# ANTI-URYSOHN SPACES

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ABSTRACT. All spaces are assumed to be infinite Hausdorff spaces. We call a space *anti-Urysohn* (*AU in short*) iff any two non-emty regular closed sets in it intersect. We prove that
- for every infinite cardinal $\kappa$ there is a space of size $\kappa$ in which fewer than $cf(\kappa)$ many non-empty regular closed sets always intersect;
- there is a locally countable AU space of size $\kappa$ iff $\omega \le \kappa \le 2^{\mathfrak{c}}$.

A space with at least two non-isolated points is called *strongly anti-Urysohn* (*SAU in short*) iff any two infinite closed sets in it intersect. We prove that
- if $X$ is any SAU space then $\mathfrak{s} \le |X| \le 2^{2^{\mathfrak{c}}}$;
- if $\mathfrak{r} = \mathfrak{c}$ then there is a separable, crowded, locally countable, SAU space of cardinality $\mathfrak{c}$;
- if $\lambda > \omega$ Cohen reals are added to any ground model then in the extension there are SAU spaces of size $\kappa$ for all $\kappa \in [\omega_1, \lambda]$;
- if GCH holds and $\kappa \le \lambda$ are uncountable regular cardinals then in some CCC generic extension we have $\mathfrak{s} = \kappa$, $\mathfrak{c} = \lambda$, and for every cardinal $\mu \in [\mathfrak{s}, \mathfrak{c}]$ there is an SAU space of cardinality $\mu$.

The questions if SAU spaces exist in ZFC or if SAU spaces of cardinality $> \mathfrak{c}$ can exist remain open.

## 1. INTRODUCTION

In this paper "space" means "infinite Hausdorff topological space".

The space $X$ is called *anti-Urysohn* (*AU*, in short) iff $A \cap B \ne \emptyset$ for any $A, B \in \mathrm{RC}^+(X)$, where $\mathrm{RC}^+(X)$ denotes the family of non-empty regular closed sets in $X$.

We call the space $X$ *strongly anti-Urysohn* (*SAU*, in short) iff $|X'| > 1$, that is $X$ has at least two non-isolated points, and $A \cap B \ne \emptyset$ for any $A, B \in \mathcal{F}^+(X)$, where $\mathcal{F}^+(X)$ denotes the family of *infinite* closed subsets of $X$. Clearly, AU spaces are crowded and a crowded SAU space is AU. Our original intention was to include crowdedness in the

definition of SAU spaces. However, we changed our minds after we realized that it seems to be just as hard to construct them with the weaker property of having at least two non-isolated points.

What led us to consider AU spaces was not just idle curiosity. Co-operating via correspondence with Alan Dow, we have recently arrived at the result that in the Cohen model any separable and sequentially compact Urysohn space has cardinality $\leq \mathfrak{c}$. (This result will be published elsewhere.) The natural question if this holds for all (Hausdorff) spaces, however, remained open. When trying to find a ZFC counterexample, it was natural to look for spaces that are as much non-Urysohn as possible.

Actually, a countable AU space, under a different name, had been constructed by W. Gustin in [3] a long time ago, as a simple(r) example of a countable connected Hausdorff space. (It is obvious that AU spaces are connected.) The first example of a countable connected Hausdorff space was constructed by Urysohn in [7], but his construction is extremely long and complicated: just the description of his space takes up three pages. (We have no idea if Urysohn's example is AU or not.) A much simpler example was obtained by Gustin in [3] where the following was proved:

[3, Theorem 4.2] There is a countably infinite Hausdorff space $X$ such that no two distinct points in $X$ have disjoint closed neighbourhoods (i.e. $X$ is AU).

An even simpler construction of a countable connected Hausdorff space, which takes up only one page, was published by Bing in [1]. This is also presented as example 6.1.6 in Engelking's book [2]. Bing's example also turns out to be AU.

In contrast to this, we are not aware of any earlier appearance of SAU spaces. In fact, we admit that when we first considered them we did not think that they could exist.

Our notation and terminology is standard. In set theory we follow [6] and in topology [2].


## 2. Existence of anti-Urysohn spaces

In this section we show that for every infinite cardinal $\kappa$ there is an AU space of cardinality $\kappa$. Actually, we prove much more that is new and of interest even for the case $\kappa = \omega$.

To do that we need the following somewhat technical lemma that provides a general method for constructing AU spaces.

**Lemma 2.1.** *Assume that $\kappa$ is an infinite cardinal and $X$ is a space with $X \cap \kappa = \emptyset$, moreover $\{K_\alpha : \alpha < \kappa\}$ are pairwise disjoint non-empty compact subsets of $X$ such that*

*(1) if $a \subset \kappa$ is cofinal then $\bigcup_{\alpha \in a} K_\alpha$ is dense in $X$;*

*(2) $Y = X \setminus \bigcup_{\alpha < \kappa} K_\alpha$ is also dense in $X$.*

*Define the topology $\varrho$ on $Z = Y \cup \kappa$ as follows:*

- *for $y \in Y$ the family $\{U \cap Y : y \in U \in \tau(X)\}$,*
- *for $\alpha \in \kappa$ the family*

$$\Big\{\{\alpha\} \cup (W \cap Y) : K_\alpha \subset W \in \tau(X)\Big\}$$

*is a $\varrho$-neighbourhood base. Then*

*(i) $\varrho$ is a Hausdorff topology on $Z$,*

*(ii) if $V \in \varrho$ is non-empty then $\overline{V}^\varrho$ includes a final segment of $\kappa$.*

*Proof of Lemma 2.1.* It is straight forward to see that the above definition of $\varrho$ is correct. That $\varrho$ is Hausdorff follows from the fact that any two disjoint compact sets have disjoint neighbourhoods in $X$.

It is obvious from the definition that the subspace topology of $Y$ inherited from $X$ is the same as $\varrho \upharpoonright Y$, moreover $Y$ is dense open in $\langle Z, \varrho \rangle$. Thus it suffices to show (ii) for $V \in \varrho$ with $V \subset Y$. But every $\varrho$-open set $V \subset Y$ is of the form $U \cap Y$ with $U \in \tau(X)$. Now, if $V$ is non-empty then (1) implies that

$$I = \{\alpha < \kappa : K_\alpha \cap U \neq \emptyset\}$$

contains a final segment of $\kappa$. But if $\alpha \in I$ and $K_\alpha \subset W \in \tau(X)$ then $W \cap U \supset K_\alpha \cap U \neq \emptyset$, hence $W \cap V = (W \cap Y) \cap V = W \cap U \cap Y \neq \emptyset$ as well because, by (2), $Y$ is dense in $X$. Thus we have $\alpha \in \overline{V}^\varrho$ for all $\alpha \in I$, which completes the proof of (ii). $\qquad\square$

Now we shall present two applications of lemma 2.1.

**Theorem 2.2.** *For any infinite cardinal $\kappa$ there is a(n AU) space $Z$ such that $|Z| = \kappa$, $d(Z) = \log \kappa$, and*

$$\bigcap \mathcal{A} \neq \emptyset \text{ whenever } \mathcal{A} \in \big[\mathrm{RC}^+(X)\big]^{<cf(\kappa)}.$$

*Proof of theorem 2.2.* Consider in the Cantor cube $\mathbb{C}_\kappa = \{0,1\}^\kappa$ the pairwise disjoint non-empty compact subsets

$$K_\alpha = \{x \in \mathbb{C}_\kappa : x(\alpha) = 1 \text{ and } x(\beta) = 0 \text{ for all } \alpha < \beta < \kappa\}.$$

It is well known that $d(\mathbb{C}_\kappa) = \log \kappa$ and we leave it to the reader to check that the standard proof of this fact (see e.g. [5]) yields a dense set $Y \subset \mathbb{C}_\kappa$ with $|Y| = \log \kappa$ such that $Y \cap \bigcup_{\alpha < \kappa} K_\alpha = \emptyset$.

Now, it is obvious that we may apply lemma 2.1 to the subspace $X = \bigcup_{\alpha < \kappa} K_\alpha \cup Y$ of $\mathbb{C}_\kappa$ to obtain the required space on $Z = Y \cup \kappa$.  $\square$

In the case $\kappa = \omega$ the countable AU, hence connected, space we obtain from theorem 2.2 has the stronger property that any intersection of finitely many non-empty regular closed sets is non-empty. We think that its construction is at least as simple as Bing's in [1]. In any case, it is certainly stronger because, as is easily checked, both Gustin's and Bing's countable AU spaces contain three non-empty regular closed sets whose intersection is empty.

