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Optimal investment under behavioural criteria in incomplete diffusion market models*

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1 Introduction

The most commonly accepted model for investors' preferences is expected utility theory, going back to [2, 20]. According to the tenets of this theory, an investor prefers a random return X to Y if $Eu(X) \geq Eu(Y)$ for some utility function $u : \mathbb{R} \to \mathbb{R}$ that is usually assumed non-increasing and concave. More recently, other theories have emerged and pose new challenges to mathematics.

The present paper treats preferences of cumulative prospect theory (CPT), [11, 19], where an "S-shaped" u is considered (i.e. convex up to a certain point and concave from there on). Also, distorted probability measures are applied for calculating the utility of a given position with respect to a (possibly random) benchmark G. We remark that techniques of the present paper easily carry over to other types of preferences, too, such as rank-dependent utility [15] or acceptability indices [8].

The theory of optimal portfolio choice for CPT preferences is in its infancy yet. Continuous-time studies almost always assume a complete market model, [3, 10, 7, 5, 17]. Only very specific types of incomplete continuous-time models have been treated to date (finite mixtures of complete models; the case where the price is a martingale under the physical measure; the case where the market price of risk is deterministic), see [18, 16]. In the present paper we make a step forward and consider incomplete models of a diffusion type where the return of the investment in consideration depends on some economic factors. Our main result asserts, under mild assumptions, the existence of an optimal strategy when the driving noise of the economic factors is independent of that of the investment and the rate of return is non-negative. The independence condition is, admittedly, rather stringent and does not allow a leverage effect (see [4]).

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We are also able to accomodate models of a specific type where the factor may have non-zero correlation with the investment. We think that our results open the door for further generalizations.

2 Optimal investment model under behavioural criteria

In this section definitions and notation related to the problem of behavioural optimal investment are presented, based on [12], [6].

Unfortunately, most of the techniques developed in the literature for finding optimal policies rely on either the Markovian nature of the problem or on convex duality. These are no longer applicable under behavioural criteria. For this reason we shall consider a weak-type formulation of the control problem associated with optimal investment (Subsection 2.1). Introducing a relaxation of the problem for which results in [12] apply, we can prove the existence of an optimal investment strategy (Subsection 2.3).

2.1 The setting: market and preferences

Fix a finite horizon $T > 0$. We consider a financial market consisting of a risky asset, whose discounted price $(S_t)_{0\leqslant t\leqslant T}$ depends on economic factors. These factors are described by a d-dimensional stochastic processes $\{Y_t\}_{t\geqslant 0}$. Without entering into rigorous definitions at this point, our market model is described by the equations

$$dY_t = \nu_t\left(Y_\cdot\right)dt + \kappa_t\left(Y_\cdot\right)dB_t, \text{ and } Y_0 = y, \tag{1}$$

$$dS_t = \theta_t\left(Y_\cdot\right)S_t dt + \lambda_t\left(Y_\cdot\right)S_t dW_t \text{ and } S_0 = s > 0, \tag{2}$$

with B, W independent standard Brownian motions of appropriate dimensions.

We also assume that there is a riskless asset of constant price equal to 1. We shall be more specific later in this section. Stochastic volatility models provide prime examples of financial market models with dynamics (1) and (2), see [9].

The investor trades in the risky and riskless assets, investing a proportion $\phi_t \in [0, 1]$ of his wealth into the risky asset at time t. This leads to the following equation for the wealth of the investor at time t:

$$dX_t = \phi_t\theta_t\left(Y_\cdot\right)X_t dt + \phi_t\lambda_t\left(Y_\cdot\right)X_t dW_t \text{ and } X_0 = x, \tag{3}$$

where $x > 0$ is the investor's initial capital.

Borrowing and short selling are not allowed, hence ϕ_t is a process taking values in $[0, 1]$. We note that, in this model, the risky asset's price has no influence on the economic factors. We will see in Section 3 below how this assumption can be weakened.

We will need certain closedness results on the laws of Itô processes from [12] hence it is necessary to work in the 'weak' setting of stochastic control theory, where the underlying probability space is not fixed.

We first set out the requirements for the coefficients in (1), (3).

Let $C([0,T];\mathbb{R}^n)$ denote the family of $\mathbb{R}^n$-valued continuous functions on $[0,T]$.

Denote by $p_t : C([0,T],\mathbb{R}^d) \to \mathbb{R}^d$ the projections $p_t(x.) = x_t$ and define the σ-algebras $\mathscr{N}_t = \sigma(\{p_s : s \leqslant t\})$, and $\mathscr{N} = \sigma(\{p_s : s \leqslant T\})$.

Definition 2.1. Let $\nu(t, y.) : [0,T] \times C([0,T],\mathbb{R}^d) \to \mathbb{R}^d$ be such that the restriction of ν to $[0,t] \times C([0,T];\mathbb{R}^d)$ is $\mathcal{B}([0,t]) \otimes \mathscr{N}_t$-measurable, for any $0 \leqslant t \leqslant T$. We shall denote this functional by either $\nu_t(y.)$ or $\nu(t, y.)$.

Similarly, we define the coefficients θ, λ, κ with the same measurability properties as ν, but with values in $\mathbb{R}$, $\mathbb{R}$ and S_+^d, respectively, where S_+^d denotes the set of real, symmetric and positive semidefinite $d \times d$ matrices.

Definition 2.2. An investment stategy π is given by the following collection:

$$\pi := \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leqslant t \leqslant T}, \mathbb{P}, X_t, Y_t, \phi_t, (B_t, W_t), (x, y)\right),$$

with $x > 0$ and $y \in \mathbb{R}^d$, where

(a) $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leqslant t \leqslant T}, \mathbb{P}\right)$ is a complete filtered probability space whose filtration satisfies the usual conditions;

(b) the process $(B_t, W_t)_{t \geqslant 0}$ is a standard $d+1$-dimensional $\mathcal{F}_t$-Wiener process;

(c) $\phi_t : \Omega \times [0,T] \to [0,1]$ is $\mathcal{F} \otimes \mathcal{B}([0,T])$-measurable and $\mathcal{F}_t$-adapted;

(d) on the filtered probability space X_t, Y_t are $\mathcal{F} \otimes \mathcal{B}([0,T])$-measurable and $\mathcal{F}_t$-adapted processes such that

$$Y_t = y + \int_0^t \nu_s(Y.)\, ds + \int_0^t \kappa_s(Y.)\, dB_s, \tag{4}$$

$$X_t = x + \int_0^t \phi_s \theta_s(Y.)\, X_s ds + \int_0^t \phi_s \lambda_s(Y.)\, X_s dW_s, \tag{5}$$

for $0 \leqslant t \leqslant T$.

In other words, $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leqslant t \leqslant T}, \mathbb{P}, X_t, Y_t, (B_t, W_t), (x, y)\right)$ is a weak solution of the system of equations (1), (3). The process ϕ_t represents a ratio of investment in the risky asset, it is measurable and $\mathcal{F}_t$-adapted. We do not consider the price process S_t from (2) at all since it is enough to work with the 'controlled dynamics' X_t.

When needed, we will use the notation X^π, Y^π, etc. to indicate that the object we mean belongs to π. Let $\Pi = \Pi(x, y)$ denote the collection of all strategies.

