MULTIPLY UNION FAMILIES IN \( \mathbb{N}^n \)

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Abstract. Let \( A \subseteq \mathbb{N}^n \) be an \( r \)-wise \( s \)-union family, that is, a family of sequences with \( n \) components of non-negative integers such that for any \( r \) sequences in \( A \) the total sum of the maximum of each component in those sequences is at most \( s \). We determine the maximum size of \( A \) and its unique extremal configuration provided (i) \( n \) is sufficiently large for fixed \( r \) and \( s \), or (ii) \( n = r + 1 \).

1. Introduction

Let \( \mathbb{N} := \{0, 1, 2, \ldots \} \) denote the set of non-negative integers, and let \([n] := \{1, 2, \ldots, n\} \). Intersecting families in \( 2^{[n]} \) or \( \{0, 1\}^n \) are one of the main objects in extremal set theory. The equivalent dual form of an intersecting family is a union family, which is the subject of this paper. In \cite{FranklTokushige} Frankl and Tokushige proposed to consider such problems not only in \( \{0, 1\}^n \) but also in \( [q]^n \). They determined the maximum size of 2-wise \( s \)-union families (i) in \([q]^n \) for \( n > n_0(q, s) \), and (ii) in \( \mathbb{N}^3 \) for all \( s \) (the definitions will be given shortly). In this paper we extend their results and determine the maximum size and structure of \( r \)-wise \( s \)-union families in \( \mathbb{N}^n \) for the following two cases: (i) \( n \geq n_0(r, s) \), and (ii) \( n = r + 1 \).

For a vector \( \mathbf{x} \in \mathbb{R}^n \), we write \( x_i \) or \((\mathbf{x})_i \) for the \( i \)th component, so \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \). Define the weight of \( \mathbf{a} \in \mathbb{N}^n \) by

\[
|\mathbf{a}| := \sum_{i=1}^{n} a_i.
\]

For a finite number of vectors \( \mathbf{a}, \mathbf{b}, \ldots, \mathbf{z} \in \mathbb{N}^n \) define the join \( \mathbf{a} \lor \mathbf{b} \lor \cdots \lor \mathbf{z} \) by

\[
(\mathbf{a} \lor \mathbf{b} \lor \cdots \lor \mathbf{z})_i := \max\{a_i, b_i, \ldots, z_i\},
\]

and we say that \( A \subseteq \mathbb{N}^n \) is \( r \)-wise \( s \)-union if

\[
|\mathbf{a}_1 \lor \mathbf{a}_2 \lor \cdots \lor \mathbf{a}_r| \leq s \text{ for all } \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r \in A.
\]

The width of \( A \subseteq \mathbb{N}^n \) is defined to be the maximum \( s \) such that \( A \) is \( s \)-union. In this paper we address the following problem.

Problem. For given \( n, r \) and \( s \), determine the maximum size \( |A| \) of \( r \)-wise \( s \)-union families \( A \subseteq \mathbb{N}^n \).

To describe candidates \( A \) that give the maximum size to the above problem, we need some more definitions. Let us introduce a partial order \( \prec \) in \( \mathbb{R}^n \). For \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \) we let \( \mathbf{a} \prec \mathbf{b} \) iff \( a_i \leq b_i \) for all \( 1 \leq i \leq n \). Then we define a down set for \( \mathbf{a} \in \mathbb{N}^n \) by

\[
D(\mathbf{a}) := \{\mathbf{c} \in \mathbb{N}^n : \mathbf{c} \prec \mathbf{a}\},
\]

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and for $A \subseteq \mathbb{N}^n$ let
\[ \mathcal{D}(A) := \bigcup_{a \in A} \mathcal{D}(a). \]

Similarly, we define an up set at distance $d$ from $a \in \mathbb{N}^n$ by
\[ \mathcal{U}(a, d) := \{ a + e \in \mathbb{N}^n : e \in \mathbb{N}^n, |e| = d \}. \]

We say that $a \in \mathbb{N}^n$ is an equitable partition, if all $a_i$'s are as close to each other as possible, more precisely, $|a_i - a_j| \leq 1$ for all $i, j$. Let $1 := (1, 1, \ldots, 1) \in \mathbb{N}^n$.

