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LIMITS OF SPIKED RANDOM MATRICES II

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The top eigenvalues of rank r spiked real Wishart matrices and additively perturbed Gaussian orthogonal ensembles are known to exhibit a phase transition in the large size limit. We show that they have limiting distributions for near-critical perturbations, fully resolving the conjecture of Baik, Ben Arous and Péché [*Duke Math. J.* (2006) **133** 205–235]. The starting point is a new $(2r+1)$-diagonal form that is algebraically natural to the problem; for both models it converges to a certain random Schrödinger operator on the half-line with $r \times r$ matrix-valued potential. The perturbation determines the boundary condition and the low-lying eigenvalues describe the limit, jointly as the perturbation varies in a fixed subspace. We treat the real, complex and quaternion ($\beta = 1, 2, 4$) cases simultaneously. We further characterize the limit laws in terms of a diffusion related to Dyson's Brownian motion, or alternatively a linear parabolic PDE; here β appears simply as a parameter. At $\beta = 2$, the PDE appears to reconcile with known Painlevé formulas for these r-parameter deformations of the GUE Tracy–Widom law.

1. Introduction. Johnstone (2001) proposed the spiked population model for simple trends in high dimensional data. One takes a data matrix X whose columns are i.i.d. vectors with (population) covariance a fixed rank perturbation of the identity, and studies the behaviour of the largest eigenvalues of the sample covariance matrix XX^* when both the dimension and the size of the sample are large. Baik, Ben Arous and Péché (2005) (hereafter *BBP*) discovered a very interesting phase transition phenomenon in the complex Gaussian setting. Small spikes do not affect the asymptotic behaviour of the top eigenvalues, which display the usual Tracy–Widom fluctuations around the upper edge of the Marchenko–Pastur law; large spikes, however,

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lead to outliers with Gaussian fluctuations. New structure emerges near the transition point with near-critical spikes deforming the soft edge limit. Understanding this transition regime in the real case remained open for some time. There is a parallel development for fixed rank additive perturbations of Wigner matrices.

In Bloemendal and Virág (2013) (hereafter **Part** I), we considered rank one spiked real/complex/quaternion Wishart matrices and additive rank one perturbations of the Gaussian orthogonal, unitary and symplectic ensembles. Our approach is based on the continuum operator limit at the general beta soft edge developed in Ramírez, Rider and Virág (2011) (hereafter *RRV*). We introduced general β analogues of the rank one spiked models, modifying the tridiagonal ensembles of Dumitriu and Edelman (2002) and extended the RRV technology to describe the soft-edge scaling limit in terms of the stochastic Airy operator

$$-\frac{d^2}{dx^2}+\frac{2}{\sqrt{\beta}}b'_x+x$$

on $L^2(\mathbb{R}_+)$ with a boundary condition depending on the spike. The boundary condition changes from Dirichlet $f(0)=0$ to Neumann/Robin $f'(0)=wf(0)$ at the onset of the BBP phase transition, with $w\in\mathbb{R}$ representing a scaling parameter for perturbations in a "critical window". The resulting largest eigenvalue laws form a one-parameter family of deformations of Tracy–Widom(β), naturally generalizing the characterization of RRV in terms of the ground state of this random Schrödinger operator.

We went on to characterize the limit laws in terms of the diffusion from RRV and in terms of an associated second-order linear parabolic PDE. We further showed that at $\beta=2,4$ the PDE is related to known Painlevé II representations originating in Baik and Rains (2000) and gave new proofs of these, finally recovering those of the undeformed Tracy–Widom laws.

Even the existence of limiting distributions in the critical regime was in general new for $\beta\neq 2$, though see the prior work of Wang (2008) on the rank one $\beta=4$ case at $w=0$, as well as the subsequent work of Mo (2012) offering a more standard treatment of the rank one $\beta=1$ case. Forrester (2013) comments on all three works and gives an alternative interpretation and construction of our general β rank one spiked model.

Here, we deal with r "spikes", or general bounded-rank perturbations of Gaussian and Wishart matrices. To do so, we introduce a new "canonical form for perturbations in a fixed subspace", a $(2r+1)$-diagonal band form that has a purely algebraic interpretation. It generalizes the Dumitriu–Edelman forms and is able to handle rank r perturbations. We then develop a generalization of the methods of RRV and Part I to a matrix-valued setting: block tridiagonal matrices converge to a half-line Schrödinger operator with matrix-valued potential, the spikes once again appearing in the boundary condition. We treat the real, complex and quaternion ($\beta = 1, 2, 4$) cases simultaneously. Once again, even the existence of a near-critical soft-edge limit is new off $\beta = 2$. Unlike in Part I, however, we do not define a general β version of either matrix model, nor of the limiting operator; in Section 2, we will see that the higher rank versions of these objects do not readily admit a β-generalization.

Dyson's Brownian motion makes a surprise appearance, providing nice SDE and PDE characterizations of the limit laws—new r parameter deformations of Tracy–Widom(β)—in which β reappears as a simple parameter. The derivation makes use of the matrix-valued version of classical Sturm oscillation theory and the Riccati transformation. In a short final section, we report on preliminary evidence that at $\beta = 2$ the PDE can be connected with a Painlevé II representation of Baik (2006) for these distributions (which appeared originally in BBP in the form of Fredholm determinants).

We highlight two more features of our approach beyond the novelty of bypassing formulas for joint eigenvalue densities and handling $\beta = 1, 2, 4$ together. First, we treat the perturbation as a *parameter*. By this, we mean that all perturbations in a fixed subspace are considered jointly (on the same probability space); this picture is carried through to the limit, which is therefore a family of point processes parameterized by an $r \times r$ matrix. Second, we allow more general scalings than those considered in BBP. Most importantly, in the Wishart case we do not require the two dimensional parameters n, p to have a positive limiting ratio but rather allow them to tend to infinity together arbitrarily.

To state our results, we introduce some objects and notation that will be used throughout the paper.

Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and $\beta = 1, 2$ or 4, respectively. A *standard* $\mathbb{F}$ *Gaussian* $Z \sim \mathbb{F}N(0, 1)$ is an $\mathbb{F}$-valued random variable described in terms of independent real Gaussians $g_1, \dots, g_\beta \sim N(0, 1)$ as g_1 for $\mathbb{F} = \mathbb{R}$, $(g_1 + g_2 i)/\sqrt{2}$ for $\mathbb{F} = \mathbb{C}$, and $(g_1 + g_2 i + g_3 j + g_4 k)/2$ for $\mathbb{F} = \mathbb{H}$. Note that in each case $\mathbf{E}|Z|^2 = 1$ and $uZ \sim \mathbb{F}N(0, 1)$ for $u \in \mathbb{F}$ with $|u|^2 = u^* u = 1$.

The space of column vectors $\mathbb{F}^n$ is endowed with the standard inner product $u^\dagger v$ and associated norm $|u|^2 = u^\dagger u$ (we reserve double bars for function spaces). Write $\mathbb{F}N_n(0, I)$ for a vector of independent standard $\mathbb{F}$ Gaussians. With $\Sigma \in M_n(\mathbb{F})$ positive definite, we write $Z \sim \mathbb{F}N_n(0, \Sigma)$ for $Z = \Sigma^{1/2} Z_0$ with $Z_0 \sim \mathbb{F}N_n(0, I)$.

Define the *unitary group* $U_n(\mathbb{F}) = \{U \in \mathbb{F}^{n\times n} : U^\dagger U = I\}$, better known as the orthogonal, unitary or symplectic group for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively. It acts on $\mathbb{F}^n$ by left multiplication, on which the distribution $\mathbb{F}N_n(0, I)$ is invariant. Write $M_n(\mathbb{F}) = \{A \in \mathbb{F}^{n\times n} : A^\dagger = A\}$ for the *self-adjoint matrices*, also known as real symmetric, complex Hermitian or quaternion self-dual. $U_n(\mathbb{F})$ acts on $M_n(\mathbb{F})$ by conjugation.

The *Gaussian orthogonal/unitary/symplectic ensemble* (GO/U/SE) is the probability measure on $M_n(\mathbb{F})$ described by $A = (X + X^\dagger)/\sqrt{2}$ where X is an $n \times n$ matrix of independent $\mathbb{F}N(0,1)$ entries. The distribution is invariant under the unitary action. Furthermore, the algebraically independent entries $A_{ij},\ i \geq j$ are statistically independent. (Together, this invariance and independence characterizes the distribution up to a scale factorr.) For an entry-wise description, the diagonal entries are distributed as $N(0, 2/\beta)$ while the off-diagonal entries are $\mathbb{F}N(0,1)$.

Fixing a positive integer r, we study *rank r additive perturbations* $A = A_0 + P$ of a GO/U/SE matrix A_0, where $P = \tilde{P} \oplus 0_{n-r}$ with $\tilde{P} \in M_r(\mathbb{F})$ nonrandom. We will be interested in the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ of A. Of course for a single P their distribution depends only on the eigenvalues of P, but we consider them jointly over all $\tilde{P}$.

