SMALL SUBSET SUMS

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Abstract. Let \( \| \cdot \| \) be a norm in \( \mathbb{R}^d \) whose unit ball is \( B \). Assume that \( V \subset B \) is a finite set of cardinality \( n \), with \( \sum_{v \in V} v = 0 \). We show that for every integer \( k \) with \( 0 \leq k \leq n \), there exists a subset \( U \) of \( V \) consisting of \( k \) elements such that \( \| \sum_{v \in U} v \| \leq \lceil d/2 \rceil \). We also prove that this bound is sharp in general. We improve the estimate to \( O(\sqrt{d}) \) for the Euclidean and the max norms. An application on vector sums in the plane is also given.

1. Definitions, notation, results

We consider the real \( d \)-dimensional vector space \( \mathbb{R}^d \) with a norm \( \| \cdot \| \) whose unit ball is \( B \). For a finite set \( U \subset \mathbb{R}^d \), \( |U| \) stands for the cardinality of \( U \), and \( s(U) \) for the sum of the elements of \( U \), so \( s(U) = \sum_{u \in U} u \), and \( s(\emptyset) = 0 \) of course.

In 1914 Steinitz [12] proved that, in the case of the Euclidean norm, for every finite set \( V \subset B \) with \( |V| = n \) and \( s(V) = 0 \), there exists an ordering \( v_1, \ldots, v_n \) of the vectors in \( V \) such that all partial sums have norm at most \( 2d \), that is

\[
\max_{k=1,\ldots,n} \left\| \sum_{i=1}^{k} v_i \right\| \leq 2d.
\]

It is important here that the bound \( 2d \) does not depend on \( n \), the size of \( V \). Steinitz’s result implies that for every norm and every finite \( V \subset B \) with \( s(V) = 0 \) there is an ordering along which all partial sums are bounded by a constant that depends only on \( B \). Let \( S(B) \) denote the smallest such constant for a given norm with unit ball \( B \), and set \( S(d) = \sup S(B) \) where the supremum is taken over all norms in \( \mathbb{R}^d \). The best known bounds on \( S(d) \) are: \( S(B) \leq d \), proved by Sevastyanov [9], and by Grinberg and Sevastyanov [7], and \( S(d) \geq \frac{d+1}{2} \), which is shown by an example coming from the \( \ell_1 \) norm [7]. For specific norms, stronger results may hold. In particular, for \( \ell_2 \) and \( \ell_\infty \), it is conjectured that the right order of magnitude of \( S(B) \) is \( \sqrt{d} \) – although not even \( o(d) \) is known.

Steinitz’s result immediately implies that for every finite set \( V \subset B \) with \( s(V) = 0 \) and every integer \( k \), \( 0 \leq k \leq |V| \), there is a subset \( U \subset V \) such that \( |U| = k \) and \( s(U) \) is not greater than a constant depending only on \( d, B, k \), for instance \( S(B) \) is such a constant. Let \( T(B, k) \) be the smallest constant with this property, set \( T(B) = \sup_k T(B, k) \), and \( T(d) = \sup T(B) \) where the supremum is taken over all norms in \( \mathbb{R}^d \). It is evident that \( T(B, k) \leq k \).

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In this paper we investigate $T(B, k), T(B)$ and $T(d)$. Here come our main results. First, the estimate for general norms.

**Theorem 1.** Let $B$ be the unit ball of an arbitrary norm on $\mathbb{R}^d$. For any finite set $V \subset B$ with $s(V) = 0$, and for any $k \leq |V|$, there exists a subset $U \subset V$ with $k$ elements, so that

$$\|s(U)\| \leq \left\lceil \frac{d^2}{2} \right\rceil.$$ 

In other words, $T(d) \leq \left\lceil \frac{d^2}{2} \right\rceil$.

**Theorem 2.** For every $d \geq 1$, there exists a norm in $\mathbb{R}^d$ with unit ball $B$, so that $T(B, k) = \left\lceil \frac{d^2}{2} \right\rceil$ for infinitely many values of $k$. Also, $T(B, k) = k$ for all $k \leq \left\lfloor \frac{d^2}{2} \right\rfloor$.

Theorems 1 and 2 imply that $T(d) = \left\lceil \frac{d^2}{2} \right\rceil$ for all integers $d \geq 1$.

One expects that for specific norms better estimates are valid. We have proved this in some cases. The unit ball of the norm $\ell^d_p$ will be denoted by $B_{p}^d$. We have the following results in the cases $p = 1, 2, \infty$.

**Theorem 3.** $\frac{d}{2} \leq T(B_1^d) \leq \left\lceil \frac{d}{2} \right\rceil$.

**Theorem 4.** $\frac{1}{2} \sqrt{d + 2} \leq T(B_2^d) \leq \frac{1 + \sqrt{5}}{2} \sqrt{d}$

**Theorem 5.** $\frac{1}{3} \sqrt{d} \leq T(B_\infty^d) \leq O(\sqrt{d})$

We mention that in Theorems 4 and 5 the order of magnitude is the same as the conjectured value of the Steinitz constant.

**Remark 1.** Note that there is a “complementary” symmetry here. Namely, for every $U \subset V$, $s(U) = -s(V \setminus U)$, hence $\|s(U)\| = \|s(V \setminus U)\|$, and the cases $k$ and $n - k$ are symmetric. Hence, we may assume $k \leq n/2$.