Our next application of lemma 2.1 will enable us to construct large AU spaces that are locally small. Before formulating it we recall that the dispersion character $\Delta(X)$ of a space $X$ is the smallest size of a non-empty open set in $X$.

**Theorem 2.3.** *For any infinite cardinal $\kappa$ there is an AU space $Z$ with a closed discrete subset $D \subset Z$ such that $|D| = |Z| = \Delta(Z) = d(Z) = \kappa$ and*

$$|D \setminus A| < \kappa \text{ for each } A \in \mathrm{RC}^+(X). \qquad (2.1)$$

*Moreover, there is a family $\{U_\alpha : \alpha < \kappa\}$ of pairwise disjoint open sets in $Z \setminus D$ such that every point $z \in Z$ has a neighborhood $W_z$ for which*

$$\{\alpha < \kappa : U_\alpha \cap W_z \neq \emptyset\} \text{ is bounded in } \kappa. \qquad (2.2)$$

*Proof of theorem 2.3.* Let $\mathbb{E}_\kappa$ be the product space $L(\kappa)^\kappa$, where $L(\kappa)$ denotes $\kappa$ with the usual ordinal topology.

For any $\alpha < \kappa$ we let

$$K_\alpha = \{p \in \mathbb{E}_\kappa : p(\zeta) \leq \alpha \text{ for all } \zeta < \alpha,$$
$$p(\alpha) = 1 \wedge p(\beta) = 0 \text{ for all } \alpha < \beta < \kappa \setminus (\alpha + 1)\}.$$

Then the $K_\alpha$ are pairwise disjoint compact subsets of $\mathbb{E}_\kappa$. Put $K_\kappa = \bigcup_{\alpha < \kappa} K_\alpha$. Then clearly $\mathbb{E}_\kappa \setminus K_\kappa$ is dense in $\mathbb{E}_\kappa$, $\Delta(\mathbb{E}_\kappa \setminus K_\kappa) = 2^\kappa$, and $w(E_\kappa) = \kappa$, hence there is a dense set $Y \subset \mathbb{E}_\kappa \setminus K_\kappa$ such that $|Y| = \Delta(Y) = \kappa$.

We may then apply lemma 2.1 to the space $X = Y \cup K_\kappa$ to obtain the space $Z = \langle Y \cup \kappa, \varrho \rangle$ with the closed discrete set $D = \kappa$. Clearly, we have $|D| = |Z| = \Delta(Z) = d(Z) = \kappa$ and property (2.1) holds.

Let us now define

$$U_\alpha = \{p \in Y : p(0) = \alpha + 1\}$$

for $\alpha < \kappa$. Since the singleton $\{\alpha + 1\}$ is open in $L(\kappa)$, we have $U_\alpha \in \varrho$, and clearly $\alpha \neq \beta$ implies $U_\alpha \cap U_\beta = \emptyset$.

Now, if $y \in Y$ then

$$W_y = \{p \in Y : p(0) \leq y(0)\} \in \varrho$$

is a neighborhood of $y$ that witnesses (2.2) because $W_y \cap U_\beta = \emptyset$ for $\beta \geq y(0)$.

If, on the other hand, $\alpha \in \kappa$, then $G_\alpha = \{p \in \mathbb{E}_\kappa : p(0) \leq \alpha + 1\}$ is an open subset of $\mathbb{E}_\kappa$ with $K_\alpha \subset G_\alpha$, hence

$$W_\alpha = \{\alpha\} \cup \{p \in Y : p(0) \leq \alpha + 1\} \in \varrho$$

is a neighborhood of $\alpha$ in $Z$ with $W_\alpha \cap U_\beta = \emptyset$ for all $\beta > \alpha$.  □

We say that a space $X$ is *locally $\kappa$* if every point of $X$ has a neighbourhood of cardinality $\leq \kappa$. The following easy result yields an upper bound for the cardinality of a locally $\kappa$ AU space. It will also be used in section 3 for SAU spaces.

**Theorem 2.4.** *Any locally $\kappa$ space $X$ contains an infinite clopen subset $Y$ of cardinality $|Y| \leq 2^{2^\kappa}$.*

*Proof.* Since $X$ is Hausdorff, we have $|\overline{A}| \leq 2^{2^\kappa}$ for all $A \in [X]^{\leq \kappa}$. We may fix for every point $p \in X$ a neighborhood $U_p$ of size $\leq \kappa$. A very simple closure procedure then yields an infinite subset $Y \subset X$ of cardinality $\leq 2^{2^\kappa}$ such that

(a) $U_p \subset Y$ for all $p \in Y$,
(b) $\overline{A} \subset Y$ for all $A \in [Y]^{\leq \kappa}$.

Then (a) implies that $Y$ is open and (b) implies that $Y$ is closed because $t(X) \leq \kappa$.  □

It is immediate from theorem 2.4 that any space which is locally $\kappa$ and connected, in particular AU, has cardinality $\leq 2^{2^\kappa}$. The following result implies that this upper bound is sharp: For every $\kappa$ there is a locally $\kappa$ AU space of cardinality $2^{2^\kappa}$.

**Theorem 2.5.** *For every infinite cardinal $\kappa$ there is a locally $\kappa$ space $X$ with a closed discrete subset $D$ such that*

 *(i) $|X| = 2^{2^\kappa}$ and $d(X) = |D| = \kappa$,*
 *(ii) $|D \setminus A| < \kappa$ holds for any $A \in \mathrm{RC}^+(X)$.*

*In particular, then $\bigcap \mathcal{A} \neq \emptyset$ whenever $\mathcal{A} \in \left[\mathrm{RC}^+(X)\right]^{<cf(\kappa)}$.*

*Proof of Theorem 2.5 .* By theorem 2.3 there is a space $Z$ with a closed discrete $D \subset Z$ such that $|D| = |Z| = \Delta(Z) = d(Z) = \kappa$ and

$$|D \setminus A| < \kappa \text{ for all } A \in \mathrm{RC}^+(Z),$$

moreover there are pairwise disjoint open sets $\{U_\alpha : \alpha < \kappa\}$ in $Z \setminus D$ so that every point $z \in Z$ has a neighborhood $W_z$ which meets $U_\alpha$ only for boundedly many $\alpha < \kappa$.

The underlying set of our space is

$$X = Z \cup S(\kappa),$$

where $S(\kappa)$ is the set of all uniform ultrafilters on $\kappa$. (Of course, we may assume that $Z \cap S(\kappa) = \emptyset$.) So we have $|X| = |S(\kappa)| = 2^{2^\kappa}$.

Next we define the topology $\tau$ on $X$ with the following stipulations:

(i) $Z \in \tau$ and the subspace topology of $Z$ inherited from $X$ is the original topology of $Z$;

(ii) for any uniform ultrafilter $x \in S(\kappa)$ the family

$$\mathcal{U}_x = \Big\{ \{x\} \cup \bigcup_{\alpha \in a} U_\alpha \ : \ a \in x \Big\}.$$

is a $\tau$-neighborhood base of $x$.

It is obvious from this definition that $Z$ is a dense open subspace of $X$, moreover $D$ remains a closed discrete set in $X$. This immediately implies (i), while (ii) follows because if $A \in \mathrm{RC}^+(X)$ then $A \cap Z \in \mathrm{RC}^+(Z)$. The only thing that is left to show is the Hausdorffness of $X$.

Since $Z$ is Hausdorff and open in $X$, it is obvious that any two points of $Z$ can be separated in $X$. If $\{x, y\} \in [S(\kappa)]^2$ then there are $a \in x$ and $b \in y$ with $a \cap b = \emptyset$, hence

$$\{x\} \cup \bigcup_{\alpha \in a} U_\alpha \quad \text{and} \quad \{y\} \cup \bigcup_{\beta \in b} U_\beta$$

are disjoint neighborhoods of $x$ and $y$ in $X$.

