# PATHS ON THE SPHERE WITHOUT SMALL ANGLES

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Abstract. It is a recent result that given a finitely many points on $\mathbb{R}^2$, it is possible to arrange them on a polygonal path so that every angle on the polygonal path is at least $\pi/9$. Here we extend this result to finite sets contained in the 2-dimensional sphere.

## 1. Introduction and results

Let $X$ be a finite set in $\mathbb{R}^2$. An ordering, $x_1, x_2, \dots, x_n$, of the points of $X$ gives rise to a polygonal path $p = x_1 x_2 \dots x_n$ on $X$: its edges are the segments connecting $x_i$ to $x_{i+1}$. The angle of $p$ at $x_i$ is just $\angle x_{i-1}x_i x_{i+1}$. The path is called $\alpha$-good if all of its angles are at least $\alpha$ where $\alpha > 0$. Answering a question of Sándor Fekete [3] from 1992, (cf [4] as well) we proved in [1] the following result.

**Theorem 1.** *If $X$ is a finite set in the plane, then there is an $\alpha_0$-good path on $X$ with $\alpha_0 = 20^\circ = \pi/9$.*

The aim of this paper is to extend this result to finite sets $X \subset S^2$, the 2-dimensional Euclidean sphere. The definitions are almost the same. Given $a, b \in S^2$ there is a shortest path $\widehat{ab} \subset S^2$ connecting $a$ and $b$ in $S^2$. This shortest path is an arc of the great circle containing $a$ and $b$, and is unique unless $a$ and $b$ are antipodal. An ordering, $x_1, x_2, \dots, x_n$, of the points of $X$ is identified with a path $x_1 x_2 \dots x_n$ on $X$ consisting of the arcs $\widehat{x_i x_{i+1}}$. The angle of this path at $x_i$ is just the spherical angle at $x_i$ of the spherical triangle with vertices $x_{i-1}, x_i, x_{i+1}$. The path is called $\alpha$-good if all of its angles are at least $\alpha$ where $\alpha > 0$.

**Theorem 2.** *There is $\alpha > 0$ such that for every finite set $X \subset S^2$ there exists an $\alpha$-good path on the points of $X$ (using every point of $X$ exactly once).*

The proof gives $\alpha = 5^\circ$ via generous computations. Slightly larger value for $\alpha$ can be reached by more careful calculations but we have not tried to find the best possible $\alpha$. The planar example consisting of the vertices of an equilateral triangle and its center shows that Theorem 1 cannot hold with $\alpha_0 > 30^\circ$. The same applies to the spherical case as shown by a small size

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spherical and equilateral triangle in $S^2$. Jan Kynčl [2] has proved recently that Theorem 1 holds with $\alpha = 30°$, the best possible bound. Using his results the bound $\alpha = 5°$ can be improved to $\alpha = 7°$.

The same question can be asked on higher dimensional spheres $S^d$. The methods of this paper work there as well, resulting in a smaller universal $\alpha$, see Section 6.

We will need a stronger version of Theorem 1 which is proved in [1]. To state it a few additional definitions are needed. The direction $\overline{xy}$ of a pair $x, y \in \mathbb{R}^2$ is the unit vector $(y-x)/|y-x|$, we suppose here that $x \neq y$. So $\overline{xy} \in S^1$, the unit circle.

Given a path $z_1 z_2 \dots z_n$ in the plane the directions $\overline{z_2 z_1}$ and $\overline{z_{n-1} z_n}$ are called the *end directions* of the path. We call a subset $R$ of $S^1$ a *restriction* if it is the disjoint union of two closed arcs $R_1, R_2 \subset S^1$ such that both have length $4\alpha_0$ and their distance from each other (along the unit circle) is larger than $2\alpha_0$. (Recall that $\alpha_0 = 20°$.) We call the path $z_1 \dots z_n$ *$R$-avoiding* if the path is $\alpha_0$-good and the two end directions are not in the same $R_i$ $(i = 1, 2)$.

**Theorem 3.** *Let $X$ be a finite set of points in the plane. For every restriction $R$ there is an $R$-avoiding path on all the points of $X$.*

## 2. Preparations

In the proofs to come we assume that our finite set $X \subset S^2$ contains no antipodal pair. The general case follows from this by a simple limit argument.

Given $a, b \in S^2$, the length of the arc $\widehat{ab}$ is simply the angle between the vectors $a$ and $b$, measured in degrees (sometimes in radians). Of course the length of $\widehat{ab}$ can be expressed by the Euclidean distance $|a-b|$. The pair $a, b \in X$ is a diameter of $X$ if it has the largest length among all pairs in $X$.