For $r, n \in \mathbb{N}$ and $a \in \mathbb{N}^n$ define a family $K$ by
\[ K = K(r, n, a, d) := \bigcup_{i=0}^{\lfloor \frac{d}{r} \rfloor} \mathcal{D}(\mathcal{U}(a + i1, d - ui)), \]
where $u = n - r + 1$. We will show that this is an $r$-wise $s$-union family, see Claim 3 in the next section.

**Conjecture.** If $A \subseteq \mathbb{N}^n$ is $r$-wise $s$-union, then
\[ |A| \leq \max_{0 \leq d \leq \lfloor \frac{s}{r} \rfloor} |K(r, n, a, d)|, \]
where $a \in \mathbb{N}^n$ is an equitable partition with $|a| = s - rd$. Moreover if equality holds, then $A = K(r, n, a, d)$ for some $0 \leq d \leq \lfloor \frac{s}{r} \rfloor$.

We first verify the conjecture when $n$ is sufficiently large for fixed $r, s$. Let $e_i$ be the $i$-th standard base of $\mathbb{R}^n$, that is, $(e_i)_j = \delta_{ij}$. Let $\tilde{e}_0 = 0$, and $\tilde{e}_i = \sum_{j=1}^{i} e_j$ for $1 \leq i \leq n$, e.g., $\tilde{e}_n = 1$.

**Theorem 1.** Let $r$ and $s$ be fixed positive integers. Write $s = dr + p$ where $d$ and $p$ are non-negative integers with $0 \leq p < r$. Then there exists $n_0(r, s)$ such that if $n > n_0(r, s)$ and $A \subseteq \mathbb{N}^n$ is $r$-wise $s$-union, then
\[ |A| \leq |\mathcal{D}(\mathcal{U}(\tilde{e}_p, d))|. \]
Moreover if equality holds, then $A$ is isomorphic to $\mathcal{D}(\mathcal{U}(\tilde{e}_p, d)) = K(r, n, \tilde{e}_p, d)$.

We mention that the case $A \subseteq \{0, 1\}^n$ of Theorem 1 is settled in [7], and the case $r = 2$ of Theorem 1 is proved in [2] in slightly stronger form. We also notice that if $A \subseteq \{0, 1\}^n$ is 2-wise $(2d + p)$-union, then the Katona’s $t$-intersection theorem [3] states that $|A| \leq |\mathcal{D}(\mathcal{U}(\tilde{e}_p, d) \cap \{0, 1\}^n)|$ for all $n \geq s$.

Next we show that the conjecture is true if $n = r + 1$. We also verify the conjecture or general $n$ if $A$ satisfies some additional properties described below.

Let $A \subseteq \mathbb{N}^n$ be $r$-wise $s$-union. For $1 \leq i \leq n$ let
\[ m_i := \max\{x_i : x \in A\}. \]
If $n - r$ divides $|m| - s$, then we define
\[ d := \frac{|m| - s}{n - r} \geq 0, \]
and for $1 \leq i \leq n$ let
\[ a_i := m_i - d, \]
and we assume that $a_i \geq 0$. In this case we have $|a| = s - rd$. Since $|a| \geq 0$ it follows that $d \leq \lfloor \frac{s}{r} \rfloor$. For $1 \leq i \leq n$ define $P_i \in \mathbb{N}^n$ by

$$P_i := a + d e_i,$$

where $e_i$ denotes the $i$th standard base, for example, $P_2 = (a_1, a_2 + d, a_3, \ldots, a_n)$.

**Theorem 2.** Let $A \subset \mathbb{N}^n$ be $r$-wise $s$-union. Assume that $P_i$’s are well-defined and

$$\{P_1, \ldots, P_n\} \subset A.$$ (1)

Then it follows that

$$|A| \leq \max_{0 \leq d' \leq \lfloor \frac{s}{r} \rfloor} |K(r, n, a', d')|,$$

where $a' \in \mathbb{N}^n$ is an equitable partition with $|a'| = s - rd'$. Moreover if equality holds, then $A = K(r, n, a', d')$ for some $0 \leq d' \leq \lfloor \frac{s}{r} \rfloor$.

We will show that the assumption (1) is automatically satisfied when $n = r + 1$.

