, z \in \mathrm{SW}(q) \cup \mathrm{NW}(q)$. It follows from Proposition 3.7 that $w \in \mathrm{SW}(q)$ and $z \in \mathrm{NW}(q)$, or vice versa.

For the other direction, assume without loss of generality that $w \in \mathrm{SW}(q)$ and $z \in \mathrm{NW}(q)$. Let $Q''$ be the square whose left side is the straight-line segment between $((q)_x - \varepsilon, (w)_y)$ and $((q)_x - \varepsilon, (z)_y)$, for some small $\varepsilon > 0$. Then $Q''$ separates $w$-$q$-$z$ and both $qw$ and $qz$ cross its left side, therefore, $w$-$q$-$z$ is a bad left 2-path. □

**Proposition 3.11.** *Every path in $\mathcal{DT}[Q']$ contains at most four bad 2-paths.*

*Proof.* Let $p := q_1\text{-}q_2\text{-}\ldots\text{-}q_k$ be a simple path in $\mathcal{DT}[Q']$ and suppose for a contradiction that $p$ contains at least five bad 2-paths. By Proposition 3.9 the path $p$ is $x$-monotone or $y$-monotone. Assume without loss of generality that $p$ is $y$-monotone and that it goes upwards, that is, $q_{i+1}$ is above $q_i$ for every $i = 1, 2, \ldots, k-1$. It follows that $p$ does not contain bad top or bad bottom 2-paths, for otherwise it would not be $y$-monotone. It is not hard to see that bad left and bad right 2-paths must alternate along $p$, that is, between every two bad left 2-paths there is a bad right 2-path and vice versa.

Consider the first five such bad 2-paths along the path $p$, and denote them by $q_{i_1-1}\text{-}q_{i_1}\text{-}q_{i_1+1}$, $q_{i_2-1}\text{-}q_{i_2}\text{-}q_{i_2+1}$, $q_{i_3-1}\text{-}q_{i_3}\text{-}q_{i_3+1}$, $q_{i_4-1}\text{-}q_{i_4}\text{-}q_{i_4+1}$ and $q_{i_5-1}\text{-}q_{i_5}\text{-}q_{i_5+1}$. By symmetry we may assume without loss of generality that $q_{i_1-1}\text{-}q_{i_1}\text{-}q_{i_1+1}$ is a bad left 2-path, and therefore $q_{i_3-1}\text{-}q_{i_3}\text{-}q_{i_3+1}$ and $q_{i_5-1}\text{-}q_{i_5}\text{-}q_{i_5+1}$ are also bad left 2-paths, whereas the 2-paths $q_{i_2-1}\text{-}q_{i_2}\text{-}q_{i_2+1}$ and $q_{i_4-1}\text{-}q_{i_4}\text{-}q_{i_4+1}$ are bad right.

Note that we may assume without loss of generality that $q_{i_1}$ is to the right of $q_{i_4}$, for otherwise $q_{i_5}$ must be to the right of $q_{i_2}$ and by reflecting about the $x$-axis and renaming the points we get the desired assumption. By Proposition 2.6, $q_{i_1}$ has a neighbor $z \notin Q'$ between $q_{i_1-1}$ and $q_{i_1+1}$ in the rotation of $q_{i_1}$. Let $Q_z$ be a square that contains $q_{i_1}$ and $z$ and no other point from $\mathcal{S}$ and let $s_z$ be its side length (refer to Figure 8). It follows from

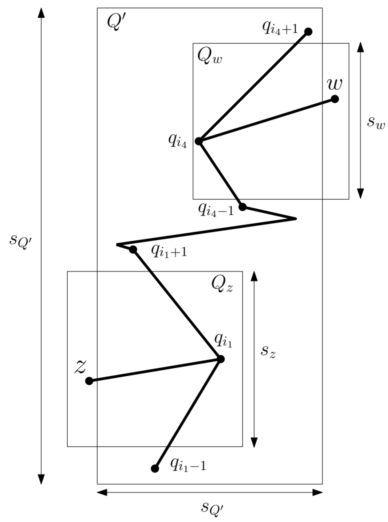



Figure 8: An illustration for the proof of Proposition 3.11.