Assumption 2.1. The functional θ is non-negative i.e. $\theta(t, y.) \geqslant 0$ for all $t \in \mathbb{R}_+$ and $y \in C([0,T];\mathbb{R}^d)$.

Remark 2.3. In other words, the return of the risky asset must be non-negative. This looks rather a harmless assumption. On the other hand, as mentioned before, (b) in Definition 2.2 is stringent. It excludes the 'leverage effect' where the volatility and the stock prices have (negative) correlation. This condition can be relaxed, see Section 3.

We now present the framework of optimal investment under CPT, as proposed in [19]. We follow [16] and [6].

The investor assesses strategies by means of utilities on gains and losses, which are described in terms of functions $u_{\pm} : \mathbb{R}_{+} \to \mathbb{R}_{+}$, by a reference point G and functions $w_{\pm} : [0,1] \to [0,1]$. The latter functions $w_{\pm}$ are introduced with the aim of explaining the distortions of her perception on the "likelihood" of her gains and losses.

According to the tenets of CPT, investors use benchmarks to asses the portfolio outcomes, this is modelled by a real-valued random variable G. The quantity G depends on economic factors as follows: let us denote by F a fixed deterministic functional $F : C\left([0,T];\mathbb{R}^d\right) \to \mathbb{R}_{+}$ which is $\mathscr{N}_T$-measurable. As the probability space is not fixed, for each $\pi \in \Pi$ we define the corresponding reference point by $G^{\pi} := F\left(Y_{\cdot}^{\pi}\right)$. That is, we assume that the benchmark is a non-negative functional of the economic factors. Results can easily be extended to the slightly more general case where $G^{\pi} := F(Y_{\cdot}^{pi}, B_{\cdot}^{\pi})$ for some functional F.

For any strategy $\pi \in \Pi$, we define the functionals

$$V_{+}(\pi) := \int_0^{\infty} w_{+}\left(\mathbb{P}^{\pi}\left(u_{+}\left(\left(X_T^{\pi} - G^{\pi}\right)_{+}\right) > t\right)\right) dt, \tag{6}$$

and

$$V_{-}(\pi) := \int_0^{\infty} w_{-}\left(\mathbb{P}^{\pi}\left(u_{-}\left(\left(X_T^{\pi} - G^{\pi}\right)_{-}\right) > t\right)\right) dt. \tag{7}$$

The optimal portfolio problem for an investor under CPT consists in maximising the following performance functional:

$$V(\pi) := V_{+}(\pi) - V_{-}(\pi), \tag{8}$$

which is defined provided that at least one of the summands is finite. Fix $x > 0$, $y \in \mathbb{R}$. Set $\Pi' := \{\pi \in \Pi(x,y) : V_{-}(\pi) < \infty\}$ and define

$$V := \sup_{\pi \in \Pi'} V(\pi). \tag{9}$$

The value V represents the maximal satisfaction achievable by investing in the stock and riskless asset in a CPT framework. Our purpose is to prove the existence of $\hat{\pi} \in \Pi'$ such that $V(\hat{\pi}) = V$.

2.2 Main result

We make the following assumptions. Recall the notation $y_t^{\star} = \sup_{s \leqslant t} |y_s|$.

Assumption 2.2. The functionals κ, λ, θ and ν are uniformly bounded on $[0,T] \times C\left([0,T];\mathbb{R}^d\right)$. Furthermore, for fixed $t \geqslant 0$ and functions $y^n, z \in C\left([0,T];\mathbb{R}^d\right)$ such that $(y^n - z)_t^\star \to 0$, $n \to \infty$ we have $\kappa_t\left(y^n_\cdot\right) \to \kappa_t\left(z_\cdot\right)$ and the same holds for the functionals λ, θ and ν. We will refer to this as the coefficients being path-continuous at any time $t \in [0,T]$.

Assumption 2.3. A (weak) solution of equation (4) exists and it is unique in law.

Assumption 2.4. We assume that $u_\pm : \mathbb{R}_+ \to \mathbb{R}_+$ and $w_\pm : [0,1] \to [0,1]$ are continuous, non-decreasing functions with $u_\pm(0) = 0$, $w_\pm(0) = 0$, $w_\pm(1) = 1$, and

$$u_+(x) \leqslant k_+\left(x^\alpha + 1\right), \text{for all } x \in \mathbb{R}_+, \tag{10}$$

$$w_+(p) \leqslant g_+ p^\gamma, \text{ for all } p \in [0,1], \tag{11}$$

with γ, $\alpha > 0$, $k_+, g_+ > 0$ fixed constants.

We denote by $L^p(\Omega,\mathbb{P})$ the usual space of p-integrable random variables on a probability space $(\Omega,\mathcal{F},\mathbb{P})$.

Assumption 2.5. There is $\vartheta > 0$ such that $\vartheta\gamma > 1$ and $G^\pi \in L^{\vartheta\gamma}\left(\Omega,\mathbb{P}^\pi\right)$ for all $\pi \in \Pi$.

Note that, under Assumption 2.3, the law of G^π is independent of π and hence Assumption 2.5 holds iff $G^\pi \in L^{\vartheta\gamma}\left(\Omega,\mathbb{P}^\pi\right)$ for one particular π.

In order to ensure that the functional V and the optimisation problem in (9) are defined over a non-empty set, we introduce the following assumption on u_-, the distortion function w_- and the reference point G^π.

Assumption 2.6. The functions w_-, u_- are such that, for all $\pi \in \Pi$,

$$\int_0^\infty w_-\left(\mathbb{P}^\pi\left(u_-\left(G^\pi\right) > y\right)\right) dy < \infty. \tag{12}$$

This assumption ensures that the set Π' is not empty. Indeed, let $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leqslant t\leqslant T},\mathbb{P}\right)$ be a filtered probability space where (1) has a solution Y_t. Then setting $\phi_t := 0$ and $X_t := x$ for all t,

$$\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leqslant t\leqslant T},\mathbb{P},x,Y_t,0,\left(B_t,W_t\right),(x,y)\right)$$

belongs to Π'. Fix $x > 0$ and $y \in \mathbb{R}$. Our main result can now be stated.

Theorem 2.4. *Under Assumptions 2.1, 2.2, 2.3, 2.4, 2.5 and 2.6 the problem (9) is well-posed, i.e. $V < \infty$. Moreover, there exists an optimal strategy $\hat{\pi} \in \Pi'$ attaining the supremum in (9), i.e. $V = V(\hat{\pi})$.*

2.3 A relaxation of the set of controls

We introduce a relaxation of the problem by extending the class of investment strategies given in Definition 2.2, we shall call this extension the class of auxiliary controls. This relaxation is introduced in order to ensure the closedness of the set of laws of the processes $(Y., X.)$.

We follow the martingale problem formulation, thus we refer to a_t and b_t as the drift/diffusion coefficients of the process (Y_t, X_t), as they appear in the martingale problem formulation of equations (4) and (5). In order to use [12], these coefficients must take values in a family of convex subsets of $S_+^{d+1} \times \mathbb{R}^{d+1}$ hence we shall consider a 'convex extension' of the set in which the coefficients in equations (4) and (5) take values.