We also consider *real/complex/quaternion Wishart matrices*. These are random nonnegative matrices in $M_p(\mathbb{F})$ given by $XX^\dagger$ where the *data matrix* X is $p \times n$ with n independent $\mathbb{F}N_p(0, \Sigma)$ columns. We speak of a *p-variate* Wishart with *n degrees of freedom* and *$p \times p$ covariance* $\Sigma > 0$. Since we are interested in the *nonzero* eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{n \wedge p}$, we can equally well consider $X^\dagger X$. The distribution of $X^\dagger X$ may also be described as $X_0^\dagger \Sigma X_0$ where X_0 is a $p \times n$ matrix of independent $\mathbb{F}N(0,1)$ entries. The case $\Sigma = I$ is referred to as the *null case*. We study the *rank r spiked case* where $\Sigma = \tilde{\Sigma} \oplus I_{p-r}$ with $\tilde{\Sigma} \in M_r(\mathbb{F})$ nonrandom. Once again the eigenvalue distribution depends only on the eigenvalues of Σ, but we consider the spectrum jointly as $\tilde{\Sigma}$ varies.

Our starting point is a new banded or multi-diagonal form introduced in Section 2, ideally suited to the types of perturbations we consider. It is defined for almost every matrix $A \in M_n(\mathbb{F})$; given vectors $v_1, \ldots, v_r \in \mathbb{F}^n$, the new basis may be obtained by applying the Gram–Schmidt process to the first n vectors of the sequence

$$v_1, \ldots, v_r, Av_1, \ldots, Av_r, A^2 v_1, \ldots, A^2 v_r, \ldots.$$

The result is a $(2r+1)$-diagonal matrix with positive outer diagonals. For Gaussian and null Wishart ensembles, the change of basis interacts well with the Gaussian structure; this observation goes back to Trotter (1984) in the $r = 1$ case. In the GO/U/SE case, we take $v_1, \ldots, v_r$ to be the initial coordinate basis vectors, while in the Wishart case we use the initial rows of the data matrix X. As in Part I, the key observation is then that the perturbations commute with the change of basis.

For the (unperturbed) Gaussian ensembles, the band form looks like

$$\begin{bmatrix} \widetilde{g} & g^* & \cdots & g^* & \chi & & & \\ g & \widetilde{g} & g^* & \cdots & g^* & \chi & & \\ \vdots & g & \widetilde{g} & g^* & \cdots & g^* & \chi & \\ g & \vdots & g & \ddots & \ddots & & \ddots & \ddots \\ \chi & g & \vdots & \ddots & & & & \\ & \chi & g & & & & & \\ & & \chi & \ddots & & & & \\ & & & \ddots & & & & \end{bmatrix},$$

where the entries are independent random variables up to the †-symmetry with $\widetilde{g} \sim N(0, 2/\beta)$, $g \sim \mathbb{F}N(0,1)$, and $\chi \sim \frac{1}{\sqrt{\beta}}\,\mathrm{Chi}((n-r-k)\beta)$, $k = 0, 1, 2, \ldots$ going down the matrix. [Recall that if $Z \sim \mathbb{R}N_m(0, I)$ then $|Z| \sim \mathrm{Chi}(m)$.] For the null Wishart ensemble, the form is best described as follows. One first obtains a lower $(r+1)$-diagonal form for the data matix X whose nonzero *singular values* are the same as those of X. It looks like

$$\begin{bmatrix} \widetilde{\chi} & & & \\ g & \widetilde{\chi} & & \\ \vdots & g & \widetilde{\chi} & \\ g & \vdots & g & \ddots \\ \chi & g & \vdots & \ddots \\ & \chi & g & \\ & & \chi & \ddots \\ & & & \ddots \end{bmatrix},$$

where the entries are independent random variables with $g \sim \mathbb{F}N(0,1)$, $\widetilde{\chi} \sim \frac{1}{\sqrt{\beta}}\,\mathrm{Chi}((n-k)\beta)$ and $\chi \sim \frac{1}{\sqrt{\beta}}\,\mathrm{Chi}((n-r-k)\beta)$, $k = 0, 1, 2, \ldots$ going down the matrix. One then forms its multiplicative symmetrization, a $(2r+1)$-diagonal matrix with the same nonzero eigenvalues as X. In both cases, the perturbations appear in the upper-left $r \times r$ block. Section 2 provides derivations. The obstacle to β-generalization at this level is the presence of $\mathbb{F}$ Gaussians in the intermediate diagonals.

Proceeding with an analogue of the RRV convergence result hinges on reinterpreting these forms as *block tridiagonal* with $r \times r$ blocks. In Section 3, we develop an $M_r(\mathbb{F})$-valued analogue of the RRV technology, providing general conditions under which the principal eigenvalues and corresponding eigenvectors of such a random block tridiagonal matrix converge to a those of a continuum half-line random Schrödinger operator with matrix-valued potential. As in Part I, we allow for a general boundary condition at the origin.

In Section 4, we apply this result to the band forms just described, proving a process central limit theorem for the potential and verifying the required tightness assumptions. The limiting operator turns out to be a multidimensional version of the stochastic Airy operator, which we now describe.

First, a *standard* $\mathbb{F}$ *Brownian motion* $\{b_t\}_{t\geq 0}$ is a continuous $\mathbb{F}$-valued random process with $b_0 = 0$ and independent increments $b_t - b_s \sim \mathbb{F}N(0, t-s)$. (It can be described in terms of $\beta = 1$, 2 or 4 independent standard real Brownian motions.) A *standard matrix Brownian motion* $\{B_t\}_{t\geq 0}$ has continuous $M_n(\mathbb{F})$-valued paths with $B_0 = 0$ and independent increments $B_t - B_s$ distributed as $\sqrt{t-s}$ times a GO/U/SE. The diagonal processes are thus $\sqrt{2/\beta}$ times standard real Brownian motions while the off-diagonal processes are standard $\mathbb{F}$ Brownian motions, mutually independent up to symmetry.

Finally, we define the *multivariate stochastic Airy operator*. Operating on the vector-valued function space $L^2(\mathbb{R}_+, \mathbb{F}^r)$ with inner product $\langle f, g\rangle = \int_0^\infty f^\dagger g$ and associated norm $\|f\|^2 = \int_0^\infty |f|^2$, it is the random Schrödinger operator

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + \sqrt{2}B'_x + rx, \tag{1.1}$$

where B'_x is "standard matrix white noise", the derivative of a standard matrix Brownian motion, and rx is scalar. (Here, again β is restricted to the classical values, as the noise term lacks a straightforward β-generalization.) The potential is thus the derivative of a continuous matrix-valued function; rigorous definitions will appear in Section 3 in a more general setting.

For now it is enough to know that, together with a general self-adjoint boundary condition

$$f'(0) = Wf(0), \tag{1.2}$$

the multivariate stochastic Airy operator is bounded below with purely discrete specturm given by a variational principle. Here, $W \in M_r(\mathbb{F})$; actually, writing the spectral decomposition $W = \sum_{i=1}^r w_i u_i u_i^\dagger$, we formally allow $w_i \in (-\infty, \infty]$. Writing $f_i = u_i^\dagger f$, (1.2) is then to be interpreted as

$$f_i'(0) = w_i f_i(0) \qquad \text{for } w_i \in \mathbb{R},$$
$$f_i(0) = 0 \qquad \text{for } w_i = +\infty.$$

We write $W \in M_r^*(\mathbb{F})$ for this extended set and $\mathcal{H}_{\beta,W}$ for (1.1) together with (1.2).

For concreteness, we record that the eigenvalues $\Lambda_0 \le \Lambda_1 \le \dots$ and corresponding eigenfunctions $f_0, f_1, \dots$ of $\mathcal{H}_{\beta,W}$ are given, respectively, by the minimum and any minimizer in the recursive variational problem

$$\inf_{\substack{f\in L^2(\mathbb{R}_+)\\ \|f\|=1, f\perp f_0,\dots,f_{k-1}}} \int_0^\infty (|f'|^2 + rx|f|^2)\,dx + f(0)^\dagger W f(0) + \frac{2}{\sqrt{\beta}}\int_0^\infty f^\dagger\, dB_x f.$$

Here, candidates f are only considered if the first integral and boundary term are finite; the stochastic integral can then be defined pathwise via integration by parts. The eigenvalues and eigenfunctions are thus jointly defined random processes indexed over W.

REMARK 1.1. We note one important property of the eigenvalue processes, namely the *pathwise monotonicity* of Λ_k in W with respect to the usual matrix partial order. This is immediate from the variational characterization and the fact that the objective functional is monotone in W. (For the higher eigenvalues, it is most apparent from the standard min–max formulation of the variational problem.)

We can now state the main convergence results. As outlined, Sections 2–4 furnish the proofs. One last shorthand: when we write that a sequence $W_n \in M_r(\mathbb{F})$ tends to $W \in M_r^*(\mathbb{F})$, we mean the following. Writing $W = \sum_{i=1}^r w_i u_i u_i^\dagger$ with $w_i \in (-\infty, \infty]$, one has $W_n = \sum_{i=1}^r w_{n,i} u_i u_i^\dagger$ with $w_{n,i} \in \mathbb{R}$ satisfying $w_{n,i} \to w_i$ for each i. In other words, the matrices are simultaneously diagonal and the eigenvalues tend to the corresponding limits.