**Corollary.** If $n = r + 1$, then Conjecture is true.

Notation: For $a, b \in \mathbb{N}^n$ we define $a \setminus b \in \mathbb{N}^n$ by $(a \lor b) - b$, in other words, $(a \setminus b)_i := \max\{a_i - b_i, 0\}$. The support of $a$ is defined by $\text{supp}(a) := \{j : a_j > 0\}$.

2. PROOF OF THEOREM 1 — THE CASE WHEN $n$ IS LARGE

Let $r, s$ be given, and let $s = dr + p, 0 \leq p < r$.

**Claim 1.** $|\mathcal{D}(U(\bar{e}_p, d))| = 2^p \binom{n+d}{d}$.

**Proof.** By definition we have

$$\mathcal{D}(U(\bar{e}_p, d)) = \{x + y \in \mathbb{N}^n : |x| \leq d, y \prec e_p\}.$$ 

The number of $x \in \mathbb{N}^n$ with $|x| \leq d$ is equal to the number of non-negative integer solutions of $x_1 + \cdots + x_n \leq d$, which is $\binom{n+d}{d}$. It is $2^p$ that the number of $y \in \mathbb{N}^n$ satisfying $y \prec e_p$.

Let $A \subset \mathbb{N}^n$ be $r$-wise $s$-union with maximal size. So $A$ is a downset. We will show that $|A| \leq 2^p \binom{n+d}{d}$. Notice that this RHS is $\Theta(n^d)$ for fixed $r, s$.

First suppose that there is $t$ with $2 \leq t \leq r$ such that $A$ is $t$-wise $(dt + p)$-union, but not $(t-1)$-wise $(dt - 1 + p)$-union. In this case, by the latter condition, there are $b_1, \ldots, b_{t-1} \in A$ such that $|b| \geq dt - 1 + p + 1$, where $b = b_1 \lor \cdots \lor b_{t-1}$. Then, by the former condition, for every $a \in A$ it follows that $|a \lor b| \leq dt + p$, so $|a \setminus b| \leq d - 1$. This gives us

$$A = \{x + y \in \mathbb{N}^n : |x| \leq d - 1, y \prec b\}.$$ 

There are $\binom{n+(d-1)}{d-1}$ choices for $x$ satisfying $|x| \leq d - 1$. On the other hand, the number of $y$ with $y \prec b$ is independent of $n$ (so it is a constant depending on $r$ and $s$ only). In fact $|b| \leq (t - 1)s < rs$, and there are less than $2^{rs}$ choices for $y$. Thus we get $|A| \leq (\binom{n+(d-1)}{d-1})2^{rs} = O(n^{d-1})$ and we are done.

Next we suppose that $A$ is $t$-wise $(dt + p)$-union for all $1 \leq t \leq r$. (2)
The case $t = 1$ gives us $|a| \leq d + p$ for every $a \in A$. If $p = 0$, then this means that $A \subset D(U(0,d))$, which finishes the proof for this case. So, from now on, we assume that $1 \leq p < r$. Then there is $u$ with $u \geq 1$ such that there exist $b_1, \ldots, b_u \in A$ satisfying

$$|b| = u(d + 1),$$

where $b := b_1 \lor \cdots \lor b_u$. In fact we have (3) for $u = 1$, if otherwise $A \subset D(U(0,d))$. If $u = p + 1$ then (3) fails. In fact setting $t = p + 1$ in (2) we see that $A$ is $(p+1)$-wise $((p+1)(d+1) - 1)$-union. We choose maximal $u$ with $1 \leq u \leq p$ satisfying (3), and fix $b = b_1 \lor \cdots \lor b_u$. By this maximality, for every $a \in A$, it follows that $|a \lor b| \leq (u + 1)(d + 1) - 1$, and

$$|a \setminus b| \leq d.$$ (4)

Using (3) we partition $A$ into $\bigsqcup_{i=0}^d A_i$, where

$$A_i := \{x + y \in A : |x| = i, y \prec b\}.$$ Then we have $|A_i| \leq \binom{n+1}{i}2^{|b|}$. Noting that $|b| \leq (d + p)u = O(1)$ it follows $\sum_{i=0}^{d-1} |A_i| = O(n^{d-1})$. So the size of $A_d$ is essential as we will see below.