For the proof of Theorem 2 we need two auxiliary results. The first one is simpler: it is essentially the planar case, that is Theorem 1 applied on $S^2$. Precisely, let $P$ be a plane touching $S^2$ at a point $z \in S^2$ and let $C = C(t)$ be the cap of $S^2$ defined by

$$C(t) = \{x \in S^2 : z \cdot x \geq t\}$$

where $t \in (0, 1)$.

**Theorem 4.** *If $X \subset C(t)$ is finite, then there is an $\alpha(t)$-good path on $X$ where $\alpha(t) \in (0, 90°)$ is given by $\sin \alpha(t) = t \sin 20°$.*

The proof is given in Section 4. The following corollary to Theorem 4 will be used in the proof of Theorem 2. Note that $\alpha(1/2) = 9.846..° > 9°$. Set $\alpha_1 = 9°$.

**Corollary 1.** *If the diameter of $X$ is at most $60°$, then there is an $\alpha_1$-good path on $X$.*

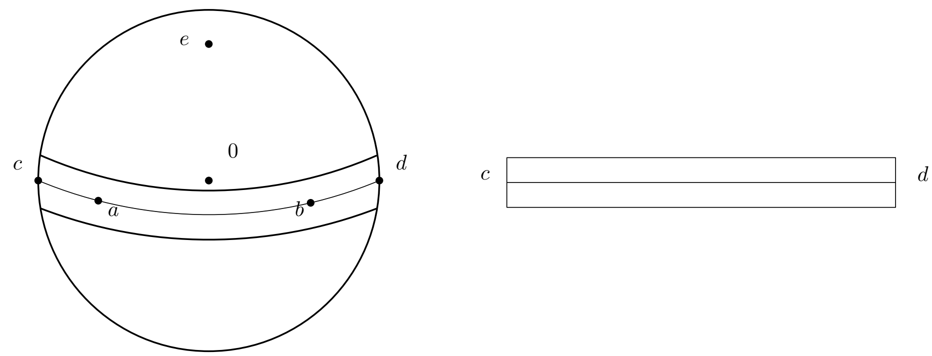


FIGURE 1. The spherical halfslab $Q(a, b)$ and its planar representation

To state the second auxiliary result we need some definitions. Let $a, b \in X$ form a diameter of $X \subset S^2$. Set $c = (a-b)/|a-b|$ so $c \in S^2$. Choose $e \in S^2$ that is orthogonal to both $c$ and $a+b$. Let $\beta = 10°$. We define the halfslab $Q = Q(a, b)$ as

$$Q = \{x \in S^2 : (a+b) \cdot x \geq 0,\ |e \cdot x| \leq \sin\beta\},$$

see Figure 1. Here is the second auxiliary result.

**Theorem 5.** *If $a, b$ form a diameter of $X$ and $X \subset Q(a, b)$, then there is an $\alpha$-good path on $X$ (where $\alpha = 5°$).*

We prove this theorem in Section 6 with some preparations in Section 5. The next section contains the proof of Theorem 2. It is essentially an induction argument reducing the problem to two cases: when $X$ lies in a cap $C(t)$ for some $t$ and when $X$ lies in the halfslab $Q(a, b)$. These two cases are covered by Theorems 4 and 5.

## 3. Proof of Theorem 2

We introduce further terminology and notation before the proof. Given $u, v \in S^2$ with $u \neq \pm v$, let $L(u; v)$ be the half of the great circle connecting $u$ to $-u$ that contains $v$. The union of $L(u; v)$ and $L(u; w)$ (when $w \notin L(u; v)$) is a closed curve without self-intersection on $S^2$ so it splits $S^2$ into two connected components to be called *sectors*. Let $E(u; v, w)$ denote the smaller one of the two. No confusion will arise here since $E(u; v, w)$ will always be much smaller than the other sector.

Note that for $x, y \in E(u; v, w)$ the arc $\widehat{xy} \subset E(u; v, w)$.

Let $L(u; z)$ be the half of the great circle exactly halving $E(u; v, w)$. Let $\gamma$ be the angle between the planes $L(u; v)$ and $L(u; z)$, we call $\gamma$ the *angle of the sector* $E(u; v, w)$. Note that this angle is at most 90° always.

We will often write $E(u; \gamma, z)$ or simply $E(u; \gamma)$ instead of $E(u; v, w)$ where $\gamma$ is the angle of this sector, especially when $v$ and $w$ are not important.