THEOREM 1.2. *Let $A = A_0 + \sqrt{n}P_n$ where A_0 is an $n \times n$ GO/U/SE matrix and $P_n = \tilde{P}_n \oplus 0_{n-r}$ with $\tilde{P}_n \in M_r(\mathbb{F})$, and let $\lambda_1 \ge \dots \ge \lambda_n$ be its eigenvalues. If*

$$n^{1/3}(1 - \tilde{P}_n) \to W \in M_r^*(\mathbb{F}) \qquad \text{as } n \to \infty$$

then, jointly for $k = 1, 2, \dots$ in the sense of finite-dimensional distributions,

$$n^{1/6}(\lambda_k - 2\sqrt{n}) \Rightarrow -\Lambda_{k-1} \qquad \text{as } n \to \infty,$$

where $\Lambda_0 \le \Lambda_1 \le \dots$ are the eigenvalues of $\mathcal{H}_{\beta,W}$. Convergence holds jointly over $\{P_n\}, W$ satisfying the condition.

THEOREM 1.3. *Consider a p-variate real/complex/quaternion Wishart matrix with n degrees of freedom and spiked covariance $\Sigma_{n,p} = \tilde{\Sigma}_{n,p} \oplus I_{p-r} > 0$ with $\tilde{\Sigma}_{n,p} \in M_r(\mathbb{F})$, and let $\lambda_1 \ge \dots \ge \lambda_{n\wedge p}$ be its nonzero eigenvalues. Writing $m_{n,p} = (n^{-1/2} + p^{-1/2})^{-2/3}$, if*

$$m_{n,p}(1 - \sqrt{n/p}(\tilde{\Sigma}_{n,p} - 1)) \to W \in M_r^*(\mathbb{F}) \qquad \text{as } n \to \infty$$

then, jointly for $k = 1, 2, \dots$ in the sense of finite-dimensional distributions,

$$\frac{m_{n,p}^2}{\sqrt{np}}(\lambda_k - (\sqrt{n} + \sqrt{p})^2) \Rightarrow -\Lambda_{k-1} \qquad \text{as } n \to \infty,$$

where $\Lambda_0 \le \Lambda_1 \le \dots$ are the eigenvalues of $\mathcal{H}_{\beta,W}$. Convergence holds jointly over $\{\Sigma_{n,p}\}, W$ satisfying the condition.

REMARK 1.4. In the band basis described above, we also have joint convergence of the corresponding eigenvectors to the eigenfunctions of $\mathcal{H}_{\beta,W}$. In detail, the eigenvectors should be embedded in $L^2(\mathbb{R}_+)$ as step functions with step width $n^{-1/3}$ in the Gaussian case and $m_{n,p}^{-1}$ in the Wishart case, and convergence is in law with respect to the L^2 norm topology. To be precise, one should use either subsequences or spectral projections; one could also formulate the joint eigenvalue-eigenvector convergence in terms of the norm resolvent topology. See Theorem 3.9 and the remark that follows.

We now give the two promised alternative characterizations of the limiting eigenvalue laws. Fix $\beta = 1, 2, 4$ and $W \in M_r^*(\mathbb{F})$ with eigenvalues $-\infty < w_1 \le \dots \le w_r \le \infty$. Writing $\mathbf{P}$ for the probability measure associated with $\mathcal{H}_{\beta,W}$ and its spectrum $\{\Lambda_0 \le \Lambda_1 \le \dots\}$, let

$$F_\beta^k(x; w_1, \dots, w_r) = \mathbf{P}(-\Lambda_k \le x)$$

for $k = 0, 1, \dots$. Write simply $F_\beta = F_\beta^0$ for the ground state distribution (limiting largest eigenvalue law). Once again, the generalization from Part I is not straightforward. The proofs are contained in Section 5.

THEOREM 1.5. *Let $\mathbf{P}_{x_0,(w_1,\dots,w_r)}$ be the measure on paths $(p_1, \dots, p_r) : [x_0, \infty) \to (-\infty, \infty]^r$ determined by the coupled diffusions*

$$dp_i = \frac{2}{\sqrt{\beta}}\,db_i + \left(rx - p_i^2 + \sum_{j \ne i} \frac{2}{p_i - p_j}\right) dx \tag{1.3}$$

with initial conditions $p_i(x_0) = w_i$ and entering into $\{p_1 < \dots < p_r\}$, where $b_1, \dots, b_r$ are independent standard Brownian motions; particles p_i may explode to $-\infty$ in finite time whereupon they are restarted at $+\infty$. Then

$$F_\beta(x; w_1, \dots, w_r) = \mathbf{P}_{x/r,(w_1,\dots,w_r)} \qquad (\textit{no explosions}). \tag{1.4}$$

More generally,

$$F_\beta^k(x; w_1, \dots, w_r) = \mathbf{P}_{x/r,(w_1,\dots,w_r)} \qquad (\textit{at most } k \textit{ explosions}). \tag{1.5}$$

We describe the diffusion more carefully in Section 5, asserting that it determines a law on paths valued in an appropriate space. Probabilistic arguments lead to the following reformulation in terms of its generator.

THEOREM 1.6. *$F_\beta(x; w_1, \dots, w_r)$ is the unique bounded function $F : \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}$ symmetric with respect to permutation of $w_1, \dots, w_r$ that satisfies the PDE*

$$r\frac{\partial F}{\partial x} + \sum_{i=1}^{r}\left(\frac{2}{\beta}\frac{\partial^2 F}{\partial w_i^2} + (x - w_i^2)\frac{\partial F}{\partial w_i}\right) + \sum_{i<j}\frac{2}{w_i - w_j}\left(\frac{\partial F}{\partial w_i} - \frac{\partial F}{\partial w_j}\right) = 0 \tag{1.6}$$

and the boundary conditions

$$F \to 1 \qquad \text{as } x \to \infty \text{ with } w_1, \dots, w_r \text{ bounded below;} \tag{1.7}$$

$$F \to 0 \qquad \text{as any } w_i \to -\infty \text{ with } x \text{ bounded above.} \tag{1.8}$$

Furthermore, F_β is "continuous to the boundary" as one or several $w_i \to +\infty$. For subsequent eigenvalue laws $F_\beta^k(x; w_1, \dots, w_r)$, (1.8) is replaced with the recursive boundary condition

$$\begin{aligned} &F^k(x; w_1, \dots, w_r) \to F^{k-1}(x^*; w_1^*, \dots, w_{r-1}^*, +\infty) \\ &\text{as } x \to x^* \in \mathbb{R}, w_i \to w_i^* \in \mathbb{R} \text{ for } i = 1, \dots, r-1, \text{ and } w_r \to -\infty. \end{aligned} \tag{1.9}$$

At $\beta = 2$, these distributions were obtained in BBP in the form of Fredholm determinants of finite-rank perturbations of the Airy kernel. Baik (2006) derived Painlevé II formulas, and by a symbolic computation with a computer algebra system we were able to verify that the latter satisfy the PDE (1.6) for $r = 2, 3, 4, 5$; details are described in Section 6. A pencil-and-paper proof for all r was found since the initial posting [Bloemendal and Baik (2013)].

We make two final remarks. From the finite n matrix models it is clear that the "rank r deformed" limiting distributions $F_{\beta,r}(x; w_1, \dots, w_r)$ reduce to those for a lower rank $r_0 < r$ in the following way:

$$F_{\beta,r}(x; w_1, \dots, w_{r_0}, +\infty, \dots, +\infty) = F_{\beta,r_0}(x; w_1, \dots, w_{r_0}).$$

Unfortunately, this reduction relation is not readily apparent from any of our characterizations (operator, SDE or PDE).

Lastly, the SDE and PDE characterizations seem to make sense for all $\beta > 0$ (although one has to be careful for $\beta < 1$). It would be interesting to find natural "general β multi-spiked models" at finite n, interpolating between those studied here at $\beta = 1, 2, 4$ and generalizing those introduced in Part I for $r = 1$. At $\beta = 2$, perhaps one could discover a relationship with formulas of Baik and Wang (2013).

2. A canonical form for perturbation in a fixed subspace. In Part I, we observed that the tridiagonal models of Gaussian and Wishart matrices were amenable to rank one perturbation. In this section, we introduce a banded (also block tridiagonal) generalization amenable to higher-rank perturbation. We first describe it as a natural object of pure linear algebra; we then show how it interacts with the structure of Gaussian and Wishart random matrices to produce the band forms displayed in the Introduction.

The basic facts of "linear algebra over $\mathbb{F}$", where $\mathbb{F}$ may be $\mathbb{R}$, $\mathbb{C}$ or the skew field of quaternions $\mathbb{H}$, are summarized in Appendix E of Anderson, Guionnet and Zeitouni (2010). Everything we need (inner product geometry, self-adjointness, eigenvalues, and the spectral theorem) simply works over $\mathbb{H}$ as expected, keeping in mind only that nonreal scalars may not commute.