Definition 2.5. Denote $\mathbb{A} = S_+^{d+1} \times \mathbb{R}^{d+1}$. For any pair of continuous functions $(x., y.) \in C([0,T]; \mathbb{R} \times \mathbb{R}^d)$ and for any $t \in [0,T]$ we define

$$A_t(x., y.) = \left\{ (a,b) \in \mathbb{A} \middle| (a,b) = \left(\begin{pmatrix} \frac{1}{2}\kappa\kappa^\star(t,y.) & 0 \\ 0 & \frac{1}{2}m\lambda^2(t,y.)x_t^2 \end{pmatrix}, \begin{pmatrix} \nu(t,y.) \\ l\theta(t,y.)x_t \end{pmatrix} \right), \begin{array}{l} 0 \leqslant m \leqslant 1, \\ 0 \leqslant l \leqslant \sqrt{m} \end{array} \right\} \tag{13}$$

Remark 2.6. Notice that, for any investment strategy π as in Definition 2.2, if $\sigma_t = \begin{pmatrix} \kappa(t,y.) & 0 \\ 0 & \phi_t\lambda(t,y.)x_t \end{pmatrix}$ and $b_t = \begin{pmatrix} \nu(t,y.) \\ \phi_t\theta(t,y.)x_t \end{pmatrix}$ then, defining $a_t = \frac{1}{2}\sigma_t\sigma_t^\star$, the pair (a_t, b_t) belongs to $A_t(x., y.)$.

The following definition describes the familiy of auxiliary controls used throughout this work. It stresses the fact of having Itô processes whose coefficients belong to the convex sets $A_t(x., y.)$ in 'a measurable way' as $t, x.$ and $y.$ vary.

Definition 2.7. We define a familiy of auxiliary controls $\overline{\Pi} = \overline{\Pi}(x,y)$. Namely, an auxiliary control $\overline{\pi} \in \overline{\Pi}$ consists of a collection

$$\overline{\pi} := \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leqslant t \leqslant T}, \mathbb{P}, X_t, Y_t, (B_t, W_t), (x,y) \right)$$

where $x > 0$, $y \in \mathbb{R}$,

(a) $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathbb{P} \right)$ is a complete filtered probability space whose filtration satisfies the usual conditions;

(b) $\xi_t := (B_t, W_t)$ is an $\mathbb{R}^{d+1}$-valued standard $\mathcal{F}_t$-Brownian motion;

(c) there exists an $\mathbb{A}$-valued, $\mathcal{F} \otimes \mathcal{B}([0,T])$-measurable and $\mathcal{F}_t$-adapted process, denoted by (a_t, b_t), such that (d) and (e) below hold;

(d) X_t and Y_t are $\mathcal{F} \otimes \mathcal{B}([0,T])$-measurable and $\mathcal{F}_t$-adapted such that a.s. for all $t \geqslant 0$;

$$\begin{pmatrix} Y_t \\ X_t \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} + \int_0^t \sqrt{2a_s}\, d\xi_s + \int_0^t b_s\, ds; \tag{14}$$

(e) for almost all $(\omega, t) \in \Omega \times [0,T]$, we have $(a_t, b_t) \in A_t(X_\cdot, Y_\cdot)$.

We will often write $X^{\overline{\pi}}$, $Y^{\overline{\pi}}$ to indicate that we mean X, Y belonging to $\overline{\pi}$.

For each $\overline{\pi} \in \overline{\Pi}$, we can define $V_\pm(\overline{\pi})$ as before and we can set $V(\overline{\pi}) := V_+(\overline{\pi}) - V_-(\overline{\pi})$ for $\overline{\pi} \in \overline{\Pi}' := \{\pi \in \overline{\Pi} : V_-(\pi) < \infty\}$.

Remark 2.8. For a pair of processes processes a_t and b_t in $A_t(X_\cdot, Y_\cdot)$ one can define the corresponding real-valued processes l_t and m_t with $0 \leqslant m_t \leqslant 1, 0 \leqslant l_t \leqslant \sqrt{m_t}$ setting

$$l_t := b_t^{d+1} 1_{\theta_t(t,Y_t)\neq 0}/(X_t \theta_t(t,Y_t)),\ m_t := a^{d+1,d+1} 1_{\lambda_t(t,Y_t)\neq 0}/(X_t^2 \lambda^2(t,Y_t)).$$

Conditions (c),(d) in Definition 2.7 together with Assumption 2.2 imply that l_t, m_t can be chosen $\mathcal{F} \otimes \mathcal{B}([0,T])$ measurable and $\mathcal{F}_t$-adapted.

Equation (14) can be rewritten as the set of equations below. Denote

$$\sigma_t = \begin{pmatrix} \kappa(t, Y_\cdot) & 0 \\ 0 & \sqrt{m_t}\lambda(t, Y_\cdot) X_t \end{pmatrix}$$

and

$$b_t = \begin{pmatrix} \nu(t, Y_\cdot) \\ l_t \theta(t, Y_\cdot) X_t \end{pmatrix}.$$

Setting $a_t := \frac{1}{2}\sigma_t \sigma_t^\star$,

$$Y_t = y + \int_0^t \nu_s(Y_\cdot)\, ds + \int_0^t \kappa_s(Y_\cdot)\, dB_s, \tag{15}$$

$$X_t = x + \int_0^t l_s \theta_s(Y_\cdot) X_s ds + \int_0^t \sqrt{m_s}\lambda_s(Y_\cdot) X_s dW_s. \tag{16}$$

Definition 2.9. Let $\overline{\pi} \in \overline{\Pi}$ be a relaxed control. We say that $X_t^{\overline{\pi}}$ is a *portfolio value* process if $l_t = \sqrt{m_t}$, i.e.

$$dX_t = \sqrt{m_t}\theta(t, Y_\cdot) X_t dt + \sqrt{m_t}\lambda(t, Y_\cdot) X_t dW_t. \tag{17}$$

Remark 2.10. If $X_t^{\overline{\pi}}$ is a portfolio value process then, taking $\phi_t = \sqrt{m_t}$, we can see that

$$\left(\Omega^{\overline{\pi}}, \mathcal{F}^{\overline{\pi}}, \{\mathcal{F}_t^{\overline{\pi}}\}_{t\geqslant 0}, \mathbb{P}^{\overline{\pi}}, X_t^{\overline{\pi}}, Y_t^{\overline{\pi}}, \phi_t, (B_t^{\overline{\pi}}, W_t^{\overline{\pi}}), (x,y)\right)$$

belongs to Π.