We naturally identify $a \in A_d$ with a subset of $[n] \times \{1, \ldots, d + p\}$. Formally let

$$\phi(a) := \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq a_i\}.$$ We say that $b' \prec b$ is rich if there exist vectors $c_1, \ldots, c_{dr}$ of weight $d$ such that $b' \lor c_j \in A$ for every $j$, and the $dr + 1$ subsets $\phi(c_1), \ldots, \phi(c_{dr}), \phi(b)$ are pairwise disjoint. Informally, $b'$ is rich if it can be extended to a $(|b'| + d)$-element subset of $A$ in $dr$ ways disjointly outside $b$. We are comparing our family $A$ with the reference family $D(U(\tilde{e}_p,d))$, and we define $b$ which plays a role of $\tilde{e}_p$ in our family, namely, let us define

$$\tilde{b} := \lor \{b' : b' \text{ is rich}\}.$$ Claim 2. $|\tilde{b}| \leq p$. Proof. Suppose the contrary, then there are distinct rich $b_1', \ldots, b_{p+1}'$. Let $c_1^{(i)}, \ldots, c_{dr}^{(i)}$ support the richness of $b_j'$. Let $a_1 := b_1' \lor c_{j_1}^{(i)} \in A$, say, $j_1 = 1$. Then choose $a_2 := b_2' \lor c_{j_2}^{(i)}$ so that $\phi(a_1)$ and $\phi(a_2)$ are disjoint. If $i \leq p$, then having $a_1, \ldots, a_i$ chosen, we only used $id$ elements as $\bigcup_{l=1}^i \phi(c_{j_l}^{(i)})$, which intersect at most $id$ of $c_1^{(i+1)}, \ldots, c_{dr}^{(i+1)}$, and since $id \leq pd < rd$ we still have some $c_{j_l+1}^{(i+1)}$ disjoint from any already chosen vectors. So we can continue this procedure until we get $a_{p+1} := b_{p+1}' \lor c_{j_{p+1}}^{(i+1)} \in A$ such that all $\phi(a_1), \ldots, \phi(a_{p+1})$ are disjoint. However, these vectors yield $|a_1 \lor \cdots \lor a_{p+1}| \geq (p + 1)(d + 1)$, which contradicts (2) at $t = p + 1$. \hfill \Box

If $y \prec b$ is not rich, then

$$\{\phi(x + y) \setminus \phi(b) : x + y \in A_d, |x| = d\}$$

is a family of $d$-element subsets on $(d + p)n$ vertices, which has no $dr$ pairwise disjoint subsets (so the matching number is $dr - 1$ or less). Thus, by the Erdős matching theorem [3], the size of this family is $O(n^{d-1})$. There are at most $2^{|b|} = O(1)$ choices
for non-rich \( y \prec b \), and we can conclude that the number of vectors in \( A_d \) coming from non-rich \( y \) is \( O\left(n^{d-1}\right) \). Then the remaining vectors in \( A_d \) comes from rich \( y \prec b \), and the number of such vectors is at most \( 2^{|b|}(n^d) \). Consequently we get

\[
|A| \leq 2^{|b|}\binom{n + d}{d} + O(n^{d-1}).
\]

Recall that the reference family is of size \( 2^p(n^d) \), and \(|b| \leq p \) from Claim 2. So we only need to deal with the case when there are exactly \( 2^p \) rich sets, in other words, \( b = \hat{e}_p \) (by renaming coordinates if necessary). We show that \( A \subseteq D(U(\hat{e}_p, d)) \).

Suppose the contrary, then there is an \( a \in A \) such that \( |a \setminus \hat{e}_p| \geq d + 1 \). Since \( \hat{e}_p \) is rich there are pairwise disjoint vectors \( c_1, \ldots, c_{r-1} \) of weight \( d \), outside \( b \). Let \( a_i := \hat{e}_p \cup c_i \in A_d \). Then we get

\[
|a \cup (a_1 \cup \cdots \cup a_{r-1})| \geq (d + 1) + p + (r - 1)d = dr + p + 1 = s + 1,
\]

which contradicts that \( A \) is \( r \)-wise \( s \)-union. This completes the proof of Theorem 1.