2.1. *The band Jacobi form as an algebraic object.* We present a natural "canonical form" for studying perturbations in a fixed subspace of dimension r. It is a $(2r+1)$-diagonal band matrix generalizing the symmetric tridiagonal Jacobi form, which is the $r=1$ case. The outermost diagonals continue to be positive; however, intermediate diagonals between the main and outermost ones are not in general real. Once again, the presence of $\mathbb{F}$ Gaussians is the obstacle to writing down a general β analogue.

We begin with a geometric, coordinate-free formulation.

THEOREM 2.1. *Let T be a self-adjoint linear transformation on a finite-dimensional inner product space V of dimension n over $\mathbb{F}$. An orthonormal sequence $\{v_1,\dots,v_r\}\subset V$ with $1\le r\le n$ can be extended to an ordered orthonormal basis $\{v_1,\dots,v_n\}$ for V such that $\langle v_i, Tv_j\rangle \ge 0$ for $|i-j|=r$ and $\langle v_i, Tv_j\rangle = 0$ for $|i-j|>r$. Furthermore, if $\langle v_i, Tv_j\rangle > 0$ for $|i-j|=r$ then the extension is unique.*

The point is that the same extension works for $T' = T+P$ provided $P\in M_n(\mathbb{F})$ satisfies $P|_{\{v_1,\dots,v_r\}^\perp}=0$. In this case $\operatorname{span}\{v_1,\dots,v_r\}$ is also an invariant subspace of P and we speak of *perturbing in this subspace*.

PROOF OF THEOREM 2.1. We give an explicit inductive construction. Along the way, we will see that the uniqueness condition holds precisely when the choice is forced at each step.

It is convenient to restate the properties of the orthonormal basis in the theorem in the following equivalent way: for $r+1\le i\le n$, we have $\langle v_i, Tv_{i-r}\rangle \ge 0$ and $Tv_{i-r}\in \operatorname{span}\{v_1,\dots,v_i\}$. Suppose inductively that $v_1,\dots,v_{k-1}$ have been obtained for some $r+1\le k\le n$, satisfying the preceding conditions for $r+1\le i\le k-1$. Let $w=Tv_{k-r}$; we must choose v_k so that $\langle v_k, w\rangle\ge 0$ and $w\in\operatorname{span}\{v_1,\dots,v_k\}$. There are two cases to consider. If $w\notin \operatorname{span}\{v_1,\dots,v_{k-1}\}$ then v_k must be a multiple of $w' = w-\sum_{i=1}^{k-1}\langle v_i, w\rangle v_i$;

the positivity condition further forces $v_k = w'/|w'|$, which gives $\langle v_k, w\rangle = |w'| > 0$. If $w \in \operatorname{span}\{v_1, \dots, v_{k-1}\}$, then any $v_k \in \{v_1, \dots, v_{k-1}\}^\perp$ will do, and in this case $\langle v_k, w\rangle = 0$. $\square$

REMARK 2.2. When uniqueness holds, as is generically the case, the basis may also be obtained by applying the Gram–Schmidt process to the first n vectors of the sequence

$$v_1, \dots, v_r, Tv_1, \dots, Tv_r, T^2 v_1, \dots, T^2 v_r, \dots.$$

We now state and prove a concrete matrix formulation in which the first r coordinate vectors play the role of $v_1, \dots, v_r$. The point of the second proof is that it emphasizes the resulting band matrix rather than the change of basis; the algorithm will be used in the next subsection.

THEOREM 2.3. *Let $A \in M_n(\mathbb{F})$ and $1 \le r \le n$. There exists $U \in U_n(\mathbb{F})$ of the form $U = I_r \oplus \tilde{U}$ with $\tilde{U} \in U_{n-r}(\mathbb{F})$ such that $B = UAU^\dagger$ satisfies*

$$B_{ij} \ge 0 \qquad \text{for } 1 \le i, j \le n \text{ with } |i-j| = r, \tag{2.1}$$

$$B_{ij} = 0 \qquad \text{for } 1 \le i, j \le n \text{ with } |i-j| > r. \tag{2.2}$$

Furthermore, if strict positivity holds in (2.1) then U and B as such are unique.

We refer to B as the *band Jacobi form* of A. The allowed perturbations here have the form $P = \tilde{P} \oplus 0_{n-r}$ for $\tilde{P} \in M_r(\mathbb{F})$; these are invariant under conjugation by U, so $U(A+P)U^\dagger = B + P$.

PROOF OF THEOREM 2.3. We prove existence by giving an explicit algorithm; it generalizes the Lanczos algorithm, which applies in the case $r = 1$.

- For the first step, let $v = [A_{i,1}]_{r+1 \le i \le n} \in \mathbb{F}^{n-r}$ and take $\tilde{U} \in U_{n-r}(\mathbb{F})$ such that $\tilde{U}v = |v|\tilde{e}_1$, where $\tilde{e}_1$ is the first standard basis vector of $\mathbb{F}^{n-r}$. A concrete choice is the Househölder reflection $\tilde{U} = I_{n-r} - 2ww^\dagger/w^\dagger w$ with $w = v - |v|\tilde{e}_1$. Set $U_1 = I_r \oplus \tilde{U}$ and $B_1 = U_1 A U_1^\dagger$.
- Continue inductively: having obtained U_k, B_k, let $v = [(B_1)_{i,(k+1)}]_{r+k+1 \le i \le n} \in \mathbb{F}^{n-r-k}$ and take $\tilde{U} \in U_{n-r-k}(\mathbb{F})$ such that $\tilde{U}v = |v|\tilde{e}_1$. Set $U_{k+1} = I_{r+k} \oplus \tilde{U}$ and $B_{k+1} = U_{k+1} B_k U_{k+1}^\dagger$.
- Stop when $k = n - r$. Let $U = U_{n-r} \cdots U_1$ and $B = B_{n-r} = UAU^\dagger$.

It is immediate that U and B have the required properties. The point is that the kth column of B_k already "looks right", that is, $(B_k)_{r+k,k} \ge 0$ and

$(B_k)_{r+l,k} = 0$ for $l > k$, and subsequent transformations $U_{k+1}, \ldots, U_{n-k} \in \{I_{r+k}\} \oplus U_{n-r-k}(\mathbb{F})$ "don't mess it up".

Toward uniqueness, suppose that $U', B' = U'AU'^{\dagger}$ also have the required properties and let $W = U'U^{-1}$ so that $B' = WBW^{\dagger}$. Assume inductively that $W \in \{I_{r+k}\} \oplus U_{n-r-k}(\mathbb{F})$, which is certainly true in the base case $k = 0$. Write $W = I_{r+k} \oplus \tilde{W}$. Let $b = [B_{i,k+1}]_{r+k+1 \le i \le n} \in \mathbb{F}^{n-r-k}$ and similarly for b'. Then $b' = \tilde{W}b$. But $b = a\tilde{e}_1$ and $b' = a'\tilde{e}_1$ with $a, a' > 0$ by assumption. It follows that $a = a'$ and $\tilde{W}\tilde{e}_1 = \tilde{e}_1$. Hence, $\tilde{W} \in \{I_1\} \oplus U_{n-r-(k+1)}(\mathbb{F})$ and $W \in \{I_{r+k+1}\} \oplus U_{n-r-(k+1)}(\mathbb{F})$, completing the induction step. We conclude that $W = I_n$. $\square$

2.2. *Perturbed Gaussian and spiked Wishart models.* The change of basis described above interacts very nicely with the Gaussian structure in Gaussian and Wishart random matrices. The $r = 1$ case of this observation is due to Trotter (1984), who described the tridiagonal forms explicitly. His forms fall into the framework of Theorem 2.1 by taking the initial vector to be fixed in the Gaussian case, and taking it to be the top row of the data matrix in the Wishart case. As we observed in Part I, the change of basis commutes with rank one additive perturbations for the Gaussian case and with rank one spiking for the Wishart case. We now extend the story to the $r > 1$ setting.

In the Gaussian case, we will be perturbing in a fixed (nonrandom) subspace; without loss of generality this may be taken as the initial r-dimensional coordinate subspace, and so we take the basis of Theorem 2.1 that begins with the first r standard basis vectors. We can therefore obtain the band form by a direct application of the algorithm from the proof of Theorem 2.3. The Wishart case is a little more complicated; here we want to perturb in the random subspace spanned by the first r rows of the data matrix. Our new basis will begin with the Gram–Schmidt orthogonalization of these initial rows. As in the $r = 1$ case, it is most transparent to construct a lower band form of the data matrix first, afterward realizing the band Jacobi form as its multiplicative symmetrization. In both the Gaussian and the Wishart cases, we will see that the uniqueness condition of Theorem 2.1 holds almost surely.

Let A be an $n \times n$ GOE matrix. Applying the algorithm from the proof of Theorem 2.3 while keeping track of the distribution of the matrix B_k at each step—the key of course being the unitary invariance of standard Gaussian vectors—yields the following band Jacobi random matrix $G=UAU^\dagger$:

$$G_{ij}=\begin{cases}\sqrt{\dfrac{2}{\beta}}\widetilde{g}_i, & i=j,\\ g_{ij}, & j<i<j+r,\\ \dfrac{1}{\sqrt{\beta}}\chi_{(n-i+1)\beta}, & i=j+r,\\ 0, & i>j+r,\\ G^*_{ji}, & i<j\end{cases}\tag{2.3}$$

for $1\le i,j\le n$, where the random variables appearing explicitly are independent, $\widetilde{g}_i\sim N(0,1)$, $g_{ij}\sim \mathbb{F}N(0,1)$, and $\chi_k\sim \mathrm{Chi}(k)$. The latter is the distribution of the length of a k-dimensional standard Gaussian vector.