When establishing Helly-type theorems for sums of vectors in a normed plane, Bárány and Jerónimo-Castro proved the following result \([3, \text{Lemma 5}]\), which matches our scheme: Given 6 vectors in the unit ball of a normed plane whose sum is 0, there always exist 3 among them, whose sum has norm at most 1. In fact, this statement served as the starting point for our current research. An application of Theorem 1 implies an extension of one of the Helly-type results \([3, \text{Theorem 3}]\), which we formulate slightly differently and prove in the last section.

**Theorem 6.** Let $k \geq 2$ be a positive integer, and $n = m(k - 1) + 1$ for some $m \geq 1$. Assume $B$ is the unit ball of a norm in $\mathbb{R}^2$, $V \subset B$ is of size $n$ and $\|s(V)\| \leq 1$. Then $V$ contains a subset $W$ of size $k$ such that $\|s(W)\| \leq 1$.

2. **Proof of Theorem 1**

We are to consider linear combinations $\sum_{v \in V} \alpha(v)v$ of the vectors in $V$. The coefficients $\alpha(v)$ form a vector $\alpha \in \mathbb{R}^V$. Define the convex polytope

$$P(V, k) = \left\{ \alpha \in \mathbb{R}^V : \sum_{v \in V} \alpha(v)v = 0, \sum_{v \in V} \alpha(v) = k, \quad 0 \leq \alpha(v) \leq 1 \quad (\forall v \in V) \right\}.$$
\( P(V, k) \) is non-empty as \( \alpha(v) \equiv k/n \) lies in it (here \( n = |V| \)). From now on let \( \alpha \) denote a fixed vertex of \( P(V, k) \). The basic idea is to choose \( U \) to be the set of vectors from \( V \) that have the \( k \) largest coefficients \( \alpha(v) \). This works directly when \( d \) is odd, and some extra care is needed for even \( d \).

We note first that \( P(V, k) \) is determined by \( d + 1 \) linear equations and \( 2n \) inequalities for the coefficients \( \alpha(v) \), so at a vertex at most \( d + 1 \) coefficients are strictly between 0 and 1. Define \( U_1 = \{ v \in V : \alpha(v) = 1 \} \) and \( Q = \{ v \in V : 0 < \alpha(v) < 1 \} \). Set \( q = \sum_{v \in Q} \alpha(v) \), \( q \) is an integer since \( q + |U_1| = k \).

Split now \( Q \) into two parts, \( E \) and \( F \), so that \( |E| = q \) and \( E \) contains the vectors with the \( q \) largest coefficients in \( Q \), and \( F \) the rest (ties broken arbitrarily). Then \( U = U_1 \cup E \) has exactly \( k \) elements and

\[
\begin{align*}
    s(U) &= \sum_{v \in U_1} v + \sum_{v \in E} v \\
          &= \sum_{v \in V} \alpha(v)v + \sum_{v \in E} (1 - \alpha(v))v - \sum_{v \in F} \alpha(v)v.
\end{align*}
\]

Here \( \sum_{v \in V} \alpha(v)v = 0 \), so by the triangle inequality

\[
\|s(U)\| \leq \sum_{v \in E} (1 - \alpha(v)) + \sum_{v \in F} \alpha(v).
\]

The average of the coefficients in \( Q \) is \( a := q/|Q| \). Thus, the average of the coefficients is at least \( a \) in \( E \), and it is at most \( a \) in \( F \). Consequently, the last sum is maximal when \( \alpha(v) = a \) for all \( v \in Q \):

\[
\|s(U)\| \leq q(1 - a) + (|Q| - q)a = \frac{2}{|Q|} q (|Q| - q) \leq \frac{|Q|}{2}.
\]

This finishes the proof when \( d \) is odd as \(|Q| \leq d + 1 \), and also when \( d \) is even and \(|Q| \leq d \).

We are left with the case when \( d \) is even and \(|Q| = d + 1 \). The vectors in \( Q \) are linearly dependent, so there is a non-zero \( \beta \in \mathbb{R}^V \) with \( \beta(v) = 0 \) when \( v \notin Q \) such that \( \sum_{v \in Q} \beta(v)v = 0 \). We can assume that \( \sum_{v \in Q} \beta(v) \leq 0 \).

Then \( \sum_{v \in V} (\alpha(v) + t\beta(v))v = 0 \) for every \( t \in \mathbb{R} \). Choose \( t > 0 \) maximal so that \( 0 \leq \gamma(v) = \alpha(v) + t\beta(v) \leq 1 \) for every \( v \in V \). This means that, for some \( v^* \in Q \), \( \gamma(v^*) = 0 \) or 1.

Assume for the time being that \( q \leq (d + 1)/2 \).