Finally, assume that $z \in Z$ and $x \in S(\kappa)$. Then, by (2.2), there is $\xi \in \kappa$ such that $W_z \cap U_\zeta = \emptyset$ for all $\xi \leq \zeta < \kappa$. But we may pick $a \in x$ with $a \cap \xi = \emptyset$, and then $W_z$ and

$$\{x\} \cup \bigcup_{\zeta \in a} U_\zeta$$

are disjoint neighborhoods of $z$ and $x$. $\qquad\qquad \square$

Since in the above construction $S(\kappa)$ is clearly closed discrete in $X$, we actually get the following result.

**Corollary 2.6.** *Given $\kappa \geq \omega$, for every cardinal $\lambda \leq 2^{2^\kappa}$ there is a locally $\kappa$ AU space of cardinality $\lambda$. In particular, for every infinite cardinal $\lambda \leq 2^{\mathfrak{c}}$ there is a locally countable AU space of cardinality $\lambda$.*

## 3. Existence of strongly anti-Urysohn spaces

We will see later in this section that, at least consistently, strongly anti-Urysohn (SAU) spaces exist. However, in strong contrast to the case of AU spaces, there are both lower and upper bounds for their possible cardinalities. Before establishing these bounds, in the following theorem we collect some simple properties of SAU spaces.

**Theorem 3.1.** *Let $X$ be any SAU space. Then*

*(1) $X$ is countably compact;*
*(2) every compact subset of $X$ is finite;*
*(3) any infinite closed set $F \subset X$ is uncountable; hence $A \in \left[X\right]^{\omega}$ implies $|A'| > \omega$;*
*(4) $\mathcal{F}^{+}(X)$ is closed under countable intersections.*

*Proof.* (1) An infinite closed discrete set breaks into two disjoint infinite closed sets.

(2) Assume that $F \subset X$ is compact and let $p_0$ and $p_1$ be two different accumulation points of $X$ with disjoint neighborhoods $U_0$ and $U_1$, respectively. Then the compact set $F \setminus U_0$ and the point $p_0$ have disjoint open neighbourhoods: $F \setminus U_0 \subset V$, $p_0 \in W$ and $V \cap W = \emptyset$. Then $F \setminus U_0$ and $\overline{W}$ are disjoint closed sets. But $p_0 \in X'$ implies that $\overline{W}$ is infinite, so $F \setminus U_0$ is finite because $X$ is SAU. A symmetrical argument yields that $F \setminus U_1$ is also finite, hence $F = (F \setminus U_0) \cup (F \setminus U_1)$ implies that $F$ is finite as well.

(3) If $F \subset X$ is countable and closed then $F$ is compact because $X$ is countably compact by (1). Consequently, $F$ is finite by (2). The second part now follows from $\overline{A} = A \cup A'$.

(4) First we show that

$$\mathcal{F}^{+}(X) \text{ is closed under finite intersections.} \qquad (3.1)$$

Otherwise we could choose $A, B \in \left[X\right]^{\omega}$ such that $n = |\overline{A} \cap \overline{B}| < \omega$ is minimal. Since $X$ is SAU, we can pick $p \in \overline{A} \cap \overline{B} \neq \emptyset$.

Then $A \cup \{p\}$ is not compact by (2), hence there is an open set $U \ni p$ such that $A \setminus U$ is infinite. But $p \notin \overline{A \setminus U}$, so

$$\overline{A \setminus U} \cap \overline{B} \subset (\overline{A} \cap \overline{B}) \setminus \{p\},$$

consequently $|\overline{A \setminus U} \cap \overline{B}| < n$, which contradicts the choice of $n$. So we proved (3.1).

Now assume that $\{F_n : n \in \omega\} \subset \mathcal{F}^+(X)$. Using (3.1) and (3) we can pick by recursion points

$$p_n \in \bigcap_{m \leq n} F_m \setminus \{p_i : i < n\}$$

for $n < \omega$, and put $P = \{p_n : n < \omega\}$. Then, by (3), $P'$ is infinite, in fact even uncountable, and we have $P' \subset \bigcap_{n \in \omega} F_n$. □

All our (consistent) examples of SAU spaces that we shall construct below have cardinality $\leq \mathfrak{c}$. We do not know if SAU spaces of size $> \mathfrak{c}$ can exist but we have the following related result.

**Theorem 3.2.** *Every SAU space $X$ has a SAU subspace of size $\leq \mathfrak{c}$.*

*Proof.* We may fix a function $\varphi : [X]^\omega \times [X]^\omega \to X$ such that $\varphi(A, B) \in \overline{A} \cap \overline{B}$ for all $\langle A, B \rangle \in [X]^\omega \times [X]^\omega$. Let us also fix $Y_0 \in [X]^\omega$. By theorem 3.1(3), $Y_0$ has two accumulation points $p$ and $q$.

Since $\mathfrak{c}^\omega = \mathfrak{c}$, there is a set $Y$ with $Y_0 \cup \{p, q\} \subset Y \in [X]^{\leq \mathfrak{c}}$ which is $\varphi$-closed, i.e. $\varphi(A, B) \in Y$ holds whenever $A, B \in [Y]^\omega$.

But then any two infinite closed subsets of $Y$ intersect, moreover, $|Y'| > 1$ because $p, q \in Y_0' \subset Y'$, hence $Y$ is SAU. □

Now we turn to giving the lower and upper bound for the possible cardinalities of SAU spaces. To do that we first prove two lemmas. The first one is purely combinatorial. To formulate it we recall that a set $A$ is said to *split* another set $B$ iff both $B \cap A$ and $B \setminus A$ are infinite. Also, a family of sets $\mathcal{A}$ is called a *splitting family for $X$* if every $B \in [X]^\omega$ is split by some member of $\mathcal{A}$.

**Lemma 3.3.** *If $\mathcal{A}$ is a splitting family for $X$ then $|X| \leq 2^{|\mathcal{A}|}$.*

*Proof.* For every $x \in X$ let us put $\mathcal{A}(x) = \{A \in \mathcal{A} : x \in A\}$. Then for each subfamily $\mathcal{B} \subset \mathcal{A}$ the set $S = \{x \in X : \mathcal{A}(x) = \mathcal{B}\}$ is finite because no element of $\mathcal{A}$ can split $S$. Thus the map $x \mapsto \mathcal{A}(x)$ is finite-to-one and hence we have $|X| \leq 2^{|\mathcal{A}|}$. □

The next lemma involves the well-known splitting number $\mathfrak{s}$ which is defined as the smallest cardinality of a splitting family for $\omega$.

**Lemma 3.4.** *If $X$ is any space of weight $w(X) < \mathfrak{s}$ then every set $A \in [X]^\omega$ has an infinite subset $B$ with at most one accumulation point in $X$, i.e. such that $|B'| \leq 1$. So, if in addition, $X$ is countably compact then actually $X$ is sequentially compact.*

*Proof.* Let $\mathcal{U}$ be a base of $X$ with $|\mathcal{U}| < \mathfrak{s}$. This implies that there is $B \in \left[A\right]^\omega$ such that no element of $\mathcal{U}$ splits $B$. But then the Hausdorff property of $X$ clearly implies $|B'| \leq 1$. Indeed, if $x$ and $y$ would be distinct accumulation points of $B$ with disjoint neighbourhoods $U, V \in \mathcal{U}$, then both $U$ and $V$ would split $B$.

The second part follows because the countable compactness of $X$ implies $|B'| \geq 1$ for all $B \in \left[X\right]^\omega$. $\qquad\square$

We are now ready to give the promised lower and upper bounds for the size of a SAU space.

**Theorem 3.5.** *If $X$ is any SAU space then*

$$\mathfrak{s} \leq |X| \leq 2^{2^{\mathfrak{c}}}.$$

*Proof.* Assume that $|X| < \mathfrak{s}$. Then there is a coarser *Hausdorff* topology $\varrho$ on $X$ of weight $w(X, \varrho) \leq |X|$, hence by lemma 3.4 there is $B \in \left[X\right]^\omega$ such that $B$ has at most one accumulation point in $(X, \varrho)$. Then $B$ has at most one accumulation point in the finer topology of $X$, as well. But this implies that $X$ is not SAU by Theorem 3.1(3).