Proposition 3.8 that $q_{i_1-1}$ lies below $Q_z$, $q_{i_1+1}$ lies above $Q_z$, and $z$ lies to the left of $Q'$. Therefore, $(q_{i_1+1})_y - (q_{i_1-1})_y > s_z$. Similarly, $q_{i_4}$ has a neighbor $w \notin Q'$ between $q_{i_4+1}$ and $q_{i_4-1}$ in the rotation of $q_{i_4}$. Let $Q_w$ be a square that contains $q_{i_4}$ and $w$ and no other point from $\mathcal{S}$ and let $s_w$ be its side length. Then $q_{i_4-1}$ lies below $Q_w$, $q_{i_4+1}$ lies above $Q_w$, and $w$ lies to the right of $Q'$. Therefore, $(q_{i_4+1})_y - (q_{i_4-1})_y > s_w$.

Note that since $q_{i_1}$ is to the right of $q_{i_4}$ and $z$ and $w$ are to the left and to the right of $Q'$, respectively, we have $s_z + s_w > ((q_{i_1})_x - (z)_x) + ((w)_x - (q_{i_4})_x) > s_{Q'}$, where $s_{Q'}$ is the side length of $Q'$. Observe also that since there are at least two other vertices between $q_{i_1}$ and $q_{i_4}$ along $p$, we have that $q_{i_1+1} \neq q_{i_4-1}$, and thus $q_{i_1+1}$ lies below $q_{i_4-1}$. This implies that $((q_{i_1+1})_y - (q_{i_1-1})_y) + ((q_{i_4+1})_y - (q_{i_4-1})_y) < s_{Q'}$. Combining the inequalities we get, $s_{Q'} > ((q_{i_1+1})_y - (q_{i_1-1})_y) + ((q_{i_4+1})_y - (q_{i_4-1})_y) > s_z + s_w > ((q_{i_1})_x - (z)_x) + ((w)_x - (q_{i_4})_x) > s_{Q'}$, a contradiction. □

To complete the proof of Lemma 3.3 we will consider a path of length 11 in $\mathcal{DT}[Q']$. It follows from Proposition 3.7 that for every $q \in \mathcal{S} \cap Q'$ we have $\deg_{\mathcal{DT}[Q']}(q) \leq 4$. This implies that if $Q'$ contains at least $1 + \sum_{i=1}^{5} 4^i = 1366$ points from $\mathcal{S}$ then $\mathcal{DT}[Q']$ contains a simple path of length at least 11. However, one can show that fewer points suffice to guarantee the existence of a path of length 11. For this we need the following.

**Proposition 3.12.** *There are at most four points in $\mathcal{S} \cap Q'$ whose degree in $\mathcal{DT}[Q']$ is greater than two.*

*Proof.* Suppose for contradiction that there are five points in $\mathcal{S} \cap Q'$ whose degrees in $\mathcal{DT}[Q']$ are greater than two. Since each of these points must have a neighbor in at least three of its four quadrants, it follows that there are two points, denote them by $q$ and $q'$, that have neighbors in the same three quadrants (they may differ in the fourth quadrant).

Since $\mathcal{DT}[Q']$ is a tree, it contains a path between $q$ and $q'$. Denote this path by $p := q\text{-}q_1\text{-}q_2\text{-}\dots\text{-}q_k\text{-}q'$. By symmetry we may assume that $q_1 \in \text{NE}(q)$. Since $\deg_{\mathcal{DT}[Q']}(q) \geq 3$, it follows from Proposition 3.7 that $q$ has a neighbor $z$ in $\text{NW}(q)$ or $\text{SE}(q)$. Again, by symmetry we may assume that $z \in \text{NW}(q)$. Therefore, the path $z\text{-}q\text{-}q_1\text{-}\dots\text{-}q'$ is not $y$-monotone and $q$ (and its neighbors in this path) is a witness for this fact.

Recall that $q_1 \in \text{NE}(q)$, or equivalently, $q \in \text{SW}(q_1)$. It follows from Proposition 3.7 that $q_2 \notin \text{SW}(q_1)$. Note that if $q_2 \in \text{NW}(q_1)$ then $z\text{-}q\text{-}q_1\text{-}q_2$ is also not $x$-monotone, which is impossible by Proposition 3.9. Thus, either $q_2 \in \text{NE}(q_1)$ or $q_2 \in \text{SE}(q_1)$. For the same arguments, it follows that for every $i = 1, \ldots, k-1$ we have either $q_{i+1} \in \mathrm{NE}(q_i)$ or $q_{i+1} \in \mathrm{SE}(q_i)$, and, $q' \in \mathrm{NE}(q_k)$ or $q' \in \mathrm{SE}(q_k)$.