Suppose first that \( \gamma(v^*) = 0 \). This time we split \( Q^* := Q \setminus v^* \) again into \( E \) and \( F \) so that \( |E| = q \) and \( E \) contains the vectors from \( Q^* \) with the \( q \) largest coefficients. Note that \( \sum_{v \in Q^*} \gamma(v) \leq \sum_{v \in Q} \alpha(v) = q \) and that \( |Q^*| = d \), so the average \( a^* \) of \( \gamma(v) \) over \( Q^* \) is at most \( q/d \). We use again \( U = U_1 \cup E \) and we have, the same way as before,

\[
\|s(U)\| \leq \sum_{v \in E} (1 - \gamma(v)) + \sum_{v \in F} \gamma(v).
\]

The right hand side is maximal again if every \( \gamma(v) \) equals their average \( a^* \), hence

\[
\|s(U)\| \leq q(1 - a^*) + (d - q)a^* = q + (d - 2q)a^* \leq q + (d - 2q)\frac{q}{d} \leq \frac{d}{2}.
\]
because \( d \) is even so \( q \leq (d + 1)/2 \) implies \( 2q \leq d \). Thus, \( ||s(U)|| \leq d/2 \).

The case when \( \gamma(v^*) = 1 \) is similar: this time \( v^* \) is added to \( U_1, Q^* = Q \setminus v^* \) is split into \( E \) and \( F \) with \( |E| = q - 1 \) so that \( E \) contains the vectors with the largest \( q - 1 \) coefficients. Now \( \sum_{v \in Q^*} \gamma(v) \leq \sum_{v \in Q} \alpha(v) - 1 = q - 1 \), and thus the average \( a^* \) of \( \gamma(v) \) over \( Q^* \) is at most \((q - 1)/d\). As above, we are led to the inequality

\[
||s(U)|| \leq (q - 1)(1 - a^*) + (d - (q - 1))a^* = (q - 1) + (d - 2(q - 1))a^*.
\]

Using that \( d - 2(q - 1) \geq 0 \) and \( a^* \leq (q - 1)/d \), we conclude that \( ||s(U)|| \leq d/2 - 2/d < d/2 \).

Finally we consider the case \( q > (d + 1)/2 \). By complementary symmetry \( s(U) = -s(V \setminus U) \). For \( q > (d + 1)/2 \), we consider the complementary problem of finding \( U \subset V \) with \( n - k \) elements so that \( ||s(U)|| \leq \lceil d/2 \rceil \).

It is easy to see that \( 1 - \alpha(.) \in \mathbb{R}^V \) is a vertex of \( P(V, n - k) \), for which \( \sum_{v \in Q}(1 - \alpha(v)) < (d + 1)/2 \).

The same proof yields a stronger statement.

**Theorem 7.** Let \( W \subset B \) finite. Then for every \( k \leq |W| \) and for every vector \( w_0 \in \text{conv} W \), there is a subset \( U \subset W \) of cardinality \( k \), so that

\[
||s(U) - kw_0|| \leq \left\lceil \frac{d}{2} \right\rceil.
\]

The proof is the same as above, except that instead of the convex polytope \( P(V, k) \), we consider the coefficient vectors \( \alpha : W \to [0, 1] \) satisfying

\[
\sum_{w \in W} \alpha(w) w = kw_0 \quad \text{and} \quad \sum_{w \in W} \alpha(w) = k.
\]

The condition \( w_0 \in \text{conv} W \) ensures that this set is a non-empty convex polytope. The rest of the argument is unchanged.

**Remark 2.** For later reference we record the fact that the linear dependence \( \alpha \) defines the sets \( U_1 \) and \( Q \), and if \( |Q| = d + 1 \), then the new linear dependence \( \gamma \) defines \( v^* \in Q \) and \( Q^* \). Note that this works for even and odd \( d \), we only need \( |Q| = d + 1 \). For later use we define

\[
A = \{v \in V : \gamma(v) = 1\} \quad \text{and} \quad C = \{v \in V : 0 < \gamma(v) < 1\}.
\]

\[ 3. \text{Proof of Theorem 2} \]

We are going to use the following fact. If the unit ball of a norm \( ||.|| \) is the convex hull of the vectors \( v_1, \ldots, v_m, -v_1, \ldots, -v_m \in \mathbb{R}^d \), then for every vector \( x \in \mathbb{R}^d \),

\[
||x|| = \min \left\{ \sum_{i=1}^m |a_i| : \sum_{i=1}^m a_i v_i = x \right\}.
\]

Let \( e_1, \ldots, e_d \) be the standard basis vectors of \( \mathbb{R}^d \), and set \( e_0 = -\sum_{i=1}^d e_i \). We define \( V \) to be \( s \) copies of \( \{e_0, e_1, \ldots, e_d\} \), where \( s \geq 1 \) is an integer. The unit ball is set to be \( B = \text{conv} \{V, -V\} \). Let \( k < n = s(d + 1) \) be a positive integer congruent to \( \left\lceil \frac{d}{2} \right\rceil \) mod \( (d + 1) \). We claim that for every \( k \)-element subset \( U \) of \( V \), \( ||s(U)|| \geq \left\lceil \frac{d}{2} \right\rceil \).
Assume that $U$ contains $b_i$ copies of $e_i$ for every $i$, so $k = \sum_0^d b_i$. We have to estimate the norm of the vector $v = \sum_0^d b_i e_i$. Assume that
\[
v = \sum_0^d a_i e_i
\]
for some $a_i \in \mathbb{R}$. Then $\sum_0^d (b_i - a_i) e_i = 0$. Since the only linear dependence of the vectors $e_0, \ldots, e_d$ is $x \sum_0^d e_i = 0$ for some constant $x \in \mathbb{R}$, we obtain that $a_i = b_i - x$ for every $i$. Set
\[
f(x) := \sum_0^d |b_i - x|.
\]
Then $\|v\| = \min f(x)$ by the fact from the beginning of this section. We are going to estimate $f(x)$. Since $b_i \in \mathbb{Z}$ for every $i$, the function $f(x)$ is piecewise linear on $\mathbb{R}$ (it is affine on all intervals $(q, q + 1)$ for $q \in \mathbb{Z}$). Therefore, there exists $c \in \mathbb{Z}$ so that the minimum of $f(x)$ is attained at $c$.