Remark 2.11. Suppose that we are given a $\overline{\pi} \in \overline{\Pi}$ i.e. there is a standard $d+1$-dimensional Brownian motion (B, W) on $\left(\Omega^{\overline{\pi}}, \mathcal{F}^{\overline{\pi}}, \{\mathcal{F}_t^{\overline{\pi}}\}_{t\geqslant 0}, \mathbb{P}^{\overline{\pi}}\right)$ and processes $X_t^{\overline{\pi}}, Y_t^{\overline{\pi}}, m_t^{\overline{\pi}}, l_t^{\overline{\pi}}$ such that equations (15) and (16) hold. Define the continuous semimartingale $M_t^{\overline{\pi}} := \int_0^t \sqrt{m_s^{\overline{\pi}_t}}\lambda_s(Y_\cdot)\, dW_s + \int_0^t l_s^{\overline{\pi}_t}\theta_s(Y_\cdot)\, ds$. Then we can rewrite equation (16) as

$$X_t = x + \int_0^t X_s dM_s^{\overline{\pi}}. \tag{18}$$

Equation (18) has a unique strong solution on the given probability space, given the stochastic exponential

$$X_t^{\overline{\pi}} = x \exp\left\{ \int_0^t \sqrt{m_s}\lambda_s\left(Y_\cdot^{\overline{\pi}}\right) dW_s^{\overline{\pi}} + \int_0^t \left[l_s\theta_s\left(Y_\cdot^{\overline{\pi}}\right) - \frac{1}{2} m_s \lambda_s^2\left(Y_\cdot^{\overline{\pi}}\right) \right] ds \right\}, \tag{19}$$

and this process is positive $\mathbb{P}^{\overline{\pi}}$-a.s.

2.4 Krylov's theorem and related results

Lemma 2.12. *Let $M = \max\{\|\kappa\|_\infty, \|\lambda\|_\infty, \|\theta\|_\infty, \|\nu\|_\infty\}$. The set $A_t(x.,y.)$ is convex, closed and bounded, where the bound depends on M and x_t only.*

Proof. Notation $|\cdot|$ will refer to Euclidean norms of varying dimensions. For simplicity, we assume $d=1$. Notice that

$$|(\sigma_t, b_t)| = \left(\kappa_t^2(y.) + m^2\lambda_t^2(y.)\,x_t^2 + \nu_t^2(y.) + l^2\theta_t^2(y.)\,x_t^2\right)^{1/2},$$

hence

$$|(\sigma_t, b_t)| \leqslant \sqrt{2}M\left(1+|x_t|\right). \tag{20}$$

It is clear that the set is closed. For a fixed t, $x.$ and $y.$ the set is bounded. Indeed, let $(a_t, b_t) \in A_t(x.,y.)$. Then we have

$$|(a_t,b_t)| = \left(\frac{1}{4}\cdot\left(\kappa_t^2(y.)\right)^2 + \frac{1}{4}\cdot\left(m\cdot\lambda_t^2(y.)\cdot x_t^2\right)^2 + \nu_t^2(y.) + l^2\theta_t^2(y.)\,x_t^2\right)^{1/2},$$

so

$$|(a_t,b_t)| \leqslant \left(\frac{1}{4}M^4 + \frac{1}{4}\cdot M^4\cdot\left(x_t^2\right)^2 + M^2 + M^2x_t^2\right)^{1/2},$$

which leads to $|(a_t,b_t)| \leqslant \frac{1}{2}(M+1)^2 + M^2x_t^2$.

In particular,

$$\|A_t(x.,y.)\| := \max\{|(a_t,b_t)| : (a_t,b_t)\in A_t(x.,y.)\} \leqslant K\left(1+|x_t|^2\right), \tag{21}$$

for some $K \geq 0$.

The set $A_t(x.,y.)$ is also convex. Indeed, let $(\alpha,b),(\gamma,c)\in A_t(x.,y.)$ then, for $0\leq\mu\leq 1$,

$$\mu(\alpha,b) + (1-\mu)(\gamma,c) =$$
$$\left(\begin{pmatrix} \frac{1}{2}\kappa_t^2(y.) & 0 \\ 0 & \frac{1}{2}(\mu m + (1-\mu)n)\,\lambda_t^2(y.)\,x_t^2 \end{pmatrix}, \begin{pmatrix} \nu_t(y.) \\ (\mu l + (1-\mu)p)\,\theta_t(y.)\,x_t \end{pmatrix}\right)$$

with $0\leqslant m,n\leqslant 1$, $0\leqslant l\leqslant\sqrt{m}$ and $0\leqslant p\leqslant\sqrt{n}$. Clearly, $\mu l + (1-\mu)p \leqslant \sqrt{\mu m + (1-\mu)\,n}$, by concavity of the square root function. □

In order to deal with (semi)continuity issues related to the family of sets defined in Definitions 2.5 and (13), the support functions of sets $A_t\left(x.,y.\right)$ are now considered. We denote for all $u\in\mathbb{R}^{(d+1)(d+1)}$, $v\in\mathbb{R}^{d+1}$ and $t\in[0,T]$,

$$F_t\left(x.,y.\right)(u,v)=\max\left\{\sum_{i,j}a_{ij}u_{ij}+\sum_j b_jv_j\ :\ (a,b)\in A_t\left(x.,y.\right)\right\}. \tag{22}$$

Under Assumption 2.2, for fixed $t\geqslant 0$ and (u,v), the support function $(x.,y.)\to F_t\left(x.,y.\right)(u,v)$ is continuous, since we are fixing t, restricting the trajectories to $[0,t]$, and thus the max is taken over a compact set by Lemma 2.12. In particular, the set $A_t\left(x.,y.\right)$ is upper-semicontinuous in the sense of Assumption 3.1 iii) in [12]. It is also clear that, for fixed $u,v\in\mathbb{A}$, $F_t\left(u,v,x.,y.\right)$ is a Borel function on $[0,T]\times C\left([0,T];\mathbb{R}^{d+1}\right)$.

We now present some moment estimates which will, in particular, guarantee tightness for the family of the laws of $(X^{\overline{\pi}},Y^{\overline{\pi}})$, $\overline{\pi}\in\overline{\Pi}$ in $C\left([0,T];\mathbb{R}^{d+1}\right)$.

Proposition 2.13. *For the ease of reference we denote $\zeta_t=(Y_t,X_t)$. Under Assumption 2.2, for any $m>0$,*

$$\sup_{\overline{\pi}\in\overline{\Pi}}\mathbf{E}_{\overline{\pi}}\left[\sup_{t\leqslant T}|\zeta_t^{\overline{\pi}}|^m\right]<\infty. \tag{23}$$

Proposition 2.14. *Under Assumption 2.2, let $\overline{\pi}\in\overline{\Pi}$ and $\left(Y_t^{\overline{\pi}},X_t^{\overline{\pi}}\right)$ its associated processes solving (15) and (16). Then, there exists a constant $K>0$ not depending on $\overline{\pi}\in\overline{\Pi}$, such that for any $\eta>0$ and $s,t\in[0,T]$,*

$$\mathbf{E}_{\overline{\pi}}\|\zeta_t-\zeta_s\|^\eta\leqslant K\left|t-s\right|^{\frac{\eta}{2}}. \tag{24}$$

See the Appendix for a standard proof of both propositions above. A well-known result on tightness of measures on $C\left([0,T];\mathbb{R}^{d+1}\right)$ gives the following corollary. This could also be obtained by the method of Theorem 3.2 in [13].

Corollary 2.15. *Let Assumption 2.2 be in force. Let $\{\overline{\pi}_n\}\subset\overline{\Pi}$. The set of laws of the process $\zeta_\cdot^{\overline{\pi}_n}$ on $C\left([0,T];\mathbb{R}^{d+1}\right)$ is relatively weakly compact.* □

Now we restate Theorem 3.2 of [12] in our setting, which will provide weak compactness of the distributions of weak controls.