To verify the upper bound of $|X|$, consider any $A \in \left[X\right]^\omega$ and put $F = \overline{A}$. Then, by Pospišil's theorem, we have $|F| \leq 2^{\mathfrak{c}}$, hence there is a family $\mathcal{A}$ of relatively open subsets of $F$ that $T_2$-separates the points of $F$ and $|\mathcal{A}| \leq |F| \leq 2^{\mathfrak{c}}$.

By Theorem 3.18(4), for every $B \in \left[X\right]^\omega$ we have $F \cap B' \in \mathcal{F}^+(X)$, hence $B$ has at least two, in fact uncountably many, accumulation points in $F$. But then some element of $\mathcal{A}$ splits $B$, i.e. $\mathcal{A}$ is a splitting family for $X$. By lemma 3.3, this implies

$$|X| \leq 2^{|\mathcal{A}|} \leq 2^{|F|} \leq 2^{2^{\mathfrak{c}}},$$

which completes the proof. $\qquad\square$

Of course, the previous results are only of interest if SAU spaces, at least consistently, exist. So now we turn to proving that they do. Our first construction will make use of the *reaping number* $\mathfrak{r}$ whose definition we recall next.

If $\mathcal{A}$ is any family of infinite sets then a set $S$ is said to reap $\mathcal{A}$ iff $S$ splits every member of $\mathcal{A}$. Now, $\mathfrak{r}$ is the minimum cardinality of a family $\mathcal{A} \subset [\omega]^\omega$ such that no $S \in [\omega]^\omega$ reaps $\mathcal{A}$. So the assumption $\mathfrak{r} = \mathfrak{c}$, that will figure in our construction of a SAU space given below, is equivalent to the statement that every subfamily of $[\omega]^\omega$ of size $< \mathfrak{c}$ can be reaped by a member of $[\omega]^\omega$.

To make the inductive construction of our SAU space easier to digest, we prove first the following lemma.

**Lemma 3.6.** *Assume that $\langle X, \tau \rangle$ is a locally countable space of weight $w(X) < \mathfrak{r}$, moreover we are given $I, J \in [X]^{\omega}$ and a family $\{\langle x_i, A_i \rangle : i < \kappa\} \subset X \times [X]^{\omega}$, where $\kappa < \mathfrak{r}$ and $x_i \in (A_i')^{\tau}$ for all $i < \kappa$. Then, for any fixed $p \notin X$, there is a locally countable Hausdorff topology $\rho$ on $X \cup \{p\}$ such that*

*(i) $\tau \subset \rho$ and $w(\rho) \leq w(\tau)$,*

*(ii) $p \in (I')^{\rho} \cap (J')^{\rho}$,*

*(iii) $x_i \in (A_i')^{\rho}$ for all $i < \kappa$.*

*Proof.* Since $\langle X, \tau \rangle$ is locally countable, there is a countable $U \in \tau$ with $I \cup J \subset U$. We fix an enumeration $U = \{u_n : n \in \omega\}$ and a base $\mathcal{B}$ of $X$ of cardinality $w(\tau)$.

Next we consider the family

$$\mathcal{C}_0 = \big(\{I, J\} \cup \{B \cap U : B \in \mathcal{B}\} \cup \{A_i \cap B \cap U : i < \kappa, \, B \in \mathcal{B}\}\big) \cap [U]^{\omega}.$$

Clearly, $|\mathcal{C}_0| < \mathfrak{r}$, hence there is $D_0 \in [U \setminus \{u_0\}]^{\omega}$ that reaps $\mathcal{C}_0$.

We continue this by recursion on $n \in \omega$: If $\mathcal{C}_n \in \big[[U]^{\omega}\big]^{<\mathfrak{r}}$ and $D_n \in [U \setminus \{u_n\}]^{\omega}$ reaping $\mathcal{C}_n$ have been defined, then we put

$$\mathcal{C}_{n+1} = \mathcal{C}_n \cup \{C \cap D_n, \, C \setminus D_n : C \in \mathcal{C}_n\}.$$

Then $\mathcal{C}_{n+1} \in [U]^{\omega}$ and $|\mathcal{C}_{n+1}| = |\mathcal{C}_n| < \mathfrak{r}$, hence we may choose $D_{n+1} \in [U \setminus \{u_{n+1}\}]^{\omega}$ that reaps $\mathcal{C}_{n+1}$.

Our topology $\rho$ on $X \cup \{p\}$ is generated by the family

$$\tau \cup \{X \setminus D_n : n \in \omega\} \cup \{\{p\} \cup D_n : n \in \omega\}. \tag{3.2}$$

Then (i) and the local countability of $\rho$ are clear. To see that $\rho$ is Hausdorff it suffices to show that any $x \in X$ and $p$ have disjoint $\rho$-neighbourhoods. But if $x = u_n$ then $X \setminus D_n$ and $\{p\} \cup D_n$ work, while for $x \notin U$ any $X \setminus D_n$ and $\{p\} \cup D_n$ will work.

To check (ii), by symmetry, it suffices to show that $p \in (I')^{\rho}$. This is clearly equivalent to the statement

$$I \cap \bigcap_{m<n} D_m \text{ is infinite for all } n < \omega, \tag{3.3}$$

which we can prove by induction on $n$. If $n = 0$, then $I \cap \bigcap_{m<n} D_m = I$ is infinite by assumption. If

$$C = I \cap \bigcap_{m<n} D_m \text{ is infinite} \tag{3.4}$$

then $C \in \mathcal{C}_n$, hence $C \cap D_n = I \cap \bigcap_{m<n+1} D_m$ is infinite as well, because $D_n$ splits $C$. This completes the proof of (ii).

For any $\varepsilon \in Fn(\omega, 2)$ let us put

$$D_\varepsilon = \bigcap_{\varepsilon(n)=1} D_n \cap \bigcap_{\varepsilon(n)=0} (X \setminus D_n).$$

It is easy to prove by induction on $|\varepsilon|$ that for every $\varepsilon \in Fn(\omega, 2)$ and for every $C \in \mathcal{C}_0$ we have $|C \cap D_\varepsilon| = \omega$.

Now, to check (iii), fix $i < \kappa$. If $x_i \in (A_i \cap U)'^\tau$, which certainly holds if $x_i \in U$, then for any $B \in \mathcal{B}$ with $x_i \in B$ we have $A_i \cap U \cap B \in \mathcal{C}_0$ and hence, by the above, $A_i \cap U \cap B \cap D_\varepsilon$ is infinite for each $\varepsilon \in Fn(\omega, 2)$. But this clearly implies that $x_i \in (A_i \cap U)'^\rho \subset (A_i')^\rho$.

If, on the other hand, we have $x_i \notin (A_i \cap U)'^\tau$, then $x_i \in (A_i \setminus U)'^\tau$. But in this case $x_i \notin U$ and then the obvious fact that $\tau$ and $\rho$ coincide on $X \setminus U$ trivially implies that $x_i \in (A_i \setminus U)'^\rho \subset (A_i')^\rho$.

Thus we have verified (iii) and completed the proof of the lemme. $\square$

We are now ready to formulate and prove our first existence result concerning SAU spaces.

**Theorem 3.7.** *If $\mathfrak{r} = \mathfrak{c}$ then there is a locally countable, separable, and crowded SAU space of cardinality $\mathfrak{c}$.*

*Proof of theorem 3.7.* The underlying set of our space will be $\mathbb{Q} \cup \mathfrak{c}$, where $\mathbb{Q}$ is the set of rational numbers. Our aim is to achieve that the closures of any two members of $[\mathbb{Q} \cup \mathfrak{c}]^\omega$ intersect, so we fix an enumeration of all these pairs:

$$\big\{\{I_\zeta, J_\zeta\} : \zeta < \mathfrak{c}\big\} = \big[[\mathbb{Q} \cup \mathfrak{c}]^\omega\big]^2,$$

where $I_\zeta \cup J_\zeta \subset \mathbb{Q} \cup \zeta$ for all $\zeta < \mathfrak{c}$.