**Proof of Theorem 2.** It goes by induction on $|X|$. Everything is easy when $|X| = 1, 2$ or 3. Suppose now that $|X| > 3$. Assume that there is a spherical triangle $\triangle$ with vertices $a, b, c$ with all of its angles at least $2\alpha$ which is not contained in any sector $E(z;\alpha)$ when $z \in X$. Then induction works as follows. Find first an $\alpha$-good path $p = x_1x_2 \dots x_n$ on $X \setminus \{a, b, c\}$. Define $E(x_1;\alpha)$ as $E(x_1;\alpha, x_2)$ if $n > 1$ (that is, $|X| > 4$), and as $E(x_1;\alpha, c)$ if $n = 1$. As some vertex of $\triangle$, say $a$, is not contained in $E(x_1;\alpha)$, $ax_1x_2 \dots x_n$ is an $\alpha$-good path. The angle of $\triangle$ at $a$ is at least $2\alpha$ so either $\angle bax_1$ or $\angle cax_1 \geq \alpha$. Suppose, say, that $\angle bax_1 \geq \alpha$. Then $cbax_1 \dots x_n$ is an $\alpha$-good path on $X$, even $\angle cba \geq 2\alpha$.

So we can assume that no such triangle $\triangle$ exists. If the diameter of $X$ is at most $60^\circ$, then Corollary 1 applies and gives an $\alpha_1$-good path on $X$ (where $\alpha_1 = 9^\circ$). So suppose that the diameter, formed by the pair $a, b \in X$ is at least $60^\circ$.

Observe now that $\widehat{ab}$ is contained in no sector $E(z;\alpha)$ with $z \in X \setminus \{a, b\}$. Indeed, $\widehat{ab}$ is the longest side of the spherical triangle with vertices $a, b, z$. Then the largest angle of this triangle is at vertex $z$, and this largest angle is more than $60^\circ > 2\alpha$.

We claim now that no point of $X$ is outside of the set

$$F := E(a; 2\alpha, b) \cup E(b; 2\alpha, a).$$

Assume the contrary and let $c \in X \setminus F$. All angles of the spherical triangle $\triangle$ with vertices $a, b, c$ are larger than $2\alpha$: the angle at $c$ is at least $60^\circ > 2\alpha$ as we just saw, while for the angles at $a, b$ this follows from $c \notin F$. The triangle $\triangle$ is not contained in any sector $E(z;\alpha)$ for $z \in X \setminus \{a, b\}$ as $\widehat{ab}$ is not contained in such a sector. Further $\triangle \subset E(a;\alpha)$ is impossible because $c \notin E(a; 2\alpha, b)$, and $\triangle \subset E(b;\alpha)$ cannot hold for the same reason. Thus $\triangle$ is not contained in any sector $E(z,\alpha)$, $z \in X$, contradicting our previous assumption.

Consequently

$$X \subset F \cap \{x \in S^2 : |x - a|, |x - b| \leq |a - b|\}.$$

We observe that the set $F \cap \{x \in S^2 : |x - a|, |x - b| \leq |a - b|\}$ is contained in the halfslab $Q(a, b)$. Then Theorem 5 applies and finishes the proof. □

## 4. Proof of Theorem 4

For $x \in C(t)$ let $x^*$ denote its radial projection (from the origin which is the center of $S^2$) to $P$. Then $X^*$, the radial projection of $X$, is a finite set in the plane $P$. So by Theorem 1 there is a polygonal path $p^* = x_1^* \dots x_n^*$ on $X^*$ with all of its angles at least $20^\circ$. The next lemma implies that the path $p = x_1 \dots x_n$ on $X$ is $\alpha(t)$-good.

**Lemma 1.** *Assume $a, b, c \in C(t)$. Let the angle of the spherical triangle $abc$ at $c$ be $\phi < 90^\circ$ and that of the (planar) triangle $a^*b^*c^*$ at $c^*$ be $\phi^*$. Then $\sin\phi \geq t \sin\phi^*$ if $\phi^* \leq 90^\circ$ and $\sin\phi \geq t$ if $\phi^* > 90^\circ$.*

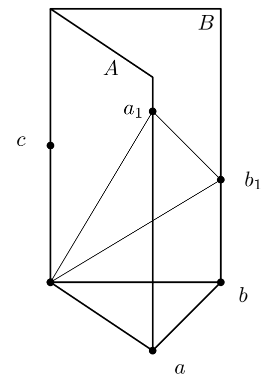


FIGURE 2. Proof of Lemma 1

**Proof.** Let $K \subset \mathbb{R}^3$ be the cone consisting of all the points of the form $\alpha a + \beta b + \gamma c$ where $\alpha, \beta \geq 0$ and $\gamma \in \mathbb{R}$. Its boundary consists of two halfplanes $A = \{\alpha a + \gamma c : \alpha \geq 0\}$ and $B = \{\beta b + \gamma c : \beta \geq 0\}$. The angle of this cone is $\phi \in (0, 180°)$, which is the same as the angle between the two halfplanes $A, B$. The plane $P$ that is tangent to $S^2$ at $z$ intersects $K$ in a 2-dimensional cone with angle $\phi^*$. Translate $P$ by $-z$. The translated copy $P_1$ contains the origin and intersects $K$ in a 2-dimensional cone whose angle is also $\phi^*$. We assume first that $\phi^* \leq 90°$.