We can introduce a rank r additive perturbation $A=A_0+\sqrt{n}P$, where $P=\tilde{P}\oplus 0_{n-r}$ with $\tilde{P}\in M_r(\mathbb{F})$; since P commutes with the change of basis $U\in\{I_r\}\oplus U_{n-r}(\mathbb{F})$, we can write

$$G=UAU^\dagger=U(A_0+\sqrt{n}P)U^\dagger=UA_0U^\dagger+\sqrt{n}P=G_0+\sqrt{n}P.\tag{2.4}$$

As expected the perturbation shows up undisturbed in the upper-left $r\times r$ corner of G.

Turning to the Wishart case, we first consider the null Wishart random matrix $X^\dagger X$, where X is $p\times n$ with independent $\mathbb{F}N(0,1)$ entries. (Remember that $X^\dagger X$ and $XX^\dagger$ have the same nonzero eigenvalues $\lambda_1,\dots,\lambda_{n\wedge p}$.) The final form can be described abstractly as given in the basis of Theorem 2.1 that extends the Gram–Schmidt orthogonalization of the first r rows of X. One cannot readily obtain a description of the resulting random matrix from here, however, so we give another way that generalizes Trotter's original procedure. It is a "singular value analogue" of the algorithm from the proof of Theorem 2.3, producing matrices $U\in U_n(\mathbb{F})$ and $V\in U_p(\mathbb{F})$ such that $L=VXU$ has a "lower band form" that is zero off the main and first r sub-diagonals and positive on the outermost of these. The key is to work alternately on rows and columns.

- Take $U_1\in U_n(\mathbb{F})$ so that the first row of XU_1 lies in the (positive) direction of the first coordinate basis vector of $\mathbb{F}^n$.
- Take $V_1=I_r\oplus U_{p-r}(\mathbb{F})$ so that $[(V_1XU_1)_{i,1}]_{r+1\le i\le p}\in\mathbb{F}^{p-r}$ lies in the direction of the first coordinate basis vector of the latter subspace.
- Take $U_2\in I_1\oplus U_{n-1}(\mathbb{F})$ so that $[(V_1XU_1U_2)_{2,j}]_{2\le j\le n}\in\mathbb{F}^{n-1}$ lies in the direction of the first coordinate basis vector of the latter subspace.
- Take $V_2\in I_{r+1}\oplus U_{p-r-1}(\mathbb{F})$ so that $[(V_2V_1XU_1U_2)_{i,2}]_{r+2\le j\le p}\in\mathbb{F}^{p-r-1}$ lies in the direction of the first coordinate basis vector of the latter subspace.
- Continue in this way until the rows and columns both run out (stop alternating if one runs out before the other).

The resulting $L = V_{n\wedge(p-r)}\cdots V_1 X U_1 \cdots U_{n\wedge p}$ has $n\wedge p$ nonzero columns and $(n+r)\wedge p$ nonzero rows, which can be described as follows:

$$L_{ij} = \begin{cases} \frac{1}{\sqrt{\beta}}\widetilde{\chi}_{(n-i+1)\beta}, & i=j, \\ g_{ij}, & j<i<j+r, \\ \frac{1}{\sqrt{\beta}}\chi_{(p-i+1)\beta}, & i=j+r, \\ 0, & i<j \text{ or } i>j+r, \end{cases} \tag{2.5}$$

where the entries are independent, $\widetilde{\chi}_k, \chi_k \sim \mathrm{Chi}(k)$, $g_{ij}\sim \mathbb{F}N(0,1)$. Truncating the remaining zero rows or columns, the matrix $S = L^\dagger L$ is $(n\wedge p)\times(n\wedge p)$ and has the same *nonzero* eigenvalues as $X^\dagger X$. It has the band form

$$S_{ij} = \begin{cases} \frac{1}{\beta}\widetilde{\chi}^2_{(n-i+1)\beta} + \sum_{i<k<i+r} |g_{k,i}|^2 + \frac{1}{\beta}\chi^2_{(p-i-r+1)\beta}, & \\ \qquad i=j, & \\ \frac{1}{\sqrt{\beta}}\widetilde{\chi}_{(n-i+1)\beta}g_{ij} + \sum_{i<k<j+r} g^*_{k,i}g_{k,j} + \frac{1}{\sqrt{\beta}}g^*_{j+r,i}\chi_{(p-j-r+1)\beta}, & \\ \qquad j<i<j+r, & \\ \frac{1}{\beta}\widetilde{\chi}_{(n-i+1)\beta}\chi_{(p-i+1)\beta}, & i=j+r, \\ 0, & i>j+r, \\ S^*_{ji}, & i<j, \end{cases} \tag{2.6}$$

where we have ignored the issue of truncation in the final r rows and columns ($g's$ and $\chi's$ with indices beyond the allowed range should simply be zero). The change of basis is thus $U_1\cdots U_{n\wedge p}$; a little thought shows that, as claimed earlier, the new basis begins with the orthogonalization of the first r rows of X. Since the form (2.6) satisfies the uniqueness condition of Theorem 2.1 a.s., the basis is indeed the one given by the theorem.

Now we consider the spiked Wishart matrix $X^\dagger X = X_0^\dagger \Sigma X_0$, with $\Sigma = \tilde{\Sigma}\oplus I_{p-r} > 0$. Here X_0 is a null Wishart matrix and $X = \Sigma^{1/2}X_0$. Notice that $X^\dagger X - X_0^\dagger X_0 = X_0^\dagger((\tilde{\Sigma}-I_r)\oplus 0)X_0$ is indeed an additive perturbation in the subspace spanned by the first r rows of X_0. Since $\Sigma^{1/2} = \tilde{\Sigma}^{1/2}\oplus I$ commutes with the inner transformation $V\in\{I_r\}\oplus U_{p-r}(\mathbb{F})$, we have

$$L^\dagger L = U^\dagger X^\dagger X U = U^\dagger X_0^\dagger \Sigma X_0 U = U^\dagger X_0^\dagger V^\dagger \Sigma V X_0 U = L_0^\dagger \Sigma L_0,$$

where $L = VXU$ and $L_0 = VX_0U$. The point is that same change of basis works in the rank r spiked case, and by the lower band structure of L_0, the perturbation shows up in the upper-left $r\times r$ corner:

$$S - S_0 = L^\dagger L - L_0^\dagger L_0 = \tilde{L}_0^\dagger(\tilde{\Sigma}-I_r)\tilde{L}_0\oplus 0. \tag{2.7}$$

Viewed in terms of the algorithm used to produce L, the point is that the first r rows of X are never "mixed" together or with the lower rows, but only "rotated" within themselves.

3. Limits of block tridiagonal matrices. The banded forms of Section 2 may also be considered as block tridiagonal matrices with $r \times r$ blocks. In this section, we give general conditions under which such random matrices, appropriately scaled, converge at the soft spectral edge to a random Schrödinger operator on the half-line with $r \times r$ matrix-valued potential and general self-adjoint boundary condition at the origin. In Section 4, we verify these assumptions for the two specific matrix models we consider.

Proposition 3.7 establishes that the limiting operator is a.s. bounded below with purely discrete spectrum via a variational principle. The main result is Theorem 3.9, which asserts that the low-lying states of the discrete models converge to those of the operator limit.

The scalar $r = 1$ case of Part I, based in turn on RRV, serves as a prototype. Care is required throughout to adapt the arguments to the matrix-valued setting, and we give a self-contained treatment.

3.1. *Discrete model and embedding.* Underlying the convergence is the embedding of the discrete half-line $\mathbb{Z}_+ = \{0, 1, \ldots\}$ into the continuum $\mathbb{R}_+ = [0, \infty)$ via $j \mapsto j/m_n$, where the scale factors $m_n \to \infty$ but with $m_n = o(n)$. Define an associated embedding of vector-valued function spaces by step functions:

$$\ell_n^2(\mathbb{Z}_+, \mathbb{F}^r) \hookrightarrow L^2(\mathbb{R}_+, \mathbb{F}^r), \qquad (v_0, v_1, \ldots) \mapsto v(x) = v_{\lfloor m_n x \rfloor},$$

which is isometric with ℓ_n^2 norm $\|v\|^2 = m_n^{-1} \sum_{j=0}^{\infty} |v_j|^2$. (Recall that $\mathbb{F}^r$ and L^2 have norms $|v|^2 = v^\dagger v$ and $\|f\|^2 = \int_0^\infty |f|^2$, respectively.) Fix a standard basis for ℓ_n^2 with lexicographic ordering

$$(e_1, 0, \ldots), (e_2, 0, \ldots), \ldots, (e_r, 0, \ldots), (0, e_1, 0, \ldots), \ldots,$$

where $e_1, \ldots, e_r$ is the standard basis for $\mathbb{F}^r$. Identify $\mathbb{F}^n$ with the n-dimensional initial coordinate subspace of ℓ_n^2, consisting of $\mathbb{F}^r$-valued step-functions supported on the interval $[0, \lceil n/r \rceil / m_n)$ and with the final step value in the subspace spanned by $e_1, \ldots, e_{r - (\lceil n/r \rceil r - n)}$. Our $n \times n$ matrices will act on $\mathbb{F}^n$ with respect to the above basis; we will generally assume the embedding $\mathbb{F}^n \subset \ell_n^2 \hookrightarrow L^2$ implicitly.