The facts $k = \sum_0^d b_i \equiv \lfloor d/2 \rfloor \mod (d + 1)$ and $c \in \mathbb{Z}$ imply that $\sum_0^d (b_i - c) \equiv \lfloor d/2 \rfloor \mod (d + 1)$. Thus,
\[
\left\lfloor \frac{d}{2} \right\rfloor \leq \left| \sum_0^d (b_i - c) \right| \leq \sum_0^d |b_i - c|,
\]
hence, $\|v\| \geq \lfloor d/2 \rfloor$.

We show next that $T(B, k) = k$ when $1 \leq k < \lfloor d/2 \rfloor$. The unit ball $B$ is the same as above and $V = \{e_0, \ldots, e_d\}$. Assume $U \subset V$ with $|U| = k$ and $\|s(U)\| < k$. Add $\lfloor d/2 \rfloor - k$ vectors from $V \setminus U$ to $U$ to obtain a subset $W$ of $\lfloor d/2 \rfloor$ elements. Every addition increases the norm of the sum by at most one (because of the triangle inequality), so we get $\|s(W)\| \leq \|s(U)\| + \lfloor d/2 \rfloor - k < \lfloor d/2 \rfloor$, contrary to what was established above. Thus $T(B, k) \geq k$, while $T(B, k) \leq k$ follows from the triangle inequality. \qed

Further examples showing $T(B, k) = \lfloor d/2 \rfloor$ will be given in the next section.

**Remark 3.** We mention that for large enough $n$, there is no vector set that works simultaneously for all $k$ with $d/2 \leq k \leq n - d/2$. This follows from Steinitz’s theorem: let $v_1, \ldots, v_n$ be the ordering where all partial sums lie in $dB$. Then necessarily two partial sums, with at least $d/2$ summands whose cardinalities differ by at least $d/2$, are close to each other: a standard volume estimate shows that their distance is bounded above by $4dn^{-1/d}$. Then their difference, which is a k-sum with some $d/2 \leq k \leq n - d/2$, must be small.

4. The $\ell_1$ norm, proof of Theorem 3

The upper bound follows from Theorem 1. For the lower bound let $V$ consist of $e_1, \ldots, e_d$ and $d$ copies of $\frac{1}{d} e_0$ (with the same notation as in the previous section). Assume $U \subset V$ has exactly $d$ elements. If $U$ contains $p$ vectors out of $e_1, \ldots, e_d$, then $s(U)$ has $p$ coordinates equal to $\frac{x}{d}$ and $d - p$ coordinates equal to $\frac{x}{d} - 1$. Thus $\|s(U)\|_1 = \frac{x}{d} (p^2 + (d - p)^2)$. The last
expression is minimal when $p = \lfloor \frac{d}{2} \rfloor$. The minimum equals $\frac{d}{2}$ when $d$ is even and $\frac{d}{2} + \frac{1}{2}$ when $d$ is odd. This is slightly better (for $d$ odd) than the stated lower bound.

This example shows that $T(B^d_1) = T(B^d_1, d) = d/2$ for even $d$. A small modification gives further examples implying $T(B^d_1, k) = d/2$ for even $d$ and for all $k \geq d$. Namely, given $d \geq 1$ and $k \geq d$, let $V$ consist of the vectors $e_1, \ldots, e_d$, and $2k - d$ copies of $\frac{1}{2k-d}e_0$. Then $V \subset B^d_1$ and $s(V) = 0$. It is not hard to check that this shows $T(B^d_1, k) = d/2$ for every $k \geq d$ ($d$ is even).

5. The $\ell_2$ norm, proof of Theorem 4

In this section, $\| \|$ stands for the Euclidean norm. For the upper bound we will need two lemmas. The first is Lemma 2.2 in Beck's paper [4]. A similar result is given in [11, Theorem 4.1]. The second is a Steinitz type statement.

**Lemma 1.** Let $Q \subset B^d_1$ be finite, and $\alpha : Q \to [0, 1]$. Then there exists $\varepsilon : Q \to \{0, 1\}$ such that $\| \sum_{v \in Q}(\varepsilon(v) - \alpha(v))v\| \leq \sqrt{d}/2$.

**Lemma 2.** Assume that $V \subset B^d_2$ is a finite set and $\|s(V)\| = \sigma$. Then there exists an ordering $v_1, \ldots, v_n$ of the elements of $V$, such that, for all $h \leq n$, \[
\left\| \sum_{i=1}^{h} v_i \right\| \leq \sqrt{\sigma^2 + h}.
\]

**Proof.** Choose $v_1 \in V$ arbitrarily. For $h \geq 2$, we select $v_h$ inductively. We set $S_h = \sum_{i=1}^{h} v_i$. Assume that $\|S_{h-1}\| \leq \sqrt{\sigma^2 + h - 1}$, and set $W = V \setminus \{v_1, \ldots, v_{h-1}\}$. We consider three cases.