Suppose that $q' \in \mathrm{NE}(q_k)$. If $q'$ has a neighbor $z' \in \mathrm{NW}(q')$, then the path $z$-$q$-$q_1$-$\ldots$-$q_k$-$q'$-$z'$ is not $x$-monotone and not $y$-monotone. It follows that $q'$ and, hence, also $q$, have neighbors in each of the other quadrants. Specifically, there are $w \in \mathrm{SE}(q)$ and $w' \in \mathrm{SE}(q')$. But then the path $w$-$q$-$q_1$-$\ldots$-$q_k$-$q'$-$w'$ is not $x$-monotone and not $y$-monotone. Therefore, $q' \in \mathrm{SE}(q_k)$. If $q'$ has a neighbor $z' \in \mathrm{SW}(q')$, then the path $z$-$q$-$q_1$-$\ldots$-$q_k$-$q'$-$z'$ is not $x$-monotone and not $y$-monotone. It follows that $q'$ and, hence, also $q$, have neighbors in each of the other quadrants. Specifically, there are $w \in \mathrm{SE}(q)$ and $w' \in \mathrm{NE}(q')$. But then the path $w$-$q$-$q_1$-$\ldots$-$q_k$-$q'$-$w'$ is not $x$-monotone and not $y$-monotone. □

It follows from Proposition 3.12 that if $Q'$ contains at least 148 points from $\mathcal{S}$, then $\mathcal{DT}[Q']$ contains a simple path of length 11. Indeed, by removing a vertex of degree at most four from a graph the number of connected components increases by at most three. Since by Proposition 3.12 there are at most four vertices of degree greater than two in $\mathcal{DT}[Q']$, by removing these vertices we get a graph with at most 13 connected components and at least 144 vertices whose degrees are at most two. Therefore, each connected component is either a path or a cycle, and one of them contains at least 12 vertices.

Let $p := q_1$-$q_2$-$\ldots$-$q_{12}$ be a simple path of length 11 in $\mathcal{DT}[Q']$. By Proposition 3.11 there are at most four bad 2-paths $q_{i-1}$-$q_i$-$q_{i+1}$ in $p$. Therefore, there is $2 \leq i \leq 10$ such that $q_{i-1}$-$q_i$-$q_{i+1}$ and $q_i$-$q_{i+1}$-$q_{i+2}$ are good 2-paths, and therefore $Q$ satisfies Property H. Lemma 3.3 is proved.

# 4 Discussion

We have presented a general framework showing that if a convex polygon $P$ satisfies a certain property (namely, Property H defined in Section 3.1), then there is an absolute constant $m$ that depends only on $P$ such that every set of points in the plane can be 2-colored such that every homothet of $P$ that contains at least $m$ points contains points of both colors.

We then used this framework for 2-coloring points with respect to squares, showing that one can 2-color any set of points in the plane such that every square that contains at least 1484 points from the point set contains points of both colors. It would be intersecting to improve this constant, as it is most likely not the best possible, and any such improvement would also improve the bound for polychromatic colorings (Corollary 1.1). The results for squares apply also for homothets of any given parallelogram by affine transformations. The main open problem related to our work is the following.

**Problem 10.** *Is it true that for every convex polygon $P$ there is a constant $m = m(P)$ such that it is possible to 2-color any set of points $\mathcal{S}$ such that every homothet of $P$ that contains at least $m$ points from $\mathcal{S}$ contains points of both colors?*

By Theorem 9 it is enough to show that every convex polygon satisfies Property H. We have demonstrated that any triangle satisfies Property H, and thus provided a new proof for the known positive answer to Problem 10 for triangles. However, for other classes of convex polygons it seems that additional ideas are needed, in light of the following.