The condition $c \in C(t)$ implies that $z \cdot c \geq t$.

Let $S$ be the unit circle centered at the origin in the plane orthogonal to $c$. We can assume that $a = S \cap A$ and $b = S \cap B$ as the angle $\phi$ remains the same. The plane $P_1$ intersects the lines $\{a + \lambda c : \lambda \in \mathbb{R}\}$ resp. $\{b + \lambda c : \lambda \in \mathbb{R}\}$ in points $a_1$ and $b_1$. Let $T$ resp. $T_1$ be the triangle with vertices $0, a, b$ and $0, a_1, b_1$, see Figure 2.

Then Area $T = \frac{1}{2} \sin \phi$, and Area $T_1 = \frac{1}{2} |a_1| \cdot |b_1| \sin \phi^* \geq \frac{1}{2} \sin \phi^*$ since $|a_1|, |b_1| \geq 1$. As $T$ is the orthogonal projection of $T_1$ to the plane orthogonal to $c$, Area $T = \cos \gamma$Area $T_1$ where $\gamma$ is the angle of the planes containing $T$ and $T_1$. Here $\cos \gamma = c \cdot z$ so we have

$$\sin \phi \geq c \cdot z \sin \phi^* \geq t \sin \phi^*$$

finishing the proof when $\phi^* \leq 90°$.

In the case $\phi^* > \frac{\pi}{2}$ fix $c$ and $b$ and rotate $a$ towards $b$ around the line through 0 and $c$. The angles $\phi$ and $\phi^*$ will continuously decrease. Rotate $a$ till $\phi_1^* = 90°$. Now $\sin \phi \geq \sin \phi_1 \geq t \sin \phi_1^* = t$ which finishes the proof. $\square$

## 5. Decreasing paths

Some preparations are needed before the proof of Theorem 5. We assume that $S^2$ is centered at the origin. For $A \subset \mathbb{R}^3$ we let lin $A$ denote the linear hull of $A$. We call the 2-dimensional plane $H = \text{lin}\,\{a, b\}$ *the horizontal*

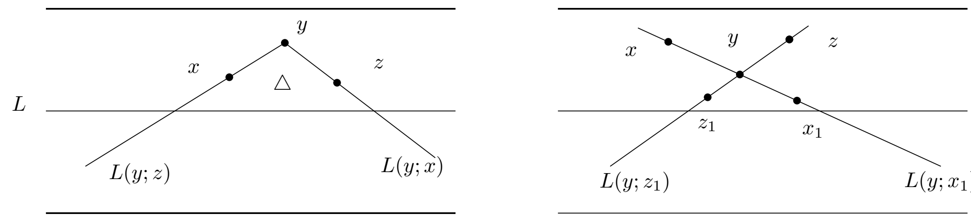


FIGURE 3. The two cases in Proposition 1

*plane*. $H$ intersects the halfslab $Q = Q(a, b)$ in the halfcircle $L = L(c; a)$ whose endpoints are $c$ and $d = -c$. Let $e$ be the unit normal vector of $H$.

The *slope* of a pair $x, y \in X$ is the angle between $H$ and the 2-plane $\text{lin}\,\{x, y\}$. We denote this angle by $\sigma(x, y)$. Note that $\sigma(x, y) \in [0, 90°]$ always. We call a pair $x, y \in X$ *steep* if $\sigma(x, y) \geq 40^0$.

If there is no steep pair in $X$, then one can construct an $\alpha_2$-good path on $X$ with $\alpha_2 = 100°$ very easily. For $x \in Q$ let $h(x) = e \cdot x$ (the height of $x$) and let $\tau(x)$ be the angle between $c$ and the midpoint of the half great circle $L(e; x)$. (Thus for instance, $\tau(c) = 0$ and $\tau(d) = 180°$). Order the points of $X$ by decreasing $\tau(x)$ and call the resulting path the *decreasing path* of $X$. The following proposition shows that all angles of the decreasing path are at least $180° - 2 \cdot 40° = 100°$.