We define some operators on L^2, all of which leave ℓ_n^2 invariant and may also be considered as infinite block matrices with $r \times r$ blocks. The translation operator $(T_n f)(x) = f(x + m_n^{-1})$ extends the left shift on ℓ_n^2. Its adjoint $T_n^\dagger$ is the right shift, where $T_n^\dagger f = 0$ on $[0, m_n^{-1})$. The difference quotient $D_n = m_n(T_n - 1)$ extends a discrete derivative. Write $\operatorname{diag}(A_0, A_1, \ldots)$ for both an $r\times r$ block diagonal matrix and its extension to a pointwise matrix multiplication on L^2. Thus $E_n = \operatorname{diag}(m_n I_r, 0, 0, \ldots)$ is scalar multiplication by $m_n \mathbf{1}_{[0,m_n^{-1})}$, a "discretized delta function at the origin". Orthogonal projection from ℓ_n^2 onto $\mathbb{F}^n$ extends to a multiplication $R_n = \operatorname{diag}(I_r, \ldots, I_r, \operatorname{diag}(1,\ldots,1,0,\ldots,0), 0, \ldots)$, in which there are $\lceil n/r\rceil$ nonzero blocks and a total of n 1's.

Let $(Y_{n,i;j})_{j\in\mathbb{Z}_+}$, $i=1,2$ be two discrete-time $r\times r$ matrix-valued random processes with $Y_{n,1;j}\in M_r(\mathbb{F})$ for all j. The processes may be embedded into continuous time as above, by setting $Y_{n,i}(x) = Y_{n,i;\lfloor m_n x\rfloor}$. Note also that T_n and $\triangle_n = m_n(1-T_n^\dagger) = -D_n^\dagger$ may be sensibly applied to such matrix-valued functions. The processes $Y_{n,i}$ are on- and off-diagonal integrated potentials, and we define a "potential operator" by

$$V_n = \operatorname{diag}(\triangle_n Y_{n,1}) + \tfrac{1}{2}\big(\operatorname{diag}(\triangle_n Y_{n,2})T_n + T_n^\dagger \operatorname{diag}(\triangle_n Y_{n,2}^\dagger)\big). \tag{3.1}$$

Fix $W_n \in M_r(\mathbb{F})$, a nonrandom "boundary term".

Finally, consider

$$H_n = R_n(D_n^\dagger D_n + V_n + W_n E_n)R_n. \tag{3.2}$$

This operator leaves the initial coordinate subspace $\mathbb{F}^n$ invariant; we shall also use H_n to denote the *matrix of its restriction to* $\mathbb{F}^n$. The matrix $H_n\in M_n(\mathbb{F})$ is self-adjoint and block tridiagonal up to a truncation in the lower-right corner. Its main- and super-diagonal processes are

$$\begin{aligned} &m_n^2 + m_n(W_n + Y_{n,1;0}), 2m_n^2 + m_n(Y_{n,1;1} - Y_{n,1;0}),\\ &\qquad 2m_n^2 + m_n(Y_{n,1;2} - Y_{n,1;1}), \ldots\\ &\qquad -m_n^2 + \tfrac{1}{2}m_n Y_{n,2;0}, -m_n^2 + \tfrac{1}{2}m_n(Y_{n,2;1} - Y_{n,2;0}), \ldots, \end{aligned} \tag{3.3}$$

respectively; the sub-diagonal process is of course the conjugate transpose of the super-diagonal process. (We could have absorbed W_n into $Y_{n,1}$ as an additive constant, but keep it separate for reasons that will soon be clear. Note also that the upper-left block has m_n^2 rather than $2m_n^2$.) We refer to H_n as a *rank r block tri-diagonal ensemble.*

As in RRV and Part I, convergence rests on a few key assumptions on the potential and boundary terms just introduced. By choice, no additional scaling will be required. The role of the convergence in the first and third assumption below will be clear as soon as we define the continuum limit. The growth and oscillation bounds of the second assumption (and the lower bound implied by the third) ensure tightness of the low-lying states; in particular, they guarantee that the spectrum remains discrete and bounded below in the limit.

ASSUMPTION 1 (Tightness and convergence). There exists a continuous $M_r(\mathbb{F})$-valued random process $\{Y(x)\}_{x\geq 0}$ with $Y(0)=0$ such that

$$\begin{aligned} &\{Y_{n,i}(x)\}_{x\geq 0}, \qquad i=1,2 \text{ are tight in law,} \\ &Y_{n,1}+\tfrac{1}{2}(Y_{n,2}+Y_{n,2}^{\dagger}) \Rightarrow Y \qquad \text{in law} \end{aligned} \tag{3.4}$$

with respect to the compact-uniform topology (defined using any matrix norm).

ASSUMPTION 2 (Growth and oscillation bounds). There is a decomposition

$$Y_{n,i;j}=m_n^{-1}\sum_{k=0}^{j}\eta_{n,i;k}+\omega_{n,i;j} \tag{3.5}$$

(so $\triangle_n Y_{n,i}=\eta_{n,i}+\triangle_n\omega_{n,i}$) with $\eta_{n,i;j}\geq 0$ (as matrices), such that for some deterministic scalar continuous nondecreasing unbounded functions $\overline{\eta}(x)>0$, $\zeta(x)\geq 1$ not depending on n, and random constants $\kappa_n\geq 1$ defined on the same probability spaces, the following hold: the κ_n are tight in distribution, and for each n we have almost surely

$$\overline{\eta}(x)/\kappa_n-\kappa_n\leq \eta_{n,1}(x)+\eta_{n,2}(x)\leq \kappa_n(1+\overline{\eta}(x)), \tag{3.6}$$

$$\eta_{n,2}(x)\leq 2m_n^2, \tag{3.7}$$

$$|\omega_{n,1}(\xi)-\omega_{n,1}(x)|^2+|\omega_{n,2}(\xi)-\omega_{n,2}(x)|^2\leq \kappa_n(1+\overline{\eta}(x)/\zeta(x)) \tag{3.8}$$

for all $x,\xi\in[0,\lceil n/r\rceil/m_n)$ with $|\xi-x|\leq 1$. Here, matrix inequalities have their usual meaning and single bars denote the spectral [or $\ell^2(\mathbb{F}^r)$ operator] norm.

ASSUMPTION 3 (Critical or subcritical perturbation). For some orthonormal basis $u_1,\dots,u_r$ of $\mathbb{F}^r$ and $-\infty<w_1\leq\cdots\leq w_r\leq\infty$ we have $W_n=\sum_{i=1}^r w_{n,i}u_iu_i^{\dagger}$, where $w_{n,i}\in\mathbb{R}$ satisfy $\lim_{n\to\infty}w_{n,i}=w_i$ for each i.

We write $r_0=\#\{i:w_i<\infty\}\in\{0,\dots,r\}$ for the "critical rank". Formally, $W_n\to W=\sum_{i=1}^r w_iu_iu_i^{\dagger}\in M_r^*(\mathbb{F})$. It is natural to view W as a parameter: that is, we will consider the joint behaviour of the model (for given $Y_{n,i},Y$) over all W_n,W satisfying Assumption 3.

3.2. *Reduction to deterministic setting.* In the next subsection, we will define a limiting object in terms of $Y(x)$ and W; we want to prove that the discrete models converge to this continuum limit in law. We reduce the problem to a deterministic convergence statement as follows. First, select any subsequence. It will be convenient to extract a further subsequence so that certain additional tight sequences converge jointly in law; Skorokhod's representation theorem [see Ethier and Kurtz (1986)] says this convergence can be realized almost surely on a single probability space. We may then proceed pathwise.

In detail, consider (3.4)–(3.8). Note in particular that nonnegativity of the $\eta_{n,i}$ and the upper bound of (3.6) give that for $i=1,2$ the piecewise linear process $\{\int_0^x \eta_{n,i}\}_{x\geq 0}$ is tight in distribution, pointwise with respect to the spectral norm and in fact compact-uniformly. Given a subsequence, we pass to a further subsequence so that the following distributional limits exist jointly:

$$\begin{aligned} Y_{n,i} &\Rightarrow Y_i, \\ \int_0 \eta_{n,i} &\Rightarrow \widetilde{\eta}_i, \\ \kappa_n &\Rightarrow \kappa, \end{aligned} \tag{3.9}$$

for $i=1,2$, where convergence in the first two lines is in the compact-uniform topology. We realize (3.9) pathwise a.s. on some probability space and continue in this deterministic setting.