Theorem 2.16. *Let Assupmtion 2.2 be in force. Denote by $\mathbb{Q}^{\overline{\pi}}$ the distribution of $\zeta_\cdot^{\overline{\pi}}$ on $C\left([0,T];\mathbb{R}^{d+1}\right)$. Then the set $\left\{\mathbb{Q}^{\overline{\pi}}:\overline{\pi}\in\overline{\Pi}\right\}$ is sequentially weakly compact: for any sequence $\overline{\pi}_n\in\overline{\Pi}$ there is a subsequence $n(m)\to\infty$ as $m\to\infty$ and a $\overline{\pi}\in\overline{\Pi}$ such that for any real-valued, bounded, continuous function $H(x.)$ on $C\left([0,T];\mathbb{R}^{d+1}\right)$ we have*

$$\lim_{m\to\infty}E^{\nu_m}H\left(\zeta_\cdot^{\nu_m}\right)=E^{\pi}H\left(\zeta_\cdot^{\pi}\right), \tag{25}$$

where $\nu_m=\pi_{n(m)}$.

Proof. It follows from the above discussions that Assumption 3.1 ii) and iii) in [12] hold in the present case. One does not have Assumption 3.1 i) of [12] though (linear growth condition on $\|A_t(X_\cdot, Y_\cdot)\|$), there is a quadratic growth instead, see (21). But, as Corollary 2.15 shows, this is still sufficient to get tightness (and hence relative weak compactness) of the sequence $\mathbb{Q}^{\overline{\pi}_n}$ in our setting. Then one can check that the proof of Theorem 3.2 in [12] goes through and we can conclude. □

The next lemma shows that, to any auxiliary control $\overline{\pi}$ in the sense of Definition 2.7, we can associate an *investment strategy* (in the sense of Definition 2.2) with higher value function.

Lemma 2.17. *Let*

$$\overline{\pi} = \left(\Omega^{\overline{\pi}}, \mathcal{F}^{\overline{\pi}}, \{\mathcal{F}_t^{\overline{\pi}}\}_{t\geqslant 0}, \mathbb{P}^{\overline{\pi}}, \left(X^{\overline{\pi}}, Y^{\overline{\pi}}\right), \left(B^{\overline{\pi}}, W^{\overline{\pi}}\right), (x,y)\right) \in \overline{\Pi}.$$

Then a solution to

$$dY_t = \nu_t(Y_\cdot)\,dt + \kappa_t(Y_\cdot)\,dB_t, \quad Y_0 = y, \tag{26}$$

$$d\hat{X}_t = \sqrt{m_t}\theta_t(Y_\cdot)\,\hat{X}_t dt + \sqrt{m_t}\lambda_t(Y_\cdot)\,\hat{X}_t dW_t, \quad X_0 = x, \tag{27}$$

exists on the same filtered probability space and $\hat{X}_T \geq X_T^{\overline{\pi}}$ a.s. Furthermore, $\hat{X}_t$ is a portfolio value process.

Proof. Let us define

$$Z_t := \exp\left(-\int_0^t \left(l_s^{\overline{\pi}} - \sqrt{m_s^{\overline{\pi}}}\right)\left(X_\cdot^{\overline{\pi}}, Y_\cdot^{\overline{\pi}}\right)\theta\left(Y_\cdot^{\overline{\pi}}\right) ds\right)$$

and set $\hat{X}_t := Z_t X_t^{\overline{\pi}}$. Itô's formula shows that $\hat{X}_t$ indeed verifies (27). Since $\theta_t \geq 0$ was assumed, we get that $Z_t \geq 1$ hence $\hat{X}_t \geq X_t^{\overline{\pi}}$, for all t. □

2.5 Proof of Theorem 2.4

Proof. Let $t > 0$. By (11) and (10),

$$w_+\left(\mathbb{Q}^\pi\left(u_+\left((X_T^\pi - G^\pi)_+\right) > t\right)\right) \leqslant g_+\left[\mathbb{Q}^\pi\left((X_T^\pi - G^\pi)_+^\alpha > \frac{t}{k_+} - 1\right)\right]^\gamma.$$

Hence,

$$\begin{aligned} V_+(\pi) \quad &\leqslant g_+ \textstyle\int_0^\infty \left[\mathbb{Q}^\pi\left((X_T^\pi - G^\pi)_+^\alpha > \frac{t}{k_+} - 1\right)\right]^\gamma = \\ &= g_+\left(1 + \textstyle\int_{k_+}^\infty \left[\mathbb{Q}^\pi\left((X_T^\pi - G^\pi)_+^\alpha > \frac{t}{k_+} - 1\right)\right]^\gamma\right), \end{aligned}$$

$$\int_{k_+}^\infty \left[\mathbb{Q}^\pi\left((X_T^\pi - G^\pi)_+^\alpha > \frac{t}{k_+} - 1\right)\right]^\gamma dy \leqslant k_+ \int_0^\infty \left[\mathbb{Q}^\pi\left((X_T^\pi - G^\pi)_+^\alpha > s\right)\right]^\gamma dx. \tag{28}$$

If $s \geq 1$, applying Chebyshev's inequality and Assumption 2.5,

$$\left[\mathbb{Q}^{\pi}\left(\left(X_T^{\pi}-G^{\pi}\right)_+^{\alpha}>s\right)\right]^{\gamma}=\left[\mathbb{Q}^{\pi}\left(\left(X_T^{\pi}-G^{\pi}\right)_+^{\alpha\vartheta}>s^{\vartheta}\right)\right]^{\gamma}\leqslant\frac{\left[\mathbf{E}_{\pi}\left(X_T^{\pi}-G^{\pi}\right)_+^{\alpha\vartheta}\right]^{\gamma}}{s^{\vartheta\gamma}}\leqslant M^{\gamma}\frac{1}{s^{\vartheta\gamma}}, \tag{29}$$

where $M=\sup_{\pi}\mathbf{E}_{\pi}\left(X_T^{\pi}\right)_+^{\alpha\vartheta}<\infty$ (note that $G\geqslant 0$), by Proposition 2.13. Note that $1/s^{\vartheta\gamma}$ is integrable on $[1,\infty)$.

Hence the problem is well-posed since $V(\pi)\leq V_+(\pi)$ for all $\pi\in\Pi'$ and we have just seen that the latter has an upper bound independent of π.

By Theorem 2.16 the set of laws $\{\mathbb{Q}^{\pi}\}$, $\pi\in\overline{\Pi}$ of the processes $\zeta_{\cdot}^{\pi}=(X_{\cdot}^{\pi},Y_{\cdot}^{\pi})$ is relatively compact in the weak topology. Let $\{\pi^n\}$ be sequence of weak controls $\pi^n\in\overline{\Pi}'$ such that

$$V\left(\pi^n\right)\to\sup_{\pi\in\overline{\Pi}'}V\left(\pi\right),\quad n\to\infty. \tag{30}$$

There is a subsequence of $\{\pi^n\}$ denoted by $\left\{\pi^k\right\}$ such that $\mathbb{Q}^{\pi_k}\Rightarrow\mathbb{Q}^{\pi^\star}$ as $k\to\infty$ and $\pi^\star\in\overline{\Pi}$.