**Proposition 1.** *Assume $x, y, z \in Q$ and let $\gamma$ be the angle of the spherical triangle with vertices $x, y, z$ at vertex $y$. If $\tau(x) \leq \tau(y) \leq \tau(z)$, then $\gamma \geq 180° - \sigma(x, y) - \sigma(y, z)$.*

**Proof.** We may assume by symmetry that $h(y) \geq 0$. To simplify the proof we also assume that $\tau(x) < \tau(y) < \tau(z)$ and $h(y) > 0$. The general case follows from this by a simple limit argument.

Observe next that $x$ can be replaced by any point (distinct from $y$) on the arc $\widehat{xy}$. The same applies to $z$. So we assume that $x$ and $z$ are close to $y$, in particular, $h(x), h(z) > 0$.

The first and basic case is when $z$ lies below the plane $\text{lin}\,\{x, y\}$. Then the half-circles $L(y; x), L(y; z)$ and the great circle $H \cap S^2$ delimit a spherical triangle $\triangle$, see Figure 3 left. The angle of $\triangle$ at $y$ coincides with $\angle xyz$, and its other two angles are $\sigma(x, y)$ and $\sigma(y, z)$. Thus $\angle xyz \geq 180° - \sigma(x, y) - \sigma(y, z)$, indeed.

The second case is when $z$ is above the plane $\text{lin}\,\{x, y\}$. Choose points $x_1$ and $z_1$ in $S^2$ close to, but distinct from, $y$ so that $y \in \widehat{xx_1}$ and $y \in \widehat{zz_1}$, see Figure 3 right. Then $\tau(z_1) < \tau(y) < \tau(x_1)$, and $h(z_1), h(y), h(x_1)$ are all positive, and $x_1$ lies below the plane $\text{lin}\,\{z_1, y\}$. The previous basic case applies now to $z_1, y, x_1$ in place of $x, y, z$. Thus $\angle z_1 y x_1 \geq 180° - \sigma(z_1, y) - \sigma(y, x_1)$. Here $\angle z_1 y x_1 = \angle xyz$ and $\sigma(z_1, y) = \sigma(y, z)$ and $\sigma(y, x_1) = \sigma(x, y)$. Consequently $\angle xyz \geq 180° - \sigma(x, y) - \sigma(y, z)$ again. $\square$

We remark that the decreasing path method is applicable to any subset, say $Y$, of $X$ that contains no steep pair. In that case the decreasing path on $Y$ is $\alpha_2$-good.

## 6. Proof of Theorem 5

For $u, v \in L$ with $\tau(u) < \tau(v)$ we define

$$T(u, v) = \{x \in Q : \tau(u) \le \tau(x) \le \tau(v)\}.$$

Now let $u, v \in L$ be two points with $\tau(v) - \tau(u) = 30^\circ$. Thus $c, u, v, d$ come on $L$ in this order.

**Proposition 2.** *If $x, y \in X$ with $x \in T(c, u)$ and $y \in T(v, d)$, then $\sigma(x, y) < 35^\circ$.*

**Proof.** The spherical cotangent formula (see spherical trigonometry on wikipedia, for instance) says that $\cos c \cos B = \cot a \sin c - \cot A \sin B$ where $a, b, c$ are the sides, and $A, B, C$ the opposite angles of the spherical triangle. With $b = 10^\circ,\ c = 15^\circ,\ B = 90^\circ$ this shows that the angle in question is at most $\operatorname{arccot}(\cot 10^\circ \sin 15^\circ) = 34.2656..$, indeed smaller than $35^\circ$. $\square$

Define now $t = \sin 15^\circ \cos 10^\circ = 0.25488..$ and set $\alpha_3 \in (0, 90^\circ)$ by

$$\sin \alpha_3 = t \sin 20^\circ = 0.087176..$$

and $\alpha_3 > 5^\circ$ follows.

**Lemma 2.** *Assume again that $u, v \in L$ with $\tau(v) - \tau(u) = 30^\circ$, and that $\tau(u) \in [90^\circ, 120^\circ]$ and further that there is no steep pair from $X$ in $T(u, v)$. Set $Y = X \cap T(c, v)$. Then there is an $\alpha_3$-good path $y_1 \dots y_m$ on $Y$ such that $\angle x y_1 y_2 > 5^\circ$ for every $x \in T(v, d)$.*

**Proof.** The conditions imply that $\tau(v) \le 150^\circ$. Then a simple computation shows that $Y$ is contained in a cap $C(t)$ with center $z \in L$ where $t = \sin 15^\circ \cos 10^\circ$. This value for $t = \cos b$ comes from the spherical cosine theorem $\cos b = \cos c \cos a + \sin c \sin b \cos B$ with $B = 90^\circ,\ c = 75^\circ,\ a = 10^\circ$. Project $Y$ radially to the plane $P$ that touches $S^2$ at $z$. We get a finite set $Y^*$ in $P$. The unit circle $S \subset P$ is centered at $z$. Let $R = R_1 \cup R_2 \subset S$ be the restriction in $P$ where the line $H \cap P$ halves both $R_1$ and $R_2$. The radial projection $c^*$ of $c$ lies in $P$ and we choose the names so that $c^* \in R_1$.