**Case 1.** If $\|S_{h-1}\| \leq \sigma - 1$, then choose $v_h \in W$ arbitrary: $\|S_h\| \leq \sigma$ holds by the triangle inequality.

**Case 2.** If $\|S_{h-1}\| \geq \sigma$, then by the assumption $\|S\| = \sigma$, there exists a vector $v_h \in W$, for which $\langle S_{h-1}, v_h \rangle \leq 0$. Therefore, \[
\|S_h\|^2 = \|S_{h-1} + v_h\|^2 \leq \|S_{h-1}\|^2 + \|v_h\|^2 \leq (\sigma^2 + h - 1) + 1 = \sigma^2 + h.
\]

**Case 3.** If $\sigma - 1 < \|S_{h-1}\| < \sigma$, define $\varepsilon = \sigma - \|S_{h-1}\|$, so $0 < \varepsilon < 1$ and $\varepsilon \leq \sigma$. Then \[
\sum_{v \in W} \langle v, S_{h-1} \rangle = \langle S_h - S_{h-1}, S_{h-1} \rangle \leq \sigma(\sigma - \varepsilon) - (\sigma - \varepsilon)^2 = \varepsilon(\sigma - \varepsilon).
\]
Thus, there exists $v_h \in W$, for which $\langle v_h, S_{h-1} \rangle \leq \varepsilon(\sigma - \varepsilon)$. Then \[
\|S_h\|^2 = \|S_{h-1} + v_h\|^2 \leq (\sigma - \varepsilon)^2 + 2\varepsilon(\sigma - \varepsilon) + 1
\]
\[
= \sigma^2 + 1 - \varepsilon^2 < \sigma^2 + h.
\]

**Proof of Theorem 4.** For the lower bound let $V$ be the set of vertices of a regular simplex inscribed in $B^d_2$. Then $s(V) = 0$. Let $U \subset V$ have $\left\lfloor \frac{d}{2} \right\rfloor$ elements. A routine computation shows that $\|s(U)\|$ equals $\frac{\sqrt{d+2}}{2}$ when $d$ is even and $\frac{d+1}{\sqrt{d}} > \frac{\sqrt{d+2}}{2}$ when $d$ is odd. This implies the lower bound $T(B^d_2) \geq \frac{\sqrt{d+2}}{2}$. 
For the upper bound we have to prove the existence of \( U \subset V \) with \(|U| = k \) and \( \|s(U)\| \leq \frac{1}{2}\sqrt{d} \). From the proof of Theorem 1 recall the definition of \( P(V,k) \) and its vertex \( \alpha \in \mathbb{R}^V \) and \( U_1 = \{ v \in V : \alpha(v) = 1 \} \) and \( Q = \{ v \in V : 0 < \alpha(v) < 1 \} \). Here \(|Q| \leq d + 1 \).

If \(|Q| = 0\), then \(|U_1| = k\) and \( s(U_1) = 0\), so we can set \( U = U_1 \). The case \(|Q| = 1\) is impossible because the sum of all \( \alpha(v) \) is an integer. From now on we assume that \( 2 \leq |Q| \) implying \(|U_1| + 1 \leq |U_1| + |Q| - 1\). Using Lemma 1 for \( \alpha \) restricted to \( Q \) we find \( \varepsilon : Q \to \{0,1\} \) such that \( \| \sum_{v \in Q}(\varepsilon(v) - \alpha(v))v \| \leq \sqrt{d}/2 \).

Define \( W = U_1 \cup \{ v \in Q : \varepsilon(v) = 1 \} \), then \( W \) has the properties that \( \|s(W)\| \leq \sqrt{d}/2 \) and \( |W| - k| \leq d \). Because of the complementary symmetry, we can assume that \( k \leq |W| \leq k + d \). Set \( h = |W| - k \). Then Lemma 2 applies to \( W \): writing \( \sigma = \|s(W)\| \) we have \( \sigma \leq \sqrt{d}/2 \) and so the elements of \( W \) can be ordered as \( w_1, w_2, \ldots \) so that \( \| \sum_{1}^{m} w_i \| \leq \sqrt{\sigma^2 + m} \) for every \( m \). In particular, with \( m = h \leq d \), \( \| \sum_{1}^{h} w_i \| \leq \sqrt{\sigma^2 + h} \leq \sqrt{d/4 + d} \). Then for \( U = W \setminus \{ w_1, \ldots, w_h \} \), we have \(|U| = k\) and \( \|s(U)\| \leq \frac{1}{2}\sqrt{d} \).

\[ \nabla \]

6. The \( \ell_\infty \) norm, proof of Theorem \[5\]

Here, \( \| \cdot \| \) denotes the maximum norm. We need two lemmas again, the first is similar to Lemma \[1\].

**Lemma 3.** If \( C \subset B_\infty^d \) consists of \( d \) linearly independent vectors, then for every point \( z \) of the parallelootope \( P = \sum_{v \in C}[0,v] \), there is a vertex \( u \) of \( P \) with \( \|z - u\|_\infty = O(\sqrt{d}) \).

This is a result of Spencer \[10\] Corollary 8], and also of Gluskin \[6\] whose work relies on that of Kashin \[8\]. Spencer’s proof gives the estimate \( \|z - u\| \leq 6\sqrt{d} \). The linear independence condition is only needed to ensure that \( P \) is a parallelootope, and so its vertices are of the form \( s(D) = \sum_{v \in D} v \) for some subset \( D \subset C \).