Then, by transfinite recursion on $\zeta \leq \mathfrak{c}$, we define locally countable Hausdorff topologies $\tau_\zeta$ on $X_\zeta = \mathbb{Q} \cup \zeta$ as follows:

(1) $\tau_0$ is the usual topology of $\mathbb{Q}$;

(2) if $\zeta < \nu \leq \mathfrak{c}$ then $w(\tau_\zeta) < \mathfrak{r}$, moreover

$$\tau_\zeta \subset \tau_\nu \text{ and } \zeta \in (I_\zeta')^{\tau_\nu} \cap (J_\zeta')^{\tau_\nu};$$

(3) if $\nu \leq \mathfrak{c}$ is a limit ordinal then $\tau_\nu$ is generated by $\bigcup_{\zeta < \nu} \tau_\zeta$ on $X_\nu$;

(4) for every $\zeta < \mathfrak{c}$ we obtain $\tau_{\zeta+1}$ by applying Lemma 3.6 to the space $\langle X_\zeta, \tau_\zeta \rangle$ with $p = \zeta$, the pair $\{I_\zeta, J_\zeta\}$, and the family

$$\big\{\langle q, \mathbb{Q} \setminus \{q\}\rangle : q \in \mathbb{Q}\big\} \cup \big\{\langle \xi, I_\xi \rangle : \xi < \zeta\big\} \cup \big\{\langle \xi, J_\xi \rangle : \xi < \zeta\big\}.$$

We claim that the space $\langle X_\mathfrak{c}, \tau_\mathfrak{c} \rangle$ is as required. Indeed, local countability and Hausdorffness is built in and, using (4), one may easily

check by transfinite induction that $\mathbb{Q}$ is both dense and crowded in each $\langle X_\zeta, \tau_\zeta \rangle$, hence $\langle X_{\mathfrak{c}}, \tau_{\mathfrak{c}} \rangle$ is separable and crowded.

Finally $\langle X_{\mathfrak{c}}, \tau_{\mathfrak{c}} \rangle$ is SAU because for any $I, J \in \left[ X \right]^\omega$ there is $\zeta < \mathfrak{c}$ such that $\{I, J\} = \{I_\zeta, J_\zeta\}$, and so $\zeta \in (I')^{\tau_{\mathfrak{c}}} \cap (J')^{\tau_{\mathfrak{c}}} \neq \emptyset$.          $\square$

Now that we know that locally countable SAU spaces may consistently exist, it makes sense to remark that they certainly cannot have cardinality $> 2^{\mathfrak{c}}$. Indeed, this is an immediate consequence of theorem 2.4 because a clopen subset of a SAU space must have finite complement. Although we do not know if this upper bound is sharp, at least we know that it is smaller than the upper bound for all SAU spaces given by theorem 3.5.

We now turn to another method of constructing consistent examples of SAU spaces. Unlike the construction in theorem 3.7, this will allow us to produce simultaneously SAU spaces of many different sizes. These constructions will make use of certain (consistent) combinatorial principles that we formulate below.

**Definition 3.8.** Let $\kappa \geq \omega$ and $\mu$ be cardinals, where $\mu = 1$ or $\mu$ is infinite. Then $\circledast_{\kappa, \mu}$ is the following statement: There is a sequence

$$\left\langle \left\langle A_\alpha^0, A_\alpha^1 \right\rangle : \alpha \in \kappa \right\rangle$$

such that

(A) each $\langle A_\alpha^0, A_\alpha^1 \rangle$ is a partition of $\alpha \times \mu$,
(B) for every $S \in \left[ \kappa \times \mu \right]^\omega$ there is $\beta < \kappa$ such that $\langle \langle A_\alpha^0, A_\alpha^1 \rangle : \alpha \in \kappa \setminus \beta \rangle$ is dyadic on $S$, i.e.

$$\left| S \cap \bigcap \{ A_\zeta^{\varepsilon(\zeta)} : \zeta \in \mathrm{dom}(\varepsilon) \} \right| = \omega,$$

whenever $\varepsilon \in Fn(\kappa \setminus \beta, 2)$.

In what follows, we shall write $A[\varepsilon] = \bigcap \{ A_\zeta^{\varepsilon(\zeta)} : \zeta \in \mathrm{dom}(\varepsilon) \}$. Also, a sequence witnessing $\circledast_{\kappa, \mu}$ will be called simply a $\circledast_{\kappa, \mu}$-sequence.

The following simple result is given just for orientation.

**Proposition 3.9.** $\circledast_{\kappa, \mu}$ *implies* $\mathfrak{s} \leq cf(\kappa) \leq \kappa \leq \mathfrak{c}$.

*Proof.* If $\langle \langle A_\alpha^0, A_\alpha^1 \rangle : \alpha < \kappa \rangle$ is a $\circledast_{\kappa, \mu}$-sequence and $I \subset \kappa$ is cofinal then, by condition 3.8(B), $\{ A_\alpha^0 : \alpha \in I \}$ is a splitting family for $\kappa \times \mu$. Thus $\mathfrak{s} \leq cf(\kappa)$.

Now, assume that we had $\kappa > \mathfrak{c}$. Then fix $S \in [\kappa \times \mu]^\omega$ and $\beta < \kappa$ such that $\langle \langle A_\alpha^0, A_\alpha^1 \rangle : \beta \leq \alpha < \kappa \rangle$ is dyadic on $S$. But $|\kappa \setminus \beta| = \kappa > \mathfrak{c}$, so there are $\beta < \zeta < \xi < \kappa$ such that $A_\zeta^0 \cap S = A_\xi^0 \cap S$, contradicting that $\langle \langle A_\alpha^0, A_\alpha^1 \rangle : \beta \leq \alpha < \kappa \rangle$ is dyadic on $S$. This proves $\kappa \leq \mathfrak{c}$.          $\square$

To obtain SAU spaces from $\circledast_{\kappa,\mu}$, we actually need $\circledast_{\kappa,\mu}$-sequences with an extra property that, luckily, we can get for free.

**Lemma 3.10.** *If $\circledast_{\kappa,\mu}$ holds then there is a $\circledast_{\kappa,\mu}$-sequence*

$$\left\langle \left\langle A_\alpha^0, A_\alpha^1 \right\rangle : \alpha < \kappa \right\rangle$$

*that, in addition to 3.8(A) and 3.8(B), satisfies condition (C) below as well:*

*(C) every pair $\{x, y\} \in \left[\kappa \times \mu\right]^2$ is separated by $\langle A_\alpha^0, A_\alpha^1 \rangle$ for cofinally many $\alpha < \kappa$, that is for any $\beta < \kappa$ there is $\alpha \in \kappa \setminus \beta$ such that*

$$\left|\{x, y\} \cap A_\alpha^0\right| = 1.$$

*Proof.* Let us start by fixing a $\circledast_{\kappa,\mu}$-sequence

$$\left\langle \left\langle B_\alpha^0, B_\alpha^1 \right\rangle : \alpha < \kappa \right\rangle.$$

Case 1: $\mu = 1$. We fix an injective map $f : \kappa \times \kappa \times \kappa \to \kappa$ such that $\max\{\zeta, \xi\} < f(\zeta, \xi, \eta)$ and then put

$$A_\alpha^0 = \begin{cases} (B_\alpha^0 \cup \{\zeta\}) \setminus \{\xi\} & \text{if } \alpha = f(\zeta, \xi, \eta) \text{ for some } \zeta, \xi, \eta < \kappa, \\ \\ B_\alpha^0 & \text{otherwise,} \end{cases}$$

and

$$A_\alpha^1 = (\alpha \times \{0\}) \setminus A_\alpha^0.$$

Then $|A_\alpha^i \bigtriangleup B_\alpha^i| \leq 2$ implies that property (B) is preserved, and (C) holds because for any $\langle \zeta, \xi \rangle \in \kappa \times \kappa$ there are cofinally many $\alpha < \kappa$ with $\alpha = f(\zeta, \xi, \eta)$.

Case 2: $\mu \geq \omega$. For any $a, b \in \kappa \times \mu$ we define

$$a \equiv b \iff \exists \gamma < \kappa \, \forall \zeta \in \kappa \setminus \gamma \, (a \in B_\zeta^0 \iff b \in B_\zeta^0).$$

Then $\equiv$ is clearly an equivalence relation and property (B) implies that every equivalence class of $\equiv$ is finite as any infinite subset of $\kappa \times \mu$ is split by $B_\alpha^0$ eventually.