3. The polytope \( P \) and proof of Theorem 2

We introduce a convex polytope \( P \in \mathbb{R}^n \), which will play a key role in our proof. This polytope is defined by the following \( n + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-r+1} \) inequalities:

\[
\begin{align*}
  x_i &\geq 0 & \text{if } 1 \leq i \leq n, \\
  \sum_{i \in I} x_i &\leq \sum_{i \in I} a_i + d & \text{if } 1 \leq |I| \leq n - r + 1, I \subset [n].
\end{align*}
\]

Namely,

\[
P := \{ x \in \mathbb{R}^n : x \text{ satisfies (5) and (6)} \}.
\]

Let \( L \) denotes the integer lattice points in \( P \):

\[
L = L(r, n, a, d) := \{ x \in \mathbb{N}^n : x \in P \}.
\]

**Lemma 1.** The two sets \( K \) and \( L \) are the same, and \( r \)-wise \( s \)-union.

**Proof.** This lemma is a consequence of the following three claims.

**Claim 3.** The set \( K \) is \( r \)-wise \( s \)-union.

**Proof.** Let \( x_1, x_2, \ldots, x_r \in K \). We show that \( |x_1 \cup x_2 \cup \cdots \cup x_r| \leq s \). We may assume that \( x_j \in U(a + i_j \mathbf{1}, d - w_{ij}) \), where \( u = n - r + 1 \). We may also assume that \( i_1 \geq i_2 \geq \cdots \geq i_r \). Let \( b := a + i_1 \mathbf{1} \). Then, informally, \(|b \cup x - b|\) counts the excess
of \( x \) above \( b \), more precisely, it is \( \sum_{j \in [n]} \max \{0, x_j - b_j\} \). Thus we have

\[
|x_1 \lor x_2 \lor \cdots \lor x_r| \leq |b| + \sum_{j=1}^{r} |b \lor x_j - b|
\]

\[
\leq |a| + ni_1 + \sum_{j=1}^{r} \left( (d - ui_j) - (i_1 - i_j) \right)
\]

\[
= a + dr + (n - r)i_1 - \sum_{j=1}^{r} (u - 1)i_j
\]

\[
= s - \sum_{j=2}^{r} j_j \leq s,
\]

as required. \( \square \)

**Claim 4.** \( K \subset L \).

**Proof.** Let \( x \in K \). We show that \( x \in L \), that is, \( x \) satisfies (3) and (4). Since (3) is clear by definition of \( K \), we show that (4). To this end we may assume that \( x \in \mathcal{U}(a + i1, d - ui) \), where \( u = n - r + 1 \) and \( i \leq \left\lfloor \frac{d}{u} \right\rfloor \). Let \( I \subset [n] \) with \( 1 \leq |I| \leq u \). Then \( i|I| \leq ui \). Thus it follows

\[
\sum_{j \in I} x_j \leq \sum_{j \in I} a_j + i|I| + (d - ui) \leq \sum_{j \in I} a_j + d,
\]

which confirms (4). \( \square \)

**Claim 5.** \( K \supset L \).

**Proof.** Let \( x \in L \). We show that \( x \in K \), that is, there exists some \( i' \) such that \( 0 \leq i' \leq \left\lfloor \frac{d}{u - r + 1} \right\rfloor \) and \( |x \setminus (a + i'1)| \leq d - (n - r + 1)i' \).

We write \( x \) as

\[
x = (a_1 + i_1, a_2 + i_2, \ldots, a_n + i_n),
\]

where we may assume that \( d \geq i_1 \geq i_2 \geq \cdots \geq i_n \). We notice that some \( i_j \) can be negative. Since \( x \in L \) it follows from (4) (a part of the definition of \( L \)) that if \( 1 \leq |I| \leq n - r + 1 \) and \( I \subset [n] \), then

\[
\sum_{j \in I} i_j \leq d.
\]

Let \( J := \{ j : x_j \geq a_j \} \) and we argue separately by the size of \( |J| \).