We can take (3.6)–(3.8) to hold with κ_n replaced with a single κ. Observe that (3.6) gives a local Lipschitz bound on the $\int \eta_{n,i}$, which is inherited by their limits $\widetilde{\eta}_i$ (the spectral norm controls the matrix entries). Thus, $\eta_i = \widetilde{\eta}_i{}'$ is defined almost everywhere on $\mathbb{R}_+$, satisfies (3.6), and may be defined to satisfy this inequality everywhere. Furthermore, one easily checks that $m_n^{-1}\sum \eta_{n,i} \to \int \eta_i$ compact-uniformly as well (use continuity of the limit). Therefore, $\omega_{n,i} = y_{n,i} - m_n^{-1}\sum \eta_{n,i}$ must have a continuous limit ω_i for $i=1,2$; moreover, the bound (3.8) is inherited by the limits. Lastly, put $\eta = \eta_1 + \eta_2$, $\omega = \omega_1 + \frac{1}{2}(\omega_2 + \omega_2^\dagger)$ and note that $Y_i = \int \eta_i + \omega_i$ and $Y = \int \eta + \omega$. For convenience, we record the bounds inherited by η, ω:

$$\overline{\eta}(x)/\kappa - \kappa \leq \eta(x) \leq \kappa(1 + \overline{\eta}(x)), \tag{3.10}$$

$$|\omega(\xi) - \omega(x)|^2 \leq \kappa(1 + \overline{\eta}(x)/\zeta(x)) \tag{3.11}$$

for $x, \xi \in \mathbb{R}_+$ with $|\xi - x| \leq 1$ (and note that $\kappa \geq 1$).

We will assume this *subsequential pathwise coupling* for the remainder of the section.

3.3. *Limiting object and variational characterization.* Formally, the limiting object is the eigenvalue problem

$$\begin{aligned} \mathcal{H}f &= \Lambda f \qquad \text{on } L^2(\mathbb{R}_+, \mathbb{F}^r), \\ f'(0) &= Wf(0), \end{aligned} \tag{3.12}$$

where

$$\mathcal{H} = -\frac{d^2}{dx^2} + Y'(x).$$

Writing the spectral decomposition $W = \sum_{i=1}^{r} w_i u_i u_i^\dagger$, recall (Assumption 3) that we actually allow $w_i \in \mathbb{R}$ for $1 \le i \le r_0$ and, symbolically, $w_i = +\infty$ for $r_0 + 1 \le i \le r$. Writing $f_i = u_i^\dagger f$, the boundary condition is then to be interpreted as

$$\begin{aligned} &f_i'(0) = w_i f_i(0) \qquad \text{for } i \le r_0, \\ &f_i(0) = 0 \qquad \text{for } i > r_0. \end{aligned} \tag{3.13}$$

We thus have a completely general homogeneous linear self-adjoint boundary condition. We refer to $\operatorname{span}\{u_i : i > r_0\}$ as the *Dirichlet subspace* and the corresponding f_i as *Dirichlet components*; they will require special treatment in what follows.

We will actually work with a symmetric bilinear form (properly, sesquilinear if $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$) associated with the eigenvalue problem (3.12). Define a space of test functions C_0^∞ consisting of smooth $\mathbb{F}^r$-valued functions φ on $\mathbb{R}_+$ with compact support; we additionally require the Dirichlet components to be supported away from the origin. Introduce a symmetric bilinear form on $C_0^\infty \times C_0^\infty$ by

$$\mathcal{H}(\varphi, \psi) = \langle \varphi', \psi' \rangle - \langle \varphi', Y\psi \rangle - \langle \varphi, Y\psi' \rangle + \varphi(0)^\dagger W \psi(0), \tag{3.14}$$

where the Dirichlet part of the last term is interpreted as zero. Formally, the form $\mathcal{H}(\cdot,\cdot)$ is just the usual one $\langle \cdot, \mathcal{H} \cdot \rangle$ associated with the operator $\mathcal{H}$; the potential term has been integrated by parts and the boundary condition "built in". See also Remark 3.5 below.

The regularity and decay conditions naturally associated with this form are given by the following weighted Sobolev norm:

$$\|f\|_*^2 = \int_0^\infty (|f'|^2 + (1 + \overline{\eta})|f|^2) + f(0)^\dagger W^+ f(0), \tag{3.15}$$

where the *positive part of* W is defined as $W^+ = \sum_{i=1}^{r} w_i^+ u_i u_i^\dagger$ with $w^+ = w \vee 0$. [Define the *negative part* similarly with $w_i^- = -(w \wedge 0)$, so that $W = W^+ - W^-$.] We refer to $\|\cdot\|_*$ as the L^* *norm* and define an associated Hilbert space L^* as the closure of C_0^∞ under this norm. (The formal Dirichlet terms are again interpreted to be zero, but they can also be thought of as imposing the Dirichlet condition.) We record some basic facts about L^*.

FACT 3.1. *Any $f \in L^*$ is uniformly Hölder$(1/2)$-continuous and satisfies $|f(x)|^2 \le 2\|f'\|\|f\| \le \|f\|_*^2$ for all x; furthermore, $f_i(0) = 0$ for $i > r_0$.*

Proof. We have $|f(y)-f(x)| = |\int_x^y f'| \le \|f'\| |y-x|^{1/2}$. For $f \in C_0^\infty$ we have $|f(x)|^2 = -\int_x^\infty 2\operatorname{Re} f^\dagger f' \le 2\|f\|\|f'\| \le \|f\|_*^2$; an L^*-bounded sequence in C_0^∞, therefore, has a compact-uniformly convergent subsequence, so we can extend this bound to $f \in L^*$ and also conclude the behaviour in the Dirichlet components. □

Fact 3.2. *Every L^*-bounded sequence has a subsequence converging in the following modes:* (i) *weakly in L^*,* (ii) *derivatives weakly in L^2,* (iii) *uniformly on compacts and* (iv) *in L^2.*

Proof. (i) and (ii) are just Banach–Alaoglu; (iii) is the previous fact and Arzelà–Ascoli again; (iii) implies L^2 convergence locally, while the uniform bound on $\int \overline{\eta}|f_n|^2$ produces the uniform integrability required for (iv). Note that the weak limit in (ii) really is the derivative of the limit function, as one can see by integrating against functions $\mathbf{1}_{[0,x]}$ and using pointwise convergence. □

By the bound in Fact 3.1 with $x=0$, the boundary term in (3.15) could be done away with. It is natural to include the term, however, when considering all W simultaneously and viewing the Dirichlet case as a limiting case. More importantly, it clarifies the role of the boundary terms in the following key bound.

Lemma 3.3. *For every $0<c<1/\kappa$ there is a $C>0$ such that, for each $b>0$, the following holds for all $W \ge -b$ and all $f \in C_0^\infty$:*

$$c\|f\|_*^2 - (1+b^2)C\|f\|^2 \le \mathcal{H}(f,f) \le C\|f\|_*^2. \tag{3.16}$$

In particular, $\mathcal{H}(\cdot,\cdot)$ extends uniquely to a continuous symmetric bilinear form on $L^* \times L^*$.

Proof. For the first three terms of (3.14), we use the decomposition $Y = \int \eta + \omega$ from the previous subsection. Integrating the $\int \eta$ term by parts, (3.10) easily yields

$$\frac{1}{\kappa}\|f\|_*^2 - \kappa\|f\|^2 \le \|f'\|^2 + \langle f, \eta f\rangle \le \kappa\|f\|_*^2.$$

Break up the ω term as follows. The moving average $\overline{\omega}_x = \int_x^{x+1}\omega$ is differentiable with $\overline{\omega}'_x = \omega_{x+1} - \omega_x$; writing $\omega = \overline{\omega} + (\omega - \overline{\omega})$, we have

$$-2\operatorname{Re}\langle f', \omega f\rangle = \langle f, \overline{\omega}' f\rangle + 2\operatorname{Re}\langle f', (\overline{\omega}-\omega) f\rangle.$$

By (3.11), $\max(|\omega_\xi - \omega_x|, |\omega_\xi - \omega_x|^2) \le C_\varepsilon + \varepsilon\overline{\eta}(x)$ for $|\xi - x| \le 1$, where ε can be made small. In particular, the first term above is bounded absolutely by $\varepsilon\|f\|_*^2 + C_\varepsilon\|f\|^2$. Averaging, we also get $|\overline{\omega}_x - \omega_x| \le (C_\varepsilon + \varepsilon\overline{\eta}(x))^{1/2}$; Cauchy–Schwarz then bounds the second term absolutely by $\sqrt{\varepsilon}\int_0^\infty |f'|^2 +$

$\frac{1}{\sqrt{\varepsilon}}\int_0^\infty (C_\varepsilon + \varepsilon\overline{\eta})|f|^2$ and thus by $\sqrt{\varepsilon}\|f\|_*^2 + C'_\varepsilon\|f\|^2$. Now combine all the terms and set ε small to obtain a version of (3.16) with the boundary terms omitted (from both the form and the norm).