This clearly implies that if $X \subset \kappa \times \mu$ contains exactly one element from each $\equiv$-equivalence class then $|X \cap (\{\alpha\} \times \mu)| = \mu$ for all $\alpha < \kappa$. Pick such an $X$ and, for every $\alpha < \kappa$, fix a bijection

$$f_\alpha : \{\alpha\} \times \mu \to X \cap (\{\alpha\} \times \mu).$$

Now it is obvious that if we put $A_\alpha^i = f_\alpha^{-1}(X \cap B_i^\alpha)$, then the sequence $\langle \langle A_\alpha^0, A_\alpha^1 \rangle : \alpha < \kappa \rangle$ has all the three properties (A), (B), and (C). $\qquad \square$

A $\circledast_{\kappa,\mu}$-sequence with the additional property (C) will be called a strong $\circledast_{\kappa,\mu}$-sequence. These, as we shall now show, yield us spaces with very strong SAU properties.

**Theorem 3.11.** *If* $\langle\langle A^0_\alpha, A^1_\alpha\rangle : \alpha < \kappa\rangle$ *is a strong* $\circledast_{\kappa,\mu}$*-sequence then there is a separable and crowded Hausdorff topology* $\tau$ *on* $\kappa \times \mu$ *such that for every infinite set* $S \subset X$ *there is* $\alpha < \kappa$ *for which*

$$\overline{S} \supset (\kappa \setminus \alpha) \times \mu.$$

*In particular, if* $\mu < \kappa$*, then*

$$|(\kappa \times \mu) \setminus \overline{S}| < \kappa \text{ for all infinite } S \subset \kappa \times \mu. \qquad (3.5)$$

*Proof.* Let $\rho$ be a topology on $Q = \omega \times \{0\}$ such that $\langle Q, \rho\rangle$ is homeomorphic to $\mathbb{Q}$ with its usual topology and fix a countable base $\mathcal{B}$ of $\langle Q, \rho\rangle$. Since $cf(\kappa) > \omega$, we can pick $\beta < \kappa$ such that, for all $B \in \mathcal{B}$, $\langle\langle A^0_\alpha, A^1_\alpha\rangle : \beta \leq \alpha < \kappa\rangle$ is dyadic on $B$.

Our topology $\tau$ is generated by

$$\mathcal{B} \cup \{A^0_\alpha : \alpha \in \kappa \setminus \beta\} \cup \{A^1_\alpha : \alpha \in \kappa \setminus \beta\}. \qquad (3.6)$$

Then $\tau$ is Hausdorff by 3.10.(C). The choice of $\beta$ implies that $Q$ is both dense and crowded with respect to $\tau$, hence $\tau$ is separable and crowded.

For every $S \in [X]^\omega$ there is $\alpha \in \kappa \setminus \beta$ such that $\langle\langle A^0_\zeta, A^1_\zeta\rangle : \zeta \in \kappa \setminus \alpha\rangle$ is dyadic on $S$. This clearly implies $(\kappa \setminus \alpha) \times \mu \subset \overline{S}$. $\qquad \square$

The above space $X = \langle \kappa \times \mu, \tau\rangle$ trivially has the following very strong SAU property: any family $\mathcal{A} \in [\mathcal{F}^+(X)]^{<cf(\kappa)}$ has non-empty intersection.

Now we turn to examining the consistency of the principles $\circledast_{\kappa,\mu}$. As it will turn out, many instances of them hold true in a generic extension obtained by adding a lot of Cohen reals to an arbitrary ground model. To see this, we need the following easy but technical lemma. First we fix some notation:

**Definition 3.12.** Assume that the cardinals $\kappa$, $\mu$ and $\alpha \in \kappa$ are given. Let $\mathcal{G}$ be $Fn((\kappa \setminus \alpha) \times \kappa \times \mu, 2)$-generic over $V$. Then in $V[\mathcal{G}]$ we put $g = \bigcup \mathcal{G}$ and define the sequence

$$\mathcal{A}^\mathcal{G}_\alpha = \langle\langle A^0_\beta, A^1_\beta\rangle : \alpha \leq \beta < \kappa\rangle$$

by

$$A^i_\beta = \{\langle\xi, \zeta\rangle \in \beta \times \mu : g(\beta, \xi, \zeta) = i\}.$$

**Lemma 3.13.** *Assume that* $\mathcal{G}$ *is* $Fn((\kappa \setminus \alpha) \times \kappa \times \mu, 2)$*-generic over* $V$*. Then* $\mathcal{A}^\mathcal{G}_\alpha$ *is dyadic on each* $S \in V \cap [\alpha \times \mu]^\omega$*.*

*Proof.* Fix $S \in V \cap [\alpha \times \mu]^\omega$, $\varepsilon \in Fn(\kappa \setminus \alpha, 2)$, and $T \in [S]^{<\omega}$. For any condition $p \in Fn((\kappa \setminus \alpha) \times \kappa \times \mu, 2)$ we can find $\langle\xi, \eta\rangle \in S \setminus T$ such that

$$(\text{dom}(\varepsilon) \times \{\langle\xi, \eta\rangle\}) \cap \text{dom}(p) = \emptyset.$$

Then we can find a condition $q \supset p$ such that for each $\zeta \in \mathrm{dom}(\varepsilon)$ we have

$$q(\zeta, \xi, \eta) = \varepsilon(\zeta),$$

hence

$$q \Vdash \langle \xi, \eta \rangle \in \mathcal{A}_\alpha^{\mathcal{G}}[\varepsilon] \cap (S \setminus T).$$

Since $p$ and $T$ were arbitrary, this implies

$$1 \Vdash \mathcal{A}_\alpha^{\mathcal{G}}[\varepsilon] \cap S \text{ is infinite,}$$

hence, as $\varepsilon$ was also arbitrary,

$$1 \Vdash \mathcal{A}_\alpha^{\mathcal{G}} \text{ is dyadic on } S.$$

$\square$

**Theorem 3.14.** *If we add $\lambda > \omega$ many Cohen reals to any ground model then in the extension $\circledast_{\omega_1, \mu}$ holds for any uncountable $\mu \leq \lambda$.*

*Proof.* Clearly, it suffices to prove that $\circledast_{\omega_1, \lambda}$ holds in the extension. Since $|\omega_1 \times \omega_1 \times \lambda| = \lambda$, we may assume that our generic extension is of the form $V[G]$ where $G$ is $Fn(\omega_1 \times \omega_1 \times \lambda, 2)$-generic over $V$.

Now, in $V[G]$, putting $g = \bigcup G$ we define for $\beta < \omega_1$ and $i < 2$

$$A_\beta^i = \{ \langle \xi, \zeta \rangle \in \beta \times \lambda : g(\beta, \xi, \zeta) = i \},$$

and we claim that

$$\left\langle \left\langle A_\beta^0, A_\beta^1 \right\rangle : \beta \in \omega_1 \right\rangle$$

is a $\circledast_{\omega_1, \lambda}$-sequence.

Then (A) is obvious and to check (B) consider any $S \in [\omega_1 \times \lambda]^\omega$. It is well-known, however, that there is some $\alpha < \omega_1$ such that $S \in V[G \cap Fn(\alpha \times \omega_1 \times \lambda, 2)]$. From this, applying lemma 3.13 to the ground model $V[G \cap Fn(\alpha \times \omega_1 \times \lambda, 2)]$, we may immediately deduce that the tail sequence $\left\langle \left\langle A_\beta^0, A_\beta^1 \right\rangle : \beta \in \omega_1 \setminus \alpha \right\rangle$ is dyadic on $S$.

Actually, it is easy to see using genericity that $\left\langle \left\langle A_\beta^0, A_\beta^1 \right\rangle : \beta < \omega_1 \right\rangle$ is a strong $\circledast_{\omega_1, \lambda}$-sequence.

$\square$

**Corollary 3.15.** *If we add $\lambda > \omega$ many Cohen reals to our ground model then in the extension for every cardinal $\mu \in [\omega_1, \lambda]$ there is a SAU space of size $\mu$.*

Theorem 3.14 and proposition 3.9 imply the well-known and trivial fact that $\mathfrak{s} = \omega_1$ holds in a generic extension obtained by adding uncountably many Cohen reals. Hence by the the above corollary it is consistent to have a gap of any possible size between $\mathfrak{s} = \omega_1$ and

$\mathfrak{c}$ and to have SAU spaces of all cardinalities between $\mathfrak{s}$ and $\mathfrak{c}$. Our next theorem implies the consistency of the analogous statement with $\mathfrak{s} > \omega_1$.