According to Theorem 3, there is a $20^\circ$-good path $y_1^* y_2^* \dots y_m^*$ on $Y^*$ which is $R$-avoiding, that is, not both end directions are in the same $R_i$. Here we choose the names so that $\overline{y_2^* y_1^*} \notin R_1$. Theorem 4 implies that $y_1 \dots y_m$ is an $\alpha_3$-good path on $Y \subset S^2$.

We have to check that $\angle x y_1 y_2 > 5^\circ$ for every $x \in T(v, d)$. We distinguish two cases.

**Case 1.** When $y_1, y_2$ is not a steep pair. Then, as is easy to check, the angle between the line spanned by $y_1^*, y_2^*$ and the line $H \cap P$ is smaller than

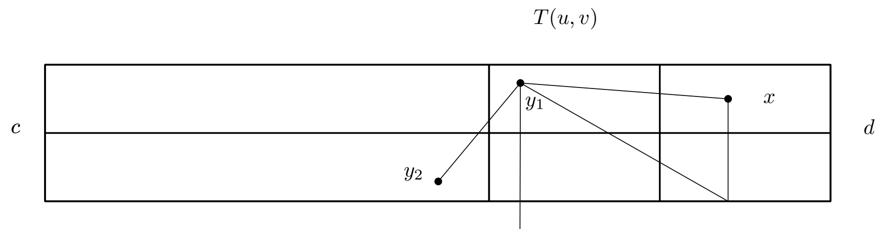


FIGURE 4. Case 2a

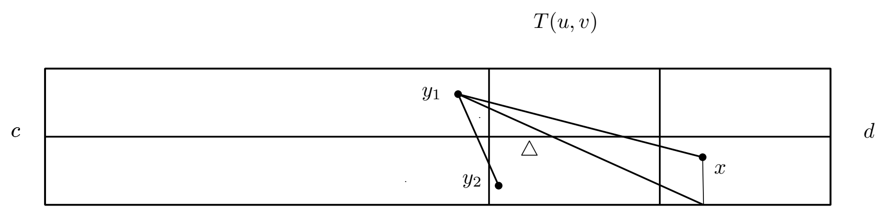


FIGURE 5. Case 2b

$40°$, so $\overline{y_2^* y_1^*} \in R_2$. Then $\tau(y_2) < \tau(y_1) \le \tau(x)$. Proposition 1 shows that $\angle xy_1y_2 \ge 180° - 90° - 40° > 5°$.

**Case 2.** When $y_1, y_2$ is a steep pair. Then at least one of $y_1$ and $y_2$ is in $T(c,u)$. We assume by symmetry that $h(y_1) \ge h(y_2)$. Clearly $\tau(x) > \tau(y_1), \tau(y_2)$. There are two subcases.

**Case 2a.** When $\tau(y_2) \le \tau(y_1)$. Then $y_2 \in T(c,u)$ and the angle in question decreases if $x$ is pushed down to $h(x) = -\sin 10°$ while keeping $\tau(x)$ the same. The halfcircle $L(e; y_1)$ cuts the angle $\angle xy_1y_2$ into two parts, see Figure 4. Assume that $\angle xy_1y_2 \le 5°$, then both parts are at most $5°$. The spherical cosine theorem implies then that $\tau(y_1) - \tau(y_2) \le 5°$ and $\tau(x) - \tau(y_1) \le 5°$, contradicting $\tau(x) - \tau(y_2) \ge 30°$.

**Case 2b**. When $\tau(y_1) \le \tau(y_2)$. Then $y_1 \in T(c,u)$. The angle in question decreases again if $x$ is pushed down to $h(x) = -\sin 10°$ while keeping $\tau(x)$ the same. Note that while $x$ is pushed down, $y_1, y_2$ and $x$ do not become coplanar as otherwise $y_1, x$ would become a steep pair contradicting Proposition 2. Let $\triangle$ be the spherical triangle delimited by $L, L(x; y_1), L(y_2; y_1)$, see Figure 5. The angle of $\triangle$ at vertex $y_1$ equals $\angle y_2y_1x$. The other two angles of $\triangle$ are $180° - \sigma(y_1, y_2) \le 140°$ because $y_1, y_2$ is a steep pair, and $\sigma(y_1, x) < 35°$ by Proposition 2. Thus $\angle y_2y_1x > 180° - (140° + 35°) = 5°$. $\square$