The next statement is the (weaker) analogue of Lemma \[2\] for the \( \ell_\infty \) norm. Note that we require the set \( W \) to contain only a few vectors. The proof is longer and it uses Chobanjan’s transference theorem (for the \( \ell_\infty \) norm) so we postpone it to Section \[7\].

**Lemma 4.** Assume \( W \subset B_\infty^d \), \(|W| = m \leq 5d \), and \( \|s(W)\|_\infty = O(\sqrt{d}) \). Then there is an ordering \( w_1, \ldots, w_m \) of the vectors in \( W \) such that

\[ \max_{h=1,\ldots,m} \left\| \sum_{1}^{h} w_i \right\|_\infty = O(\sqrt{d}). \]

**Proof of Theorem \[5\]** The lower bound uses Hadamard matrices and is given in \[1\]. For the upper bound we assume, rather for convenience than necessity, that the set \( V \subset \mathbb{R}^d \) is in general position, for instance, no \( d \) vectors from \( V \) are linearly dependent. The general case follows from this by a limit argument. We assume further that \(|V| = n > 5d \) since for \( n \leq 5d \) the result is a consequence of Lemma \[4\]. Set \( m = \lceil n/(2d) \rceil \).
We are going to define linear dependencies $\gamma_i$, for $i = 1, 2, \ldots, m - 1$ so that the sets

\[ A_i = \{ v \in V : \gamma_i(v) = 1 \}, \quad C_i = \{ v \in V : 0 < \gamma_i(v) < 1 \} \]

satisfy the conditions

\[ A_i \subset A_{i+1}, \quad (2i - 1)d \leq |A_i| < h_i := \sum_{v \in V} \gamma_i(v) \leq 2di, \quad |C_i| = d. \]

The construction is recursive and is similar to how $\alpha$ and $\gamma \in \mathbb{R}^V$ were constructed. For $i = 1$ we take an arbitrary vertex $\alpha$ of the convex polytope $P(V, 2d)$, then $|Q| = d + 1$ (because of the general position assumption) and $d \leq |U_1| < 2d$ follows. We construct $\gamma$ as specified in Remark 2 and (I).

Then define $\gamma_1 = \gamma$, set $A_1 = \{ v \in V : \gamma_1(v) = 1 \}$, $C_1 = \{ v \in V : 0 < \gamma_1(v) < 1 \}$. General position implies that $|C_1| = d$ and then $d \leq |A_1| < h_1 = \sum_{v \in V} \gamma_1(v) \leq 2d$.

Assume next that $\gamma_1, \ldots, \gamma_i$ have been constructed ($1 < i < m - 1$), and the sets $A_j, C_j$ for $j \leq i$ satisfy the required conditions. Define the convex polytope

\[ P_{i+1} = \{ \alpha \in P(V, 2d(i+1)) : \alpha(v) = 1 \ (\forall v \in A_i) \} \]

We check that $P_{i+1}$ is non-empty. As $|A_i| < h_i \leq 2di$, the linear dependence $\alpha = \gamma_i + t(1 - \gamma_i)$ lies in $P_{i+1}$ for a suitable $t$, we only have to check that $0 < t < 1$ as this implies $0 \leq \alpha(v) = \gamma_i(v) + t(1 - \gamma_i(v)) \leq 1$. To fulfill the condition $\sum_{v \in V} \alpha(v) = 2d(i + 1)$, we must set

\[ t = \frac{2d(i+1) - h_i}{n-h_i} = 1 - \frac{n - 2d(i+1)}{n-h_i}. \]

Thus $0 < t < 1$ indeed as $h_i \leq 2di$.

Next, let $\alpha_{i+1}$ be a fixed vertex of $P_{i+1}$. The method recorded in Remark 2 gives another linear dependence $\gamma_{i+1}$ with $|C_{i+1}| = d$. $A_i \subset A_{i+1}$ by the construction. All $v \in V$ with $\alpha_{i+1}(v) = 1$ are in $A_{i+1}$, and there are at least $2d(i+1) - d$ of them. Thus $(2i + 1)d \leq |A_{i+1}|$. Further $|A_{i+1}| < h_{i+1}$ follows since $\gamma_{i+1}(v) = 1$ for every $v \in A_{i+1}$ and $h_{i+1} \leq 2d(i + 1)$ because $h_{i+1} = \sum_{v \in V} \gamma_{i+1}(v) \leq \sum_{v \in V} \alpha_{i+1}(v) = 2d(i + 1)$.

The construction is almost finished, as a last step we define $A_0 = C_0 = \emptyset$.

We use Lemma 3 next. The parallelopotope $P := \sum_{v \in C_i} [0, v]$ contains the point $-s(A_i)$, since $0 = s(A_i) + \sum_{v \in C_i} \gamma_i(v)v$. A vertex of $P$ is of the form $s(D) = \sum_{v \in D} v$, where $D$ is a subset of $C_i$. By Lemma 3 there is a $D_i \subset C_i$ such that the vertex $s(D_i)$ is at distance $O(\sqrt{d})$ from $-s(A_i)$. Thus the vector $z_i = s(A_i \cup D_i)$ is short, namely, $\|z_i\| = O(\sqrt{d})$. Note that by setting $D_0 = \emptyset$, we have $z_0 = 0$ which is again of norm $O(\sqrt{d})$.