**Lemma 4.1.** *For every integer $n \geq 4$ there exists a convex polygon with $n$ vertices that does not satisfy property H.*

*Proof.* Assume first that $n = 4$ and let $m$ be a constant. We will construct a set of points $\mathcal{S}$, a convex quadrilateral $P$, and a homothet of $P$, denote it by $P'$, such that $|\mathcal{S} \cap P'| = m$,

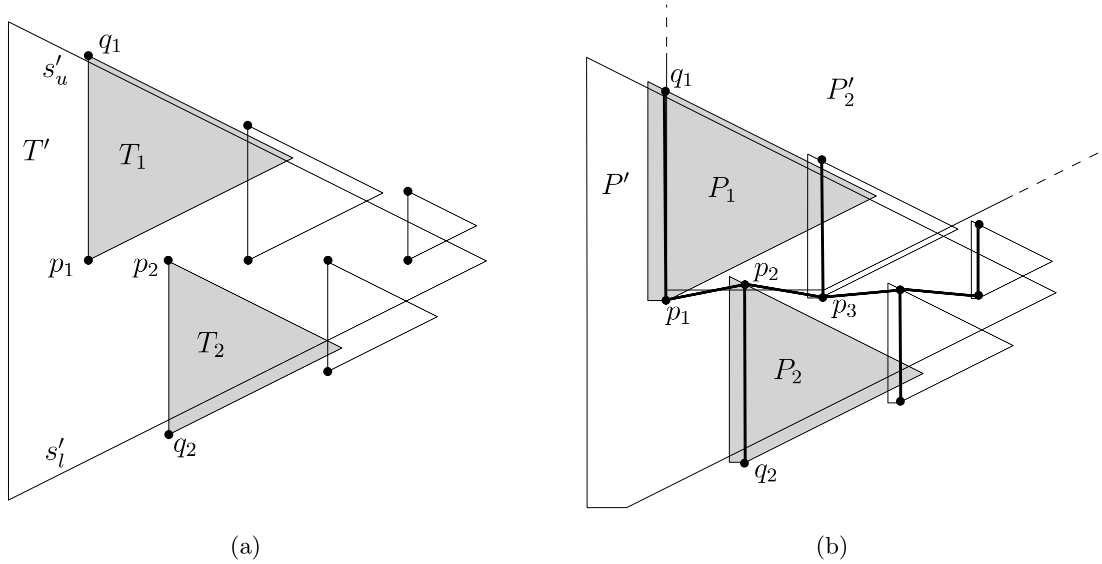



Figure 9: Constructing a convex quadrilateral that does not satisfy Property H.

$\mathcal{DT}(P, \mathcal{S})[P']$ is a path, and $P'$ does not contain two good 2-paths whose midpoints are connected by a Delaunay edge. This will imply that $P$ does not satisfy Property H.

Let $T$ be the triangle whose vertices are $(0, 1)$, $(0, -1)$, and $(2, 0)$. Let $\mathcal{S}'$ be the set of points $p_i := (i, 0)$, for $i = 1, \ldots, m$, and let $T'$ be a homothet of $T$ such that the vertical side of $T'$ lies on the $y$-axis and its right vertex is at $(m + 1, 0)$. Note that $T'$ contains $\mathcal{S}'$. Denote by $s'_u$ and $s'_l$ the upper and lower sides of $T'$, respectively. For every point $p_i$ define a homothet of $T$, denote it by $T_i$, such that if $i$ is odd (resp., even) the lower (resp., upper) vertex of $T_i$ is $p_i$ and its upper (resp., lower) vertex lies at a vertical distance $\frac{i}{1000m}$ above $s'_u$ (resp., below $s'_l$). For every triangle $T_i$ add a point $q_i$ at the endpoint of its vertical side that is different from $p_i$. See Figure 9a for an example. Let $\mathcal{S}''$ be the point set that consists of the points $p_i$ and $q_i$, for $i = 1, \ldots, m$. Note that for every $i$ the triangle $T_i$ contains only the points $p_i$ and $q_i$ from $\mathcal{S}''$.