By Skorokhod's theorem there is a probability space, that will be denoted by $\left(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}}\right)$ and random variables $\tilde{X}_{\cdot}^k,\tilde{Y}_{\cdot}^k:\left(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}}\right)\to C\left([0,T];\mathbb{R}\right), C\left([0,T];\mathbb{R}^d\right)$, respectively, such that the law of $\left(\tilde{X}_{\cdot}^k,\tilde{Y}_{\cdot}^k\right)$ equals $\mathbb{Q}^{\pi_k}$ and $\tilde{X},\tilde{Y}:\left(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}}\right)$ $\to C\left([0,T];\mathbb{R}\right)$, $C\left([0,T];\mathbb{R}^d\right)$ with law equal to $\mathbb{Q}^{\pi^\star}$ such that $\tilde{X}^k\to\tilde{X}$, $\tilde{Y}^k\to\tilde{Y}$ a.s. in the uniform norm.

By Assumption 2.3, $\tilde{Y}^k$ and $\tilde{Y}$ have the same law and $\tilde{Y}^k\to\tilde{Y}$ in probability (even a.s.). By Théorème 1 in [1] $F\left(\tilde{Y}^k\right)\to F\left(\tilde{Y}\right)$ in probability.

By continuity of $u_{\pm}$ and the projection $p_T(f):=f(T)$, $f\in C\left([0,T];\mathbb{R}\right)$, we also have $u_{\pm}\left(\left(\tilde{X}_T^k-F\left(\tilde{Y}^k\right)\right)_{\pm}\right)\to u_{\pm}\left(\left(\tilde{X}_T-F\left(\tilde{Y}\right)\right)_{\pm}\right)$ in probability.

It follows that, denoting by D the set of discontinuity points of the cumulative distribution functions of $u_{\pm}\left(\left(\tilde{X}_T-F\left(\tilde{Y}\right)\right)_{\pm}\right)$, for any $y\in\mathbb{R}\setminus D$ we have

$$\mathbb{Q}^{\pi_k}\left(u_{\pm}\left(\left(X_T^{\pi_k}-G^{\pi_k}\right)_{\pm}\right)>y\right)\to\mathbb{Q}^{\pi^\star}\left(u_{\pm}\left(\left(X_T^{\pi^\star}-G^{\pi^\star}\right)_{\pm}\right)>y\right)$$

as $k\to\infty$.

Since $w_{\pm}$ are continuous, also

$$w_{\pm}\left(\mathbb{Q}^{\pi_k}\left(u_{\pm}\left(\left(X_T^{\pi_k}-G^{\pi_k}\right)_{\pm}\right)>y\right)\right)\to w_{\pm}\left(\mathbb{Q}^{\pi^\star}\left(u_{\pm}\left(\left(X_T^{\pi^\star}-G^{\pi^\star}\right)_{\pm}\right)>y\right)\right),$$

for $y\notin D$. By Fatou's lemma,

$$\int_0^{\infty}w_-\left(\mathbb{Q}^{\pi^\star}\left(u_-\left(\left(X_T^{\pi^\star}-G^{\pi^\star}\right)_-\right)>y\right)\right)dy\leqslant$$
$$\underline{\lim}_k\int_0^{\infty}w_-\left(\mathbb{Q}^{\pi_k}\left(u_-\left(\left(X_T^{\pi_k}-G^{\pi_k}\right)_-\right)>y\right)\right)dy,$$

and, by (29) and the Fatou lemma,

$$\int_0^\infty w_+\left(\mathbb{Q}^{\pi^\star}\left(u_+\left(\left(X_T^{\pi^\star}-G^{\pi^\star}\right)_+\right)>y\right)\right)dy \geqslant$$
$$\overline{\lim}_k \int_0^\infty w_+\left(\mathbb{Q}^{\pi_k}\left(u_+\left((X_T^{\pi_k}-G^{\pi_k})_+\right)>y\right)\right)dy,$$

It follows that $V(\pi^\star)=\sup_{\pi\in\overline{\Pi}'}V(\pi)$. It is also clear that $\pi^\star\in\overline{\Pi}'$. Let (a_t,b_t) be the $\mathbb{A}$- valued processes associated to $\pi^\star\in\overline{\Pi}$ as in Definition 2.7. By Lemma 2.17 there is

$$\pi'=\left(\Omega^{\pi^\star},\mathcal{F}^{\pi^\star},(\mathcal{F}_t^{\pi^\star})_{0\leqslant t\leqslant T},\mathbb{P}^{\pi^\star},X^{\pi'},Y^{\pi^\star},\left(B^{\pi^\star},W^{\pi^\star}\right),(x,y)\right)$$

which is a portfolio value process in the sense of Definition 2.9 and for which

$$u_+\left(\left(X_T^{\pi'}-G^{\pi'}\right)_+\right)\geqslant u_+\left(\left(X_T^{\pi^\star}-G^{\pi^\star}\right)_+\right),$$

notice that $G^{\pi'}=G^{\pi^\star}$ and $u_-\left((X_T^{\pi^\star}-G^{\pi^\star})_-\right)\geqslant u_-\left(\left(X_T^{\pi'}-G^{\pi'}\right)_-\right)$ also. Hence $V(\pi')\geq\sup_{\pi\in\overline{\Pi}'}V(\pi)$. Thus, recalling Remark 2.10, the investment strategy

$$\hat{\pi}=\left(\Omega^{\pi\star},\mathcal{F}^{\pi\star},\mathbb{P}^{\pi\star},\{\mathcal{F}_t^{\pi\star}\}_{0\leqslant t\leqslant T},X_\cdot^{\pi'},Y_\cdot^{\pi\star},\sqrt{m_\cdot^{\pi'}},(B^{\pi\star},W^{\pi\star}),(x,y)\right)$$

is optimal i.e. $\sup_{\pi\in\Pi'}V(\pi)\leq V(\pi^\star)\leq V(\hat{\pi})=V(\hat{\pi})$ and, obviously, $\hat{\pi}\in\Pi'$. □

3 Extensions

Based on economic considerations, we extend the model that was developed in the last section, by allowing the portfolio value process to influence the factor modelled by Y_t, the influence being 'additive'. This may be an appropiate model for e.g. a large investor. Furthermore, a riskless asset with deterministic interest rate r_t at time t is included. For the sake of simplicity we will assume that the factor process Y is one-dimensional, the results can be extended to the multidimensional case in a trivial way.

Definition 3.1. Let $\nu(t,y_\cdot)$ be a $\mathbb{R}$-valued process, such that the restriction of ν to $[0,t]\times C([0,T];\mathbb{R})$ is $\mathcal{B}([0,t])\otimes\mathscr{N}_t$-measurable, for any $0\leqslant t\leqslant T$.

Similarly, we define the $\mathbb{R}$-valued coefficients $\theta,\lambda,\rho,\kappa$ to have the same measurability.