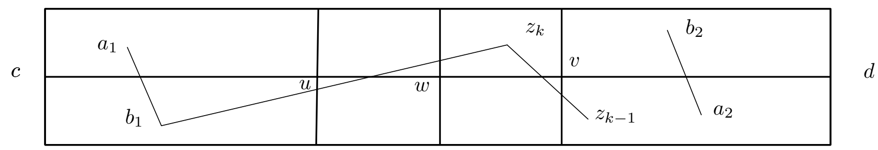


FIGURE 6. Part of the constructed path

**Proof of Theorem 5**. We have to consider two cases.

**Case 1.** There is no steep pair in $T(u,d)$ where $\tau(u) = 120°$. We can apply Lemma 2 to $T(u,v)$: setting $Y = X \cap T(c,v)$ there is no steep pair from $Y$ in $T(u,v)$. We get an $\alpha_3$-good path $y_1 \dots y_m$ on $Y$. Let $x_1 \dots x_k$ (where $m+k=n$) be the decreasing path on the points of $X \setminus Y$ which is an $\alpha_2$-good path on $X \setminus Y$ ($\alpha_2 = 100°$). We claim that $x_1 \dots x_k y_1 \dots y_m$ is an $\alpha$-good path on $X$. We only have to check its angles at $x_k$ and $y_1$. The angle at $y_1$ is at least $\alpha$ by Lemma 2. The pair $x_{k-1}x_k$ is not steep and $\tau(y_1) \le \tau(x_k) < \tau(x_{k-1})$. Then Proposition 1 shows that the angle at $x_k$ is at least $180° - 90° - 40° > 5°$.

The same method works when there is no steep pair in $T(c,v)$ where $\tau(v) = 60°$.

Define now $u, v, w \in L$ by $\tau(u) = 60°$, $\tau(w) = 90°$, and $\tau(v) = 120°$. We are left with the following case.

**Case 2.** There is a steep pair $a_1, b_1 \in X \cap T(c,u)$ and a steep pair $a_2, b_2 \in X \cap T(v,d))$. By swapping names if necessary we may assume that $\tau(a_1) \le \tau(b_1)$ and that $\tau(b_2) \le \tau(a_2)$. Set $Y = T(c,w) \cap X \setminus \{a_1, b_1\}$ and $Z = T(w,d) \cap X \setminus \{a_2, b_2\}$. Lemma 2 applies to $T(w,v)$ and $Y$ because there is no steep pair from $Y$ in $T(w,v)$ (actually, no point of $Y$ there at all). We get an $\alpha_3$-good path $y_1 \dots y_m$ on $Y$ such that $\angle b_2 y_1 y_2 > 5°$. The same lemma applies to $Z$ and $T(u,w)$ giving an $\alpha_3$-good path $z_1 \dots z_k$ on $Z$ with $\angle b_1 z_k z_{k-1} > 5°$. Here $m + 4 + k = n$, and the case when either $Y$ or $Z$ is empty or singleton is easy.

We claim finally that $z_1 \dots z_k b_1 a_1 a_2 b_2 y_1 \dots y_m$ is an $\alpha$-good path, see Figure 6. We have to check the angles at $a_1, b_1$ and also at $a_2, b_2$ but the latter would follow by symmetry. Observe that $\tau(a_1) < \tau(b_1) \le \tau(z_k)$ and $\sigma(z_k, b_1) < 35°$. Then Proposition 1 shows that the angle at $b_1$ is at least $180° - 90° - 35° > 5°$. Finally, the pair $a_1, b_1$ is steep and $\tau(a_1) \le \tau(b_1)$, and $\tau(a_2) - \tau(a_1) \ge 60° > 30°$. The spherical triangle with vertices $b_1, a_1, a_2$ satisfies the same conditions as the triangle $y_2, y_1, x$ in Case 2b in the proof of Lemma 2. The same argument shows then that the angle at $a_1$ is larger than $5°$. $\square$

## 7. Higher dimensions

In the paper [1] we proved the higher dimension analogue of Theorem 1 in the following form.

**Theorem 6.** *For every $d \geq 2$ there is a positive $\alpha_d$ such that for every finite set of points $X \subset \mathbb{R}^d$ there exists an $\alpha_d$-good path on $X$.*

Here the value of $\alpha_d$ is $\pi/80$ (for $d > 2$), see [1]. The proof of Theorem 2 goes through in higher dimensions without any real difficulty, and gives the following result.