Next we apply a small perturbation of the points in $\mathcal{S}'$ and slightly scale and translate the triangles $T_i$ such that: (1) $T_i$ still contains the points $p_i$ and $q_i$ and no other point from $\mathcal{S}''$, for every $i = 1, \ldots, m$; and (2) for every odd (resp., even) $i$ the point $p_i$ lies slightly below (resp., above) the $x$-axis. We now replace every homothet of $T$ with a 'similar' convex quadrilateral $P$ as follows. We obtain a convex quadrilateral $P$ by 'trimming' the lower vertex of $T$. That is, we replace a small neighborhood of the lower vertex of $T$ by a very short horizontal edge that connects the left and lower sides of $T$ (see Figure 9b). Every homothet of $T$ is then replaced by a "very similar" homothet of $P$. Let $P'$ be the homothet that replaces $T'$ and let $P_i$ be the homothet of $P$ that replaces $T_i$, for $i = 1, \ldots, m$. We set the length of the horizontal side of $P$ such that the length of its corresponding side in $P'$ is, say, 1/10000. Note that we can replace the triangles with quadrilaterals such that every quadrilateral contains the same set of points from $\mathcal{S}''$ that belong to the triangle that it replaces. To complete the construction we add some points to $\mathcal{S}''$, as in the proof of Theorem 9, to obtain a nice set of points with respect to $P$ and perturb the points to obtain a set of points in very general position with respect to $P$. Denote this set of points by $\mathcal{S}$.

Let $\mathcal{DT} = \mathcal{DT}(P, \mathcal{S})$ be the generalized Delaunay triangulation of $\mathcal{S}$ with respect to $P$. Observe that $\mathcal{DT}[P']$ consists of the path $p_1$-$p_2$-$\ldots$-$p_m$. Indeed, for every $i = 1, \ldots, m - 1$ there is a homothet of $P$ that contains $p_i$ and $p_{i+1}$ and no other point from $\mathcal{S}$. Furthermore, for every $i = 2, \ldots, m - 1$ there is an edge $p_iq_i$ that lies between the edges $p_ip_{i-1}$ and $p_ip_{i+1}$

in the rotation of $p_i$ in $\mathcal{DT}$. Therefore, if $P'$ contains two good 2-paths whose midpoints are connected by an edge, then they must be of the form $p_{i-1}$-$p_i$-$p_{i+1}$ and $p_i$-$p_{i+1}$-$p_{i+2}$ for some $2 \le i \le m-2$. Thus, it is enough to prove that for every even $i$ the 2-path $p_{i-1}$-$p_i$-$p_{i+1}$ is not good. Suppose for contradiction that $p_{i-1}$-$p_i$-$p_{i+1}$ is a good 2-path for some even $i$. Since $i$ is even, $p_i$ lies slightly above the $x$-axis whereas $p_{i-1}$ and $p_{i+1}$ lie slightly below the $x$-axis. Let $P'_i$ be a homothet of $P$ such that the endpoints of its bottom side are $(i-1,0)$ and $(i+1,0)$. It follows that $p_i$ lies inside $P'_i$ whereas $p_{i-1}$ and $p_{i+1}$ are outside of $P'_i$. Moreover, the edges $p_ip_{i-1}$ and $p_ip_{i+1}$ both cross the bottom side of $P'_i$. Therefore, the 2-path $p_{i-1}$-$p_i$-$p_{i+1}$ is not a good 2-path, a contradiction.

Note that if $n > 4$, then we can replace the right vertex of $P$ (and all of its homothets) with a very small convex chain of $n-3$ vertices, and again obtain a convex $n$-gon and a construction that shows that this $n$-gon does not satisfy Property H. □

We conclude with two interesting related open problems. Considering coloring of points with respect to disks, recall that Pálvölgyi [26] recently proved that there is no constant $m$ such that any set of points in the plane can be 2-colored such that any (unit) disk that contains at least $m$ points from the given set is non-monochromatic (that is, contains points of both colors). Coloring the points with four colors such that any disk that contains at least two points is non-monochromatic is easy since the (generalized) Delaunay graph is planar. Therefore, it remains an interesting open problem whether there is a constant $m$ such that any set of points in the plane can be 3-colored such that any disk that contains at least $m$ points is non-monochromatic.

Perhaps the most interesting and challenging problem of coloring geometric hypergraphs is to color a planar set of points $\mathcal{S}$ with the minimum possible number of colors, such that every axis-parallel rectangle that contains at least two points from $\mathcal{S}$ is non-monochromatic. It is known that $\Omega(\log(|\mathcal{S}|)/\log^2\log(|\mathcal{S}|))$ colors are sometimes needed [10], and it is conjectured that $polylog(|\mathcal{S}|)$ colors always suffice. The latter holds when considering rectangles that contain at least three points [1], however, for the original question only polynomial upper bounds are known [3, 9, 12, 23].

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