In this case, the stochastic differential equations of the optimal investment model are given by

$$dY_t=\nu(t,Y_\cdot)\,dt+\kappa(t,Y_\cdot)\,dB_t+\rho(t,X_\cdot)\,dX_t, \tag{31}$$

$$dX_t = \phi_t \theta(t, Y.) X_t dt + \phi_t \lambda(t, Y.) X_t dW_t + (1-\phi_t) r_t X_t dt, \tag{32}$$

where $\phi_t \in [0,1]$ represents the proportion of wealth invested in the stock, Y is an economic factor, X is the value process of the given portfolio strategy ϕ. The set Π can be defined analogouly to Definition 2.2.

Assumption 3.1. For all $t \geqslant 0$ the growth rate of the stock is greater than the growth rate of the bond, i.e. for all $t, y.$,

$$\theta(t, y.) \geqslant r_t \geqslant 0, \ \ \mathbb{P}^\pi - \text{a.s.} \tag{33}$$

The functionals $\nu, \theta, \lambda, \kappa, \rho$ are bounded and path-continuous in the sense of Assumption 2.2.

Assumption 3.2. The reference point G is a constant.

As in Subsection 2.3, we consider a relaxed setting. With this purpose in mind, we define $\theta^r(t, y.) = \theta(t, y.) - r_t$. In what follows, E is the 2×2 matrix such that $E^{11} = 1$ and $E^{ij} = 0$ otherwise.

Definition 3.2. We define the following family of sets.

$$A_t(x., y.) = \left\{ (a,b) \in \mathbb{A} \middle| a = \frac{1}{2}\kappa^2(t, y.) E + \frac{1}{2} m \lambda^2(t, y.) x_t^2 \begin{pmatrix} \rho^2(t, x.) & \rho(t, x.) \\ \rho(t, x.) & 1 \end{pmatrix} \right., \tag{34}$$

$$\left. b = \begin{pmatrix} \nu(t, y.) \\ 0 \end{pmatrix} + (l x_t \theta^r(t, y.) + r_t x_t) \begin{pmatrix} \rho(t, x.) \\ 1 \end{pmatrix}, \ \ \begin{matrix} 0 \leqslant m \leqslant 1 \\ 0 \leqslant l \leqslant \sqrt{m} \end{matrix} \right\} \tag{35}$$

The following lemma is crucial: it enables us to use results of [12].

Lemma 3.3. *The set $A_t(x., y.)$ is closed, convex and bounded for each $(x., y.) \in C([0,T]; \mathbb{R}^2)$ and each $t \geqslant 0$*

Proof. Only convexity needs to be checked. Let $0 \leqslant \mu \leqslant 1$ and (a,b), $(\alpha, \beta) \in A_t(x., y.)$ then the convex linear combination $\mu a + (1-\mu)\alpha$ is equal to

$$\frac{1}{2}\kappa^2(t, y.) E + \frac{1}{2}(\mu m + (1-\mu) m') \lambda^2(t, y.) x_t^2 \begin{pmatrix} \rho^2(t, x.) & \rho(t, x.) \\ \rho(t, x.) & 1 \end{pmatrix}$$

and $\mu b + (1-\mu)\beta$ equals

$$\begin{pmatrix} \nu(t, y.) \\ 0 \end{pmatrix} + ((\lambda l + (1-\lambda) l') x_t \theta^r(t, y.) + r_t x_t) \begin{pmatrix} \rho(t, x.) \\ 1 \end{pmatrix}$$

As $\mu l + (1-\mu) l' \leqslant \mu\sqrt{m} + (1-\mu)\sqrt{m'} \leqslant \sqrt{\mu m + (1-\mu) m'}$,we have $\mu(a,b) + (1-\mu)(\alpha, \beta) \in A_t(x., y.)$. □

The estimates of Lemma 2.12 apply to this case as well, for some $K > 0$,

$$\|A_t(x., y.)\| \leqslant K\left(1 + |X_t|^2\right).$$

This allows to apply the results of [12] just as above, using the class of relaxed controls defined below.

Definition 3.4. We say that $\overline{\pi} \in \overline{\Pi}$ if

$$\overline{\pi} := \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, \mathbb{P}, X_t, Y_t, (B_t, W_t), (x, y)\right)$$

with

(a) $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, \mathbb{P}\right)$ a complete filtered probability space whose filtration satisfies the usual conditions;

(b) the 2-dimensional process $\xi_t := (B_t, W_t)$ is a standard $\mathcal{F}_t$-Brownian motion;

(c) the vector $(x, y) \in (0, \infty) \times \mathbb{R}$ is the initial endowment of the portfolio process X_t and the initial state of the economic factors Y_t, respectively;

(d) there exists an $\mathbb{A}$-valued, $\mathcal{F} \otimes \mathcal{B}([0, T])$ measurable and $\mathcal{F}_t$-adapted process denoted by (a_t, b_t) such that

$$\begin{pmatrix} Y_t \\ X_t \end{pmatrix} = \int_0^t \sqrt{2a_s} d\xi_s + \int_0^t b_s ds \tag{36}$$

(e) for almost all $(\omega, t) \in \Omega \times [0, T]$, we have $(a_t, b_t) \in A_t(X_\cdot, Y_\cdot)$ (i.e. we can choose a pair (m_t, l_t) in a "measurable way").

The vectorial form of the equations (4) and (5) can be rewritten. Define

$$\sigma_t := \begin{pmatrix} \kappa(t, y_\cdot) & \rho(t, x_\cdot)\sqrt{m_t} x_t \lambda(t, y_\cdot) \\ 0 & \sqrt{m_t}_t \lambda(t, y_\cdot) x_t \end{pmatrix}$$

and we have that the drift is given by

$$b_t = \begin{pmatrix} \nu(t, y_\cdot) \\ 0 \end{pmatrix} + (x_t l_t \theta^r(t, Y_\cdot) + x_t r_t) \cdot \begin{pmatrix} \rho(t, x_\cdot) \\ 1 \end{pmatrix}$$

Given a relaxed control $\overline{\pi}$, X_t, Y_t are $\mathcal{F} \otimes \mathcal{B}([0, T])$-measurable and $\mathcal{F}_t$-adapted such that for all $t \geqslant 0$

$$dY_t = \nu(t, Y_\cdot) dt + \kappa(t, Y_\cdot) dB_t + \rho(t, X_\cdot) dX_t, \tag{37}$$

$$dX_t = [l_t(\theta(t, Y_\cdot) - r_t) X_t + r_t \cdot X_t] dt + \sqrt{m_t}\lambda(t, Y_\cdot) X_t dW_t. \tag{38}$$

The proof of the next result follows closely that of Theorem 2.4.

Theorem 3.5. *Let Assumptions 2.3, 2.4, 3.1 and 3.2 hold. The problem* (9) *is well-posed and* $\Pi' \neq \emptyset$ *(the identically zero strategy belongs to* Π'*, where* Π' *is defined analogously to Subsection 2.2). There is* $\hat{\pi} \in \Pi'$ *such that the supremum in (9) is attained.* □

4 Appendix

Some proofs of auxiliary results are included in this section.