If \( |J| \leq n - r + 1 \), then we may choose \( i' = 0 \). In fact,

\[
|x \setminus a| = \max \{0, i_1\} + \max \{0, i_2\} + \cdots + \max \{0, i_{n-r+1}\}
\]

\[
= \max \left\{ \sum_{j \in I} i_j : I \subset 2^{n-r+1} \right\} \leq d.
\]
If $|J| \geq n - r + 2$, then we may choose $i' = i_{n-r+2}$. In fact, by letting $i' := i_{n-r+2}$, we have

$$|x \setminus (a + i'1)| = (i_1 - i') + (i_2 - i') + \cdots + (i_{n-r+1} - i') \leq d - (n - r + 1)i'.$$

We need to check $0 \leq i' \leq \lfloor \frac{d}{n-r+1} \rfloor$. It follows from $|J| \geq n - r + 2$ that $i' \geq 0$. Also, $d \geq i_1 \geq i_2 \geq \cdots \geq i_{n-r+2}$ and $i_1 + i_2 + \cdots + i_{n-r+1} \leq d$ yield $i' \leq \lfloor \frac{d}{n-r+1} \rfloor$.  

This completes the proof of Lemma \textbf{1}.

Let

$$\sigma_k(a) := \sum_{K \in \binom{[n]}{k}} \prod_{i \in K} a_i$$

be the $k$th elementary symmetric polynomial of $a_1, \ldots, a_n$.

**Lemma 2.** The size of $K(r, n, a, d)$ is given by

$$|K(r, n, a, d)| = \sum_{j=0}^{n} \binom{d+j}{j} \sigma_{n-j}(a)$$

$$+ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=i+1}^{n} \left( \binom{d-u_i+j}{j} - \binom{d-u_i+u}{j} \right) \sigma_{n-j}(a+i1),$$

where $u = n - r + 1$. Moreover, for fixed $n, r, d$ and $|a|$, this size is maximized if and only if $a$ is an equitable partition.

**Proof.** For $J \subset [n]$ let $x| J$ be the restriction of $x$ to $J$, that is, $(x|J)_i$ is $a_i$ if $i \in J$ and 0 otherwise.

First we count the vectors in the base layer $\mathcal{D}(\mathcal{U}(a, d))$. To this end we partition this set into $\bigsqcup_{J \subset [n]} A_0(J)$, where

$$A_0(J) = \{a| J + e + b : \text{supp}(e) \subset J, |e| \leq d, \text{supp}(b) \subset [n] \setminus J, b_i < a_i \text{ for } i \notin J \}.$$  

The number of vectors $e$ with the above property is equal to the number of non-negative integer solutions of the inequality $x_1 + x_2 + \cdots + x_{|J|} \leq d$, which is $\binom{d+|J|}{|J|}$. The number of vectors $b$ is clearly $\prod_{t \in [n] \setminus J} a_t$. Thus we get

$$\sum_{J \in \binom{[n]}{j}} |A_0(J)| = \sum_{J \in \binom{[n]}{j}} \binom{d+|J|}{|J|} \prod_{l \in [n] \setminus J} a_l = \binom{d+j}{j} \sigma_{n-j}(a),$$

and $|\mathcal{D}(\mathcal{U}(a, d))| = \sum_{j=0}^{n} \binom{d+j}{j} \sigma_{n-j}(a)$.

Next we count the vectors in the $i$th layer:

$$\mathcal{D}(\mathcal{U}(a + i1, d - ui)) \setminus \left( \bigcup_{j=0}^{i-1} \mathcal{D}(\mathcal{U}(a + j1, d - uj)) \right).$$
For this we partition the above set into $\bigcup_{J \subset [n]} A_i(J)$, where

$$A_i(J) = \{(a + i\mathbf{1})|_J + e + b : \text{supp}(e) \subset J, \ d - u(i - 1) - |J| < |e| \leq d - ui, \ \text{supp}(b) \subset [n] \setminus J, \ b_l < a_l + i \text{ for } l \notin J\}.$$  