**Theorem 3.16.** *Assume that GCH holds and $\nu \leq \lambda$ are uncountable cardinals in our ground model $V$ such that $\nu$ is regular and $cf(\lambda) > \omega$. Then there is a CCC, hence cardinal and cofinality preserving, forcing notion $P$ such that in the extension $V^P$ we have $\mathfrak{s} = \nu$, $\mathfrak{c} = \lambda$, moreover $\circledast_{\kappa,\mu}$ holds whenever $\nu \leq \kappa = cf(\kappa) \leq \lambda$ and either $\omega \leq \mu \leq \lambda$ or $\mu = 1$.*

*Proof.* Let $P = Fn(\lambda, 2) * \dot{Q}$, where $\dot{Q}$ is a name in $V^{Fn(\lambda,2)}$ for the standard finite support iteration

$$\langle Q_\zeta : \zeta \leq \lambda, R_\xi : \xi < \lambda \rangle$$

that forces $\mathfrak{p} = \nu$ in such a way that each $R_\xi$ is a CCC, even $\sigma$-centered, poset of size $< \nu$ in $V^{Fn(\lambda,2)*Q_\xi}$. So in $V^P$ we have $\mathfrak{c} = \lambda$ and $\mathfrak{p} = \nu$.

Let us now fix $\kappa$ and $\mu$ as indicated and check that $\circledast_{\kappa,\mu}$ holds. This will imply $\mathfrak{s} = \nu$ because on one hand $\nu = \mathfrak{p} \leq \mathfrak{s}$, while, by proposition 3.9, $\circledast_{\nu,\mu}$ implies $\mathfrak{s} \leq \nu$.

Because of $\kappa \cdot \mu \leq \lambda$ the forcings $Fn(\lambda, 2) \times Fn(\kappa \times \kappa \times \mu, 2)$ and $Fn(\lambda, 2)$ are equivalent, hence we may assume that actually

$$P = Fn(\lambda, 2) \times Fn(\kappa \times \kappa \times \mu, 2) * \dot{Q},$$

and from now on we will work in the intermediate model

$$W = V^{Fn(\lambda,2)}.$$

Now, let $\mathcal{G}$ be $Fn(\kappa \times \kappa \times \mu, 2)$-generic over $W$ and $\mathcal{H}$ be $Q$-generic over $W[\mathcal{G}]$. Our result then follows from the following claim.

**Claim 3.16.1.** *The sequence*

$$\mathcal{A}_0^{\mathcal{G}} = \left\langle \left\langle A_\alpha^0, A_\alpha^1 \right\rangle : \alpha < \kappa \right\rangle,$$

*defined in $W[\mathcal{G}]$ following 3.12 with the choice $\alpha = 0$, is a $\circledast_{\kappa,\mu}$-sequence in the final generic extension $W[\mathcal{G}][\mathcal{H}]$.*

*Proof of the claim.* Let us fix $S \in \left[ \kappa \times \mu \right]^\omega$ in $W[\mathcal{G}][\mathcal{H}]$. It is easy to see that there is a regular suborder $Q' \lessdot Q$ such that $|Q'| < \kappa$ and

$$S \in W[\mathcal{G}][\mathcal{H}']$$

where $\mathcal{H}' = \mathcal{H} \cap Q'$. Since $\kappa$ is regular, then there is $\alpha < \kappa$ such that $Q' \in W[\mathcal{G}_\alpha]$, where $\mathcal{G}_\alpha = \mathcal{G} \cap Fn(\alpha \times \kappa \times \mu, 2)$. But $W[\mathcal{G}_\alpha]$ contains both $Q'$ and $Fn((\kappa \setminus \alpha) \times \kappa \times \mu, 2)$, hence, putting

$$\mathcal{G}^\alpha = \mathcal{G} \cap Fn((\kappa \setminus \alpha) \times \kappa \times \mu, 2),$$

we have

$$W[\mathcal{G}][\mathcal{H}'] = W[\mathcal{G}_\alpha][\mathcal{G}^\alpha][\mathcal{H}'] = W[\mathcal{G}_\alpha][\mathcal{H}'][\mathcal{G}^\alpha]. \qquad (3.7)$$

By lemma 3.13 then we have

$$W[\mathcal{G}][\mathcal{H}'] = W[\mathcal{G}_\alpha][\mathcal{H}'][\mathcal{G}^\alpha] \models \text{``}\mathcal{A}_\alpha^{\mathcal{G}^\alpha} \text{ is dyadic on } S\text{''},$$

while $\mathcal{A}_\alpha^{\mathcal{G}^\alpha}$ is clearly just the final segment of $\mathcal{A}_0^{\mathcal{G}}$ starting at $\alpha$. But then by $W[\mathcal{G}][\mathcal{H}'] \subset W[\mathcal{G}][\mathcal{H}]$ we also have

$$W[\mathcal{G}][\mathcal{H}] \models \text{``}\mathcal{A}_\alpha^{\mathcal{G}^\alpha} \text{ is dyadic on } S\text{''},$$

hence $\mathcal{A}_0^{\mathcal{G}}$ is indeed a $\circledast_{\kappa,\mu}$-sequence in $W[\mathcal{G}][\mathcal{H}]$.                    □

This completes the proof of our theorem.                    □

We have seen in theorem 3.11 that if $\mu < \kappa$ then our construction from $\circledast_{\kappa,\mu}$ yields a space $X$ of size $\kappa$ with the very strong SAU property that every infinite closed set $F \in \mathcal{F}^+(X)$ is "co-small", i.e. $|X \setminus F| < |X| = \kappa$. Our following result shows that this strong property cannot be pushed any further.

**Theorem 3.17.** *If $X$ is any space then*

$$|X| \le \sup\{|X \setminus F|^+ : F \in F \in \mathcal{F}^+(X)\}.$$

*Proof.* Assume, on the contrary, that

$$\kappa = \sup\{|X \setminus F|^+ : F \in F \in \mathcal{F}^+(X)\} < |X|. \qquad (3.8)$$

Then every point $p \in X$ has an open neighborhood $U(p)$ with $|U(p)| < \kappa$. But then, by Hajnal's Set Mapping Theorem from [4], there is a set $Y \in [X]^{|X|}$ such that $q \notin U(p)$ for any distinct $p, q \in Y$. Fix any $Z \subset Y$ such that $|Z| = |Y| = |X|$ and $Y \setminus Z$ is infinite. Then for $U = \bigcup_{p \in Z} U(p)$ we have $|U| = |X|$ while $X \setminus U \in \mathcal{F}^+(X)$, contradicting (3.8).                    □

Our results below show that, consistently, the existence of a SAU space of a given size does not imply the existence of a space of the same size in which every infinite closed set is "co-small".

**Theorem 3.18.** *Assume that $\mathfrak{r} = \mathfrak{c} = \omega_2$ and $\clubsuit$ holds. Then SAU spaces of cardinality $\mathfrak{c}$ exist but in every space $X$ of cardinality $\mathfrak{c}$ there is an infinite closed set $F$ such that $|X \setminus F| = \mathfrak{c}$.*

*Proof.* By theorem 3.7 $\mathfrak{r} = \mathfrak{c}$ implies that SAU spaces of cardinality $\mathfrak{c}$ exist .

Now assume that $X = \langle \omega_2, \tau \rangle$ is a space such that $|\omega_2 \setminus F| \leq \omega_1$ for all $F \in \mathcal{F}^+(X)$ and fix a ♣-sequence $\langle T_\alpha : \alpha \in Lim(\omega_1) \rangle$. Then

$$F = \bigcap_{\alpha \in Lim(\omega_1)} \overline{T_\alpha}$$

contains a final segment of $\omega_2$. So we may pick two points $x, y \in F \setminus \omega_1$ with disjoint open neighborhoods $U$ and $V$, respectively. Clearly, then $U \cap T_\alpha$ is infinite for all $\alpha \in Lim(\omega_1)$, consequently $|U \cap \omega_1| = \omega_1$. But $\langle T_\alpha : \alpha \in Lim(\omega_1) \rangle$ is a ♣-sequence, hence there is $\alpha \in Lim(\omega_1)$ with $T_\alpha \subset U$. Thus we get $V \cap T_\alpha = \emptyset$, which contradicts $y \in F \subset \overline{T_\alpha}$. □

The consistency of the assumptions of theorem 3.18 follows from a result that had been proved by the first author back in 1983 but has never been published. So we decided to include it here. For that we need some preparation.