Proof of Proposition 2.13. We shall write $\xi_s = (W_s, B_s)$. Suppose $m \geqslant 2$. The notation $|\cdot|$ will be used to denote Euclidean norm in spaces of various dimensions. Then

$$|\zeta_t|^m = \left[(X_t)^2 + |Y_t|^2\right]^{\frac{m}{2}} \leqslant 2^{\frac{m}{2}-1} \cdot \left[|X_t|^m + |Y_t|^m\right], \tag{39}$$

so it is enough to obtain that the moments of each of the processes Y_t and X_t satisfy (23). Set $b_{s,2} = (l_t \theta_t (Y.) X_t)$ and $\sigma_{s,2} = (0, m_t \lambda_t (Y.) X_t)$

$$\mathbf{E}_\pi \left[\sup_{t \leqslant T} |X_t|^m\right] \leqslant 3^{m-1} \left(|X_0|^m + \mathbf{E}_\pi \left(\int_0^T |b_{s,2}| ds\right)^m + \mathbf{E}_\pi \left[\sup_{t \leqslant T} \left|\int_0^t \sigma_{s,2} d\xi_s\right|^m\right]\right), \tag{40}$$

By Jensen's inequality and Burkholder-Davis-Gundy inequality

$$\mathbf{E}_\pi \left[\sup_{t \leqslant T} |X_t|^m\right] \leqslant 3^{m-1} \left(|X_0|^m + \mathbf{E}_\pi \left(\int_0^T |b_{s,2}| ds\right)^m + C_m \mathbf{E}_\pi \left[\left|\int_0^T |\sigma_{s,2}|^2 ds\right|^{\frac{m}{2}}\right]\right),$$

for some $C_m > 0$.

We can apply again Jensen's inequality (now with respect the "uniform density" on $[0, T]$)

$$\left(T \int_0^T \frac{|b_{s,2}|}{T} ds\right)^m \leqslant T^{m-1} \cdot \int_0^T |b_{s,2}|^m ds, \text{ and } \left|T \int_0^T \frac{\|\sigma_{s,2}\|^2}{T} ds\right|^{\frac{m}{2}} \leqslant T^{m/2-1} \cdot \int_0^T \|\sigma_{s,2}\|^m ds, \tag{41}$$

here $\|\sigma_{s,2}\|^m = m_t^m |\lambda_t (Y.)|^m \cdot X_t^m$ and $|b_{s,2}|^m = l_t^m \theta_t^m (Y.) X_t^m$, however, we can use the estimate (20) above,

$$\mathbf{E}_\pi \left[\sup_{t \leqslant T} |X_t|^m\right] \leqslant 3^{m-1} \left(|X_0|^m + K \mathbf{E}_\pi \left[\int_0^T \left(1 + \sup_{t \leqslant s} \|\zeta_t\|\right)^m ds\right]\right),$$

similarly,

$$\mathbf{E}_\pi \left[\sup_{t \leqslant T} |Y_t|^m\right] \leqslant 3^{m-1} \left(|Y_0|^m + K' \mathbf{E}_\pi \left[\int_0^T \left(1 + \sup_{t \leqslant s} \|\zeta_t\|\right)^m ds\right]\right),$$

for constants K, K'. Then we have

$$\mathbf{E}_\pi \left[\sup_{t \leqslant T} \|\zeta_t\|^m\right] \leqslant K(m) \left(\|\zeta_0\|^m + \left[\int_0^T 1 + \mathbf{E}_\pi \left(\sup_{t \leqslant s} \|\zeta_t\|\right)^m ds\right]\right), \tag{42}$$

for some $K(m) > 0$ so by Gronwall's lemma,

$$\mathbf{E}_\pi \left[\sup_{t \leqslant T} \|\zeta_t\|^m \right] \leqslant L(m),$$

with a fixed constant $L(m)$, for all $\pi \in \Pi$. The case $0 < m < 2$ follows from the monotonicity of the norms. □

Proof of Proposition 2.14. As in Proposition 2.13, it is enough to show a similar estimate (24) for each of the coordinates X_t and Y_t. That means

$$\mathbf{E}\left|Y_t - Y_s\right|^\eta \leqslant K_1 \left|t-s\right|^{\eta/2} \quad \text{and} \quad \mathbf{E}\left|X_t - X_s\right|^\eta \leqslant K_2 \left|t-s\right|^{\eta/2}. \tag{43}$$

By Assumption 2.2 the first inequality is a simple consequence of B-D-G's and Jensen's inequality:

$$\mathbf{E}\left|Y_t - Y_s\right|^\eta \leqslant 2^{\eta-1} \cdot \left[\mathbf{E}\left(\int_s^t \left|\nu\left(Y_{[0,r]}\right)\right| dr \right)^\eta + \mathbf{E}\left(\sup_{s \leqslant r \leqslant t} \left| \int_s^r \kappa_r\left(Y_{[0,r]}\right) dB_r \right|^\eta \right) \right],$$

$$\mathbf{E}\left|Y_t - Y_s\right|^\eta \leqslant 2^{\eta-1} \cdot \left[M^\eta \left|t-s\right|^\eta + C_\eta \cdot \mathbf{E}\left(\int_s^t \kappa_r^2\left(Y_{[0,r]}\right) dr \right)^{\eta/2} \right],$$

thus

$$\mathbf{E}\left|Y_t - Y_s\right|^\eta \leqslant 2^{\eta-1} \cdot \left[M^\eta \left|t-s\right|^\eta + C_\eta M^\eta \left|t-s\right|^{\eta/2} \right] \leqslant 2^{\eta-1} \cdot \left[M^\eta T^{\eta/2} + C_\eta M^\eta \right] \cdot \left|t-s\right|^{\eta/2}.$$

The second estimate relies upon Proposition 2.13:

$$\mathbf{E}\left|X_t - X_s\right|^\eta \leqslant 2^{\eta-1} \cdot \left[\mathbf{E}\left(\int_s^t \left|l_r \theta_r\left(Y_{[0,r]}\right) X_r\right| dr \right)^\eta + \mathbf{E}\left(\sup_{s \leqslant r \leqslant t} \left| \int_s^r \lambda_r\left(Y_{[0,r]}\right) \sqrt{m_r}\, X_r dW_r \right|^\eta \right) \right],$$

$$\leqslant 2^{\eta-1} \cdot M^\eta \cdot \left[\mathbf{E}\left(\int_s^t \left|X_r\right| dr \right)^\eta + C_\eta \mathbf{E}\left(\left| \int_s^t \left|X_r\right|^2 dr \right|^{\eta/2} \right) \right],$$

$$\mathbf{E}\left|X_t - X_s\right|^\eta \leqslant 2^{\eta-1} \cdot M^\eta \left[(t-s)^\eta \cdot \mathcal{N}(\eta, T) + (t-s)^{\eta/2} \cdot \mathcal{N}(\eta, T) \right] = K_3 \left|t-s\right|^{\eta/2},$$

where $K_3 = 2^{\eta-1} M^\eta \mathcal{N}(\eta, T) \cdot \left(T^{\eta/2} + 1\right)$ and $\mathcal{N}(\eta, T)$ is the upper bound of $\sup_{\overline{\pi} \in \overline{\Pi}} \mathbf{E}_\pi \left[\sup_{t \leqslant T} \left|X_t\right|^\eta\right]$ in Proposition 2.13. Note that the constants do not depend on $\overline{\pi}$ as neither M nor $\mathcal{N}(\eta, T)$ do. □

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