For the next step of the proof we first check that the size of the symmetric difference $(A_{i+1} \cup D_{i+1}) \Delta (A_i \cup D_i)$ is at most $5d$. This holds for $i = 0$. For larger $i$, $D_{i+1}$ and $A_{i+1}$ are disjoint, and $A_{i+1}$ contains $A_i$, so the symmetric difference is the same a $X \Delta D_i$, where $X = (A_{i+1} \backslash A_i) \cup D_{i+1}$. Here $|A_{i+1} \backslash A_i| < 3d$, and both $D_i$ and $D_{i+1}$ have at most $d$ elements, which gives the upper bound $5d$. 

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Now \( z_i - s(D_i) + s(X) = z_{i+1} \). Thus, adding at most \( 5d \) vectors from \( B_\infty^d \) to \( z_i \) one arrives at \( z_{i+1} \), and both \( z_i, z_{i+1} \) are short. Define

\[
W = \{ -u : u \in D_i \setminus X \} \bigcup (X \setminus D_i).
\]

Then \( W \) is a subset of \( B_\infty^d \), of at most \( 5d \) elements, such that \( s(W) = \sum_{w \in W} w = z_{i+1} - z_i \). Thus \( \| s(W) \| = O(\sqrt{d}) \). By applying Lemma 3 to \( W \) we get an ordering \( w_1, \ldots, w_m \) such that every partial sum along this ordering is \( O(\sqrt{d}) \). Then for every \( h = 1, \ldots, m \),

\[
\| z_i + \sum_{1}^{h} w_j \| \leq \| z_i \| + \| \sum_{1}^{h} w_j \| = O(\sqrt{d}).
\]

In the original problem we have to show that for every \( k \leq n \) there is a set \( U \subset V \) of size \( k \) with \( \| s(U) \| = O(\sqrt{d}) \). This is clear when \( k \) equals the size of some \( A_i \cup D_i \), but what is to be done for the other \( k \)? Well, such a \( k \) lies between \( |A_i \cup D_i| \) and \( |A_{i+1} \cup D_{i+1}| \) for some \( i \). Note that \( z_i = s(A_i \cup D_i) \). Moreover, each sum \( z_i + w_1 + \ldots + w_k \) is the sum of vectors in a subset of \( V \). This can be seen by induction on \( h \). The case \( h = 0 \) is clear. The induction step \( h - 1 \rightarrow h \) is clear again when \( w_h \) does not come from \( D_i \), simply one more term appears in the sum. If however \( w_h \) comes from \( D_i \), then it cancels the previous \( -w_h \) that is a unique term in \( s(A_i \cup D_i) \). So each partial sum is a subset-sum. The number of elements in these subsets increases or decreases by one when the next \( w_h \) is added. Then for every \( k \) between \( |A_i \cup D_i| \) and \( |A_{i+1} \cup D_{i+1}| \) there is a partial sum containing exactly \( k \) terms.

\[\square\]

**Remark 4.** The above proof yields a slightly stronger statement: we construct a chain of subsets of \( V \), each with sum of order of magnitude \( O(\sqrt{d}) \), so that the cardinality of two consecutive subsets differ by one, and the chain traverses from the empty set to \( V \). We have hoped to give a better value for the Steinitz constant \( S(B_2^d) \) or \( S(B_\infty^d) \) by a suitable modification of the argument (we would need an increasing chain of subsets with the previous properties), but our efforts have failed so far.

**Remark 5.** A simpler proof may be given if one only aims for the existence a \( k \)-element subset with small sum. We may assume that \( k \leq n - d \). Starting from a vertex of \( P(v, k - d) \) and using Lemma 3 similarly to the proof of Theorem 3 we can construct a set \( W \) so that \( \| s(W) \| \leq 6\sqrt{d} \), and \( k - 2d \leq |W| \leq k \). Let \( \alpha \) be the characteristic function of \( W \), i.e. \( \alpha(v) = 1 \) if \( v \in W \), and 0 otherwise. Let \( l = |W| \), and set \( t \) so that \( l + t(n - l) = k + d \). Then \( t \leq 1 \).

Next, consider the set \( P \) of the linear dependencies \( \beta : V \rightarrow [0, 1] \) with

\[
\sum_{v \in V} \beta(v)v = (1 - t)s(W), \quad \sum_{v \in V} \beta(v) = k + d, \quad \beta(v) = 1(\forall v \in W).
\]

Then \( P \) is a non-empty convex polytope, since \( \alpha + t(1 - \alpha) \) satisfies all the above conditions. Take an arbitrary a vertex of \( P \). As before, invoking Lemma 3 we find a set \( Y \) so that \( \| s(Y) - (1 - t)s(W) \|_\infty = O(\sqrt{d}) \), and
\[|Y| - (k + d)| \leq d.\] Furthermore, the construction implies that \(W \subset Y\). Hence,
\[k - 2d \leq |W| \leq k \leq |Y| \leq k + 2d,
\] and \(\|s(W)\| = O(\sqrt{d})\) as well as \(\|s(Y)\| = O(\sqrt{d})\). We finish the proof by applying Lemma 4 to the set \(Y \setminus W\).