**Theorem 7.** *There exists a constant $\alpha > 0$ such that for every $d \geq 2$ and for every finite set of points $X \subset S^d$ there exists an $\alpha$-good path on $X$.*

We omit the details.

## 8. Open problems

The same question comes up in more general settings. For instance when $X$ is a finite subset of the boundary of a convex body (compact convex set with nonempty interior) $K \subset \mathbb{R}^3$ (and $\mathbb{R}^d$, $d \geq 2$). Again there is a shortest path $\widehat{ab}$, the geodesic connecting $a, b$ in $\partial K$. So an ordering $x_1, \ldots, x_n$ of the elements of a finite set $X \subset \partial K$ gives rise to a path on $\partial K$. The angle at $x_i$ is defined in the usual way. Extending Theorem 2 would mean that there is $\alpha > 0$ such that for every convex body $K \subset \mathbb{R}^3$ and for every finite $X \subset \partial K$ there is an ordering such that every angle of the corresponding path is at least $\alpha$. We suspect that such a universal $\alpha$ exists.

The same problem can be considered on a smooth or piecewise linear manifold. We remark however that in the hyperbolic plane there are triangles with all three angles very small. The same thing occurs on other 2-dimensional manifolds for instance when they have three long "tentacles".

Here comes an abstract or combinatorial version of the same problem. Let $X$ be a finite set. For every three elements $a, b, c$ in $X$ the *combinatorial angle* is a real number $\angle abc \in [0, 1]$ satisfying the following conditions:

- $\angle abc = \angle cba$ for all $a, b, c \in X$, (symmetry),
- $\angle abc + \angle cbd \geq \angle abd$ for all $a, b, c, d \in X$, (triangle inequality),
- $\angle abc + \angle bca + \angle cab \geq 1$ for all $a, b, c \in X$, (no small triangle).

The question is now whether there exists an $\varepsilon > 0$ such that for every finite set $X$ with angles satisfying these three conditions there is an ordering $x_1, \ldots, x_n$ of the elements of $X$ such that $\angle x_{i-1}x_ix_{i+1} \geq \varepsilon$ for every $2 \leq i \leq n-1$.

It turns out that for every $n$ there exists a largest number $\varepsilon = \varepsilon(n)$ such that if $|X| = n$ there exist an $\varepsilon$-good path on $X$. In Lemma 3 below we show that if $\varepsilon(n)$ is not zero, then $\varepsilon(n) = \frac{1}{k}$ for some integer $k$. One can check the case $n = 4$ directly and show that $\varepsilon(4) = \frac{1}{6}$.

Let $X$ be a finite set and let $S$ be a subset of the combinatorial angles of $X$. We say that $S$ is *blocking* if any path on $X$ has an angle in $S$. Let $\angle_S abc$ be the smallest number $t$ such that there are $b_0, \dots, b_t \in X$, where $b_0 = a, b_t = c$ and all the combinatorial angles $\angle b_i b b_{i+1}$ for $i = 0, \dots, t-1$ are in $S$. It is possible that $\angle_S abc = \infty$. Let $\alpha(S) = \min_{a,b,c \in X}(\angle_S abc + \angle_S bca + \angle_S cab)$. It is possible that $\alpha(S) = \infty$. Define $\alpha(n) = \max_{|X|=n, S \text{ is blocking}} \alpha(S)$. Clearly $\alpha(n)$ is an integer, or $\infty$.

**Lemma 3.** *If $\alpha(n)$ is an integer, then $\varepsilon(n) = \frac{1}{\alpha(n)}$.*

**Proof.** Let $S$ be the blocking set where $\alpha(S) = \alpha(n)$. Then the abstract combinatorial geomery with $\angle abc = \frac{\angle_S abc}{\alpha(S)}$ shows that $\varepsilon(n) \le \frac{1}{\alpha(S)}$ since all the angles in $S$ have that size, and $S$ is blocking each path.

Assume that $\varepsilon(n) < \frac{1}{\alpha(n)}$. Then for some abstract geometry $X$ every path contains an angle smaller than $\frac{1}{\alpha(n)}$. Let $S$ be the set of all angles smaller than $\frac{1}{\alpha(n)}$. By definition $S$ is blocking. By the triangle inequality $\angle abc < \frac{\angle_S abc}{\alpha(n)}$ for any angle. Let $abc$ be the triangle where $\alpha(S) = \angle_S abc + \angle_S bca + \angle_S cab$. Obviously $\angle abc + \angle bca + \angle cab < \frac{\alpha(S)}{\alpha(n)} \le 1$ which is a contradiction. $\square$

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