For any cardinal $\mu$ we shall write

$$S_\mu^\omega = \{\alpha < \mu : cf(\alpha) = \omega\}.$$

We also need the following definition.

**Definition 3.19.** For any given set $X$ we define the forcing notion $\mathcal{J}_X = \langle J_X, \leq \rangle$ as follows:

$$J_X = \{f \in Fn(X \times \omega, 2\,;\omega_1) :$$
$$\mathrm{dom}(f) = A \times n \text{ for some } A \in [X]^{\leq \omega} \text{ and } n \in \omega\}.$$

For $p, q \in J_X$ we let $p \leq q$ iff $p \supset q$.

We now present some properties of this forcing.

**Theorem 3.20.** *Let $\kappa$ be any infinite cardinal in our ground model $V$.*

*(1) $\mathcal{J}_\kappa$ is $\mathfrak{c}^+$-CC; in fact, for any $\{p_\alpha : \alpha < \mathfrak{c}^+\} \subset J_\kappa$ there is $I \in [\mathfrak{c}^+]^{\mathfrak{c}^+}$ such that $\bigcup_{\alpha \in K} p_\alpha \in J_\kappa$ whenever $K \in [I]^\omega$. Consequently, the forcing $\mathcal{J}_\kappa$ preserves all cardinals $> \mathfrak{c}$.*

*(2) $\mathfrak{c}$ becomes countable in $V^{\mathcal{J}_\kappa}$, hence $(\mathfrak{c}^+)^V = (\omega_1)^{V^{\mathcal{J}_\kappa}}$.*

*(3) If ♣$(S_{\mathfrak{c}^+}^\omega)$ holds in $V$ then ♣ holds in $V^{\mathcal{J}_\kappa}$.*

*(4) If $\kappa = \kappa^{\mathfrak{c}}$ then $\mathfrak{c}^{V^{\mathcal{J}_\kappa}} = \kappa$ and $V^{\mathcal{J}_\kappa} \models \mathrm{MA}(countable)$.*

*Proof.* (1) Assume that $\mathrm{dom}(p_\alpha) = A_\alpha \times n_\alpha$ for $\alpha < \mathfrak{c}^+$. Clearly we can find $H \in [\mathfrak{c}^+]^{\mathfrak{c}^+}$ and $n \in \omega$ such that $n_\alpha = n$ for all $\alpha \in H$. A simple $\Delta$-system and counting argument then yields $I \in [H]^{\mathfrak{c}^+}$ such that the functions $\{p_\alpha : \alpha \in I\}$ are pairwise compatible. It is obvious then that $I$ is as required.

(2) Let $\mathcal{G}$ be $\mathcal{J}_\kappa$-generic over $V$, then $g = \bigcup \mathcal{G} : \kappa \times \omega \to 2$. For each $n \in \omega$ we define the function $d_n \in {}^\omega 2$ by putting for all $i < n$

$$d_n(i) = g(i, n).$$

It is straight forward to check that if $r : \omega \to 2$ is in the ground model then

$$D_r = \{p \in J_\kappa : \exists n \in \omega \, \forall i \in \omega \, [r(i) = p(i, n)]\}.$$

is dense in $\mathcal{J}_\kappa$, consequently we have

$$V^{\mathcal{J}_\kappa} \models \{d_n : n < \omega\} \supset {}^\omega 2 \cap V.$$

But this clearly implies that $\mathfrak{c}$ becomes countable in $V^{\mathcal{J}_\kappa}$. Then $(\mathfrak{c}^+)^V = (\omega_1)^{V^{\mathcal{J}_\kappa}}$ follows because $(\mathfrak{c}^+)^V$ remains a cardinal by (1).

(3) To aid readability, we write $\mu = \mathfrak{c}^+$ and $S = S_\mu^\omega$. Then we fix a $\clubsuit(S)$-sequence $\langle A_\zeta : \zeta \in S \rangle$ in $V$. By (2) we have $S \subset Lim(\omega_1)$ in the generic extension $V^{\mathcal{J}_\kappa}$.

Let us assume now that $p \Vdash \dot{X} \in [\mu]^\mu$ for a condition $p$ in $\mathcal{J}_\kappa$. We can then define in $V$ a strictly increasing map $\varphi : S \to \mu$ and for each $\zeta \in S$ a condition $p_\zeta \leq p$ such that $p_\zeta \Vdash \varphi(\zeta) \in \dot{X}$. Applying (1) we can find $I \in [\mu]^\mu$ such that $p_K = \bigcup_{\zeta \in K} p_\zeta \in J_\kappa$ holds whenever $K \in [I]^\omega$.

Now, $\varphi[I] = \{\varphi(\zeta) : \zeta \in I\} \in [\mu]^\mu$, hence there is some $\eta \in S$ such that $A_\eta \subset \phi[I]$. But then for $K = \phi^{-1}[A_\eta] \in [I]^\omega$ we have $p_K \in J_\kappa$ and $p_K \leq p$, moreover we clearly have $p_K \Vdash A_\eta \subset \dot{X}$. Thus, no matter how we define $A_\zeta$ for $\zeta \in Lim(\omega_1) \setminus S$, the sequence $\langle A_\zeta : \zeta \in Lim(\omega_1) \rangle$ will be a $\clubsuit$-sequence in the generic extension $V^{\mathcal{J}_\kappa}$.

(4) For each $\alpha < \kappa$ we define the real $q_\alpha \in {}^\omega 2$ in $V^{\mathcal{J}_\kappa}$ by stipulating $q_\alpha(n) = g(\alpha, n)$ for all $n \in \omega$. Then, by genericity, $\{q_\alpha : \alpha < \kappa\}$ are pairwise distinct, hence we have $\mathfrak{c}^{V^{\mathcal{J}_\kappa}} \geq \kappa$. On the other hand, by (1) $\mathcal{J}_\kappa$ satisfies the $\mathfrak{c}^+$-chain condition, hence the standard calculation using nice names and the condition $\kappa = \kappa^\mathfrak{c}$ yield us that $\mathfrak{c}^{V^{\mathcal{J}_\kappa}} \leq \kappa$. Thus indeed $\mathfrak{c}^{V^{\mathcal{J}_\kappa}} = \kappa$.

Now suppose that $\mathfrak{c}^V \leq \lambda < \kappa$ and $\mathcal{D} = \{D_\alpha : \alpha < \lambda\}$ is a family of dense subsets of $Fn(\omega, 2)$ in $V^{\mathcal{J}_\kappa}$. Then there is $I \in [\kappa]^\lambda$ such that $\mathcal{D} \in V^{\mathcal{J}_I}$. Pick any $\alpha \in \kappa \setminus I$. Then, as $\mathcal{D} \in V^{\mathcal{J}_{\kappa \setminus \{\alpha\}}}$ and

$$\mathcal{J}_\kappa \approx \mathcal{J}_{\kappa \setminus \{\alpha\}} \times Fn(\omega, 2),$$

$q_\alpha$ is generic over $\mathcal{D}$. This clearly implies MA(*countable*) in the generic extension $V^{\mathcal{J}_\kappa}$. $\qquad\square$

In the constructible universe $L$ we have $\mathfrak{c} = \omega_1$, $(\omega_2)^{\omega_1} = \omega_2$, moreover $\clubsuit(S_{\mathfrak{c}^+}^\omega)$ holds. Also, it is well-known and easy to prove that

MA(*countable*) implies $\mathfrak{r} = \mathfrak{c}$. Consequently, it is an immediate corollary of theorem 3.20 that $L^{\mathcal{J}_{\omega_2}}$ satisfies all the assumptions of theorem 3.18.

## 4. Problems

In this section we formulate the most intriguing questions concerning SAU spaces that are left open.

**Problem 4.1.** *Is there a SAU space in ZFC?*

**Problem 4.2.** *Is it consistent that there is a SAU space of cardinality $> \mathfrak{c}$? Is it consistent that there is a locally countable SAU space of cardinality $> \mathfrak{c}$?*

**Problem 4.3.** *Does the existence of a SAU space imply the existence of a crowded SAU space?*

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