**Remark 6.** The above proofs translate for arbitrary norms as long as the analogues of Lemmas 1 and 2 (or Lemmas 3 and 4) may be established.

### 7. **Proof of Lemma 4**

For this lemma it is natural to use Chobanyan’s transference theorem [5] (see also [6]), which connects Steinitz’s theorem with sign assignments to vectors in a sequence.

Assume \(v_1, \ldots, v_n\) is a sequence of vectors from the unit ball \(B\) of an arbitrary norm on \(\mathbb{R}^d\). It is proved in [2] that there are signs \(\varepsilon_1, \ldots, \varepsilon_n = \pm 1\) such that

\[(2) \quad \max_{k=1,\ldots,n} \left\| \sum_{i=1}^{k} \varepsilon_i v_i \right\| \leq 2d - 1.
\]

This is a general bound that does not depend on \(n\) and the norm. But better estimates are valid for specific norms and some (small) values of \(n\). For fixed \(B\) and \(n\) let \(F(B, n)\), the sign sequence constant of \(B\), be defined as the smallest number that one can write on the right hand side of (2), and let \(F(B) = \sup_n F(B, n)\). It is quite easy to see for instance that \(F(B_2^d, n) \leq \sqrt{n}\) for all \(n\) (but we don’t need this). What we need is a result of Spencer [11, Theorem 1.4]:

**Fact 1.** \(F(B_\infty^d, d) \leq K \sqrt{d}\) where \(K\) is a universal constant.

Chobanyan’s transference theorem [5] says that, for every norm with unit ball \(B\), \(S(B) \leq F(B)\), that is, the Steinitz constant is at most as large as the sign sequence constant. We need a slightly stronger variant, so we define \(S(B, n)\) as the smallest number \(R\) such that for every set \(V \subset B\) with \(s(V) = 0\) and \(|V| = n\) there is an ordering \(v_1, \ldots, v_n\) of the elements in \(V\) such that

\[\max_{k=1,\ldots,n} \left\| \sum_{i=1}^{k} v_i \right\| \leq R.
\]

Of course, \(S(B) = \sup_n S(B, n)\). Here comes the stronger version of Chobanyan’s theorem, and comes without proof as the proof is identical with the original one.

**Theorem 8.** For every norm in \(\mathbb{R}^d\) with unit ball \(B\), \(S(B, n) \leq F(B, n)\).

Theorem 8 and Fact 1 imply the following.

**Fact 2.** Given \(V \subset B_\infty^d\) with \(|V| = m\) where \(m \leq 5d\) and \(s(V) = 0\), there is an ordering \(v_1, \ldots, v_m\) of \(V\) such that \(\max_{h=1,\ldots,m} \left\| \sum_{i=h}^{m} v_i \right\|_\infty \leq K_1 \sqrt{d}\), where \(K_1\) is a universal constant.
Proof. We note first that for $m \leq d$ this follows directly from Fact 1 and Theorem 8, with $K_1 = K$. For $m \geq d$, take the natural embedding of $\mathbb{R}^d$ into $\mathbb{R}^m$, so that $V$ lies in the ball of radius $\sqrt{d}$ in $\mathbb{R}^m$. Apply Fact 1 and Theorem 8 there, and you get an ordering of $V$ in $\mathbb{R}^d$ along which all partial sums have norm at most $K \sqrt{m} \leq K \sqrt{5d}$. Thus Fact 2 holds with $K_1 = \sqrt{5K}$.

\[ \square \]

\textbf{Proof of Lemma 4}. We need a concrete bound on $\|s(W)\|_\infty$, so suppose that $\|s(W)\|_\infty \leq K_2 \sqrt{d}$. For $w \in W$ define $w^* = w - \frac{1}{m}s(W)$. Then $\|w^*\|_\infty \leq \|w\|_\infty + \frac{1}{m}\|s(W)\|_\infty \leq 2s(W)$, being the sum of $m$ vectors from $B_\infty^d$. Thus Fact 2 holds with $K_2 \sqrt{d}$.

\[ \square \]

8. An application: proof of Theorem 8

We proceed by induction on $m$. For $m = 1$, the assertion is clearly true. For the induction step $(m - 1) \rightarrow m$, let $V \subset B$ with $|V| = (k - 1)m + 1$ and $\|s(V)\| \leq 1$. Set $v_0 = -s(V)$ so $\|v_0\| \leq 1$. Define $V_0 = V \setminus \{v_0\}$. Then $V_0 \subset B$ and $s(V_0) = 0$. So by Theorem 8 there exists a subset $U$ of $V_0$ of size $k$, with $\|s(U)\| \leq 1$. We are done if $v_0 \notin U$. So suppose that $v_0 \in U$. Then $v_0 \notin W := V \setminus U$, and $\|s(W)\| \leq 1$ because

$\quad s(U) = -s(W).$

Here $W$ is of size $(m - 1)(k - 1) + 1$, so the induction hypothesis implies that $W$ contains a subset $U$ of size $k$ with $\|s(U)\| \leq 1$.

\[ \square \]

We mention finally that Theorem 8 is equivalent to the following Helly type statement. If $V \subset B$ and $|V| = (k - 1)m + 1$, and $\|s(U)\| > 1$ for every set $U \subset V$ of size $k$, then $\|s(V)\| > 1$.

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