In this case we need $d - u(i - 1) < |J| + |e|$ because the vectors satisfying the opposite inequality are already counted in the lower layers $\bigcup_{j<i} A_j(J)$. We also notice that $d - u(i - 1) - |J| < d - ui$ implies that $|J| > u$. So $A_i(J) = \emptyset$ for $|J| \leq u$. Now we count the number of vectors $e$ in $A_i(J)$, or equivalently, the number of non-negative integer solutions of

$$d - u(i - 1) - |J| < x_1 + x_2 + \cdots + x_{|J|} \leq d - ui.$$

This number is $\binom{d-ui+j}{j} - \binom{d-ui+u}{j}$, where $j = |J|$. On the other hand, the number of vectors $b$ in $A_i(J)$ is $\prod_{i \in [n] \setminus J} (a_i + i)$. Consequently we get

$$\sum_{J \subset [n]} |A_i(J)| = \sum_{j=u+1}^{n} \left( \binom{d-ui+j}{j} - \binom{d-ui+u}{j} \right) \sigma_{n-j}(a + i\mathbf{1}).$$

Summing this term over $1 \leq i \leq \lfloor \frac{d}{u} \rfloor$ we finally obtain the second term of the RHS of $|K|$ in the statement of this lemma. Then, for fixed $|a|$, the size of $K$ is maximized when $\sigma_{n-1}(a)$ and $\sigma_{n-1}(a + i\mathbf{1})$ are maximized. By the property of symmetric polynomials, this happens if and only if $a$ is an equitable partition. \hfill \Box

**Proof of Theorem 3.** Let $A \subset \mathbb{N}^n$ be an $r$-wise $s$-union with (II). For $I \subset [n]$ let

$$m_I := \max \left\{ \sum_{i \in I} x_i : x \in A \right\}.$$  

**Claim 6.** If $I \subset [n]$ and $1 \leq |I| \leq n - r + 1$, then

$$m_I = \sum_{i \in I} a_i + d.$$  

*Proof.* Choose $j \in I$. By (II) we have $P_j \in A$ and

$$m_I \geq \sum_{i \in I} (P_j)_i = \sum_{i \in I} a_i + d. \quad (7)$$

We need to show that this inequality is actually an equality. Let $[n] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_r$ be a partition of $[n]$. Then it follows that

$$s \geq m_{I_1} + m_{I_2} + \cdots + m_{I_r} \geq \sum_{i \in [n]} a_i + rd = s,$$

where the first inequality follows from the $r$-wise $s$-union property of $A$, and the second inequality follows from (II). Since the left-most and the right-most sides are the same $s$, we see that all inequalities are equalities. This means that (II) is equality, as needed. \hfill \Box
By this claim if $x \in A$ and $1 \leq |I| \leq n - r + 1$, then we have

$$\sum_{i \in I} x_i \leq m_I = \sum_{i \in I} a_i + d.$$ 

This means that $A \subset L$. Finally the theorem follows from Lemmas 1 and 2. □

**Proof of Corollary.** Let $n = r + 1$ and we show that (II) is satisfied. Let $A \subset \mathbb{N}^{r+1}$ be $r$-wise $s$-union with maximum size.

We first check that $P_i$’s are well-defined. For this, we need (i) $(n-r)(|m| - s)$, and (ii) $a_i \geq 0$ for all $i$. Since $n - r = 1$ we have (i). To verify (ii) we may assume that $m_1 \geq m_2 \geq \cdots \geq m_{r+1}$. Then $a_i \geq a_{r+1} = m_{r+1} - d$, so it suffices to show $m_{r+1} \geq d$. Since $A$ is $r$-wise $s$-union it follows that $m_1 + m_2 + \cdots + m_r \leq s$. This together with the definition of $d$ implies $d = |m| - s \leq m_{r+1}$, as needed.

Next we check that $x \in A$ satisfies (I) and (III). By definition we have $x_i \leq m_i = a_i + d$, so we have (III). Since $A$ is $r$-wise $s$-union, we have

$$(x_1 + x_2) + m_3 + \cdots + m_{r+1} \leq s,$$

or equivalently,

$$(x_1 + x_2) + (a_3 + d) + \cdots + (a_{r+1} + d) \leq s = |a| + rd.$$ 

Rearranging we get $x_1 + x_2 \leq a_1 + a_2 + d$, and we get the other cases similarly, so we obtain (I). Thus $A \subset L$. But by the maximality of $|A|$ we have $A = L$. Now noting that every $P_i$ satisfies (II) and (I), namely, $P_i$ is in $L$, and thus (II) is satisfied. □

**References**