We break the boundary term in (3.14) into its positive and negative parts. For the negative part, Fact 3.2 gives $|f(0)|^2 \le (\varepsilon/b)\|f'\|^2 + (b/\varepsilon)\|f\|^2$; $W^- \le b$ then implies that

$$0 \le f(0)^\dagger W^- f(0) \le \varepsilon\|f\|_*^2 + C''_\varepsilon b^2\|f\|^2,$$

which may be subtracted from the inequality already obtained. For the positive part $f(0)^\dagger W^+ f(0)$, use the fact that $c \le 1 \le C$ to simply add it in. We thus arrive at (3.16).

For the L^* bilinear form bound, begin with the quadratic form bound $|\mathcal{H}(f,f)| \le C_{c,b}\|f\|_*^2$; it is a standard Hilbert space fact that it may be polarized to a bilinear form bound [see, e.g., Section 18 of Halmos (1951)]. $\square$

DEFINITION 3.4. We say $f \in L^*$ is an *eigenfunction* with *eigenvalue* Λ if $f \neq 0$ and for all $\varphi \in C_0^\infty$ we have

$$\mathcal{H}(\varphi, f) = \Lambda\langle\varphi, f\rangle. \tag{3.17}$$

Note that (3.17) then automatically holds for all $\varphi \in L^*$, by L^*-continuity of both sides.

REMARK 3.5. This definition represents a weak or distributional version of the problem (3.12). As further justification, integrate by parts to write the definition

$$\langle\varphi', f'\rangle - \langle\varphi', Yf\rangle - \langle\varphi, Yf'\rangle + \varphi(0)^\dagger W f(0) = \Lambda\langle\varphi, f\rangle$$

in the form

$$\langle\varphi', f'\rangle - \langle\varphi', Yf\rangle + \left\langle\varphi', \int_0 Yf'\right\rangle - \langle\varphi', Wf(0)\rangle = -\Lambda\left\langle\varphi', \int_0 f\right\rangle,$$

which is equivalent to

$$f'(x) = Wf(0) + Y(x)f(x) - \int_0^x Yf' - \Lambda\int_0^x f \qquad \text{a.e. } x. \tag{3.18}$$

(For a Dirichlet component f_i the restriction on test functions implies that $\langle\varphi'_i, 1\rangle = 0$, so the first boundary term on the right-hand side is replaced with an arbitrary constant.) Now (3.18) shows that f' has a continuous version, and the equation may be taken to hold everywhere. In particular, f satisfies the boundary condition of (3.12) classically. [For a Dirichlet component, we just find that the arbitrary constant is $f'_i(0)$.] One can also view

(3.18) as a straightforward integrated version of the eigenvalue equation in which the potential term has been interpreted via integration by parts. This equation will be useful in Lemma 3.6 below and is the starting point for the development in Section 5.

We now characterize the eigenvalues and eigenfunctions variationally. As usual, it follows from the symmetry of the form that eigenvalues are real (and eigenfunctions with distinct eigenvalues are L^2-orthogonal). The L^2 part of the lower bound in (3.16) says the spectrum is bounded below. The rest of (3.16) implies that there are only finitely many eigenvalues below any given level: a sequence of normalized eigenfunctions with bounded eigenvalues must have an L^2-convergent subsequence by Fact 3.2. At a given level, more is true.

LEMMA 3.6. *For each $\Lambda \in \mathbb{R}$, the corresponding eigenspace is at most r-dimensional.*

PROOF. By linearity, it suffices to show a solution of (3.18) with $f'(0) = f(0) = 0$ must vanish identically. Integrate by parts to write

$$f'(x) = Y(x)\int_0^x f' - \int_0^x Y f' - \Lambda x \int_0^x f' + \Lambda \int_0^x t f'(t)\,dt,$$

which implies that $|f'(x)| \le C(x)\int_0^x |f'|$ with some $C(x) < \infty$ increasing in x. Gronwall's lemma then gives $|f'(x)| = 0$ for all $x \ge 0$. $\square$

PROPOSITION 3.7. *There is a well-defined $(k+1)$st lowest eigenvalue Λ_k, counting with multiplicity. The eigenvalues $\Lambda_0 \le \Lambda_1 \le \dots$ together with an orthonormal sequence of corresponding eigenvectors $f_0, f_1, \dots$ are given recursively by the variational problem*

$$\Lambda_k = \inf_{\substack{f \in L^*, \|f\|=1, \\ f \perp f_0, \dots, f_{k-1}}} \mathcal{H}(f, f)$$

in which the minimum is attained and we set f_k to be any minimizer.

REMARK 3.8. Since we must have $\Lambda_k \to \infty$, $\{\Lambda_0, \Lambda_1, \dots\}$ exhausts the spectrum and the resolvent operator is compact. We do not make this statement precise.

PROOF. First taking $k = 0$, the infimum $\tilde{\Lambda}$ is finite by (3.16). Let f_n be a minimizing sequence; it is L^*-bounded, again by (3.16). Pass to a subsequence converging to $f \in L^*$ in all the modes of Fact 3.2. In particular,

$1 = \|f_n\| \to \|f\|$, so $\mathcal{H}(f,f) \geq \tilde{\Lambda}$ by definition. But also

$$\begin{aligned}\mathcal{H}(f,f) &= \|f'\|^2 + \langle f, \eta f\rangle + \langle f, \overline{\omega}' f\rangle + 2\,\mathrm{Re}\langle f', (\overline{\omega} - \omega) f\rangle + f(0)^\dagger W f(0)\\ &\leq \liminf_{n\to\infty} \mathcal{H}(f_n, f_n)\end{aligned}$$

by a term-by-term comparison. Indeed, the inequality holds for the first term by weak convergence, and for the second term by pointwise convergence and Fatou's lemma; the remaining terms are just equal to the corresponding limits, because the second members of the inner products converge in L^2 by the bounds from the proof of Lemma 3.3 together with L^*-boundedness and L^2-convergence. Therefore, $\mathcal{H}(f,f) = \tilde{\Lambda}$.

A standard argument now shows $(\tilde{\Lambda}, f)$ is an eigenvalue–eigenfunction pair: taking $\varphi \in C_0^\infty$ and ε small, put $f^\varepsilon = (f + \varepsilon\varphi)/\|f + \varepsilon\varphi\|$; since f is a minimizer, $\frac{d}{d\varepsilon}|_{\varepsilon=0}\mathcal{H}(f^\varepsilon, f^\varepsilon)$ must vanish; the latter says precisely (3.17) with $\tilde{\Lambda}$. Finally, suppose (Λ, g) is any eigenvalue–eigenfunction pair; then $\mathcal{H}(g,g) = \Lambda$, and hence $\tilde{\Lambda} \leq \Lambda$. We are thus justified in setting $\Lambda_0 = \tilde{\Lambda}$ and $f_0 = f$.

Proceed inductively, minimizing now over the orthocomplement $\{f \in L^* : \|f\| = 1, f \perp f_0, \dots, f_{k-1}\}$. Again, L^2-convergence of a minimizing sequence guarantees that the limit remains admissible; as before, the limit is in fact a minimizer; conclude by applying the arguments of the previous paragraph with φ, g also restricted to the orthocomplement. □

3.4. *Statement.* We are finally ready to state the main result of this section. Recall that we consider eigenvectors of a matrix $H_n \in M_n(\mathbb{F})$ in the embedding $\mathbb{F}^n \subset \ell_n^2(\mathbb{Z}_+, \mathbb{F}^r) \hookrightarrow L^2(\mathbb{R}_+, \mathbb{F}^r)$ above.

THEOREM 3.9. *Let H_n be a rank r block tr-diagonal ensemble as in (3.2) satisfying Assumptions 1–3, and let $\lambda_{n,k}$ be its $(k+1)$st lowest eigenvalue. Define the associated form $\mathcal{H}$ as in (3.14) and let Λ_k be its a.s. defined $(k+1)$st lowest eigenvalue. In the deterministic setting of subsequential path-wise coupling, $\lambda_{n,k} \to \Lambda_k$ for each $k = 0, 1, \dots$. Furthermore, a sequence of normalized eigenvectors corresponding to $\lambda_{n,k}$ is precompact in L^2 norm, and every subsequential limit is an eigenfunction corresponding to Λ_k. Finally, convergence holds uniformly over possible $W_n, W \geq -b > -\infty$. One recovers the corresponding distributional tightness and convergence statements for the full sequence, jointly for $k = 0, 1, \dots$ in the sense of finite-dimensional distributions and jointly over W_n, W.*

REMARK 3.10. The eigenvector convergence statement requires subsequences for two reasons: possible multiplicity of the limiting eigenvalues, and the sign or phase ambiguity of the eigenvectors. It is possible to formulate the conclusion of the theorem very simply using spectral projections. [If H has purely discrete spectrum, the spectral projection $\mathbf{1}_A(H)$ is simply orthogonal projection of L^2 onto the span of those eigenvectors of H whose eigenvalues lie in $A \subset \mathbb{R}$.] The joint eigenvalue-eigenvector convergence may be restated in the deterministic setting as follows: *For all* $a \in \mathbb{R} \setminus \{\Lambda_0, \Lambda_1, \ldots\}$, *the spectral projections* $\mathbf{1}_{(-\infty,a)}(H_n) \to \mathbf{1}_{(-\infty,a)}(\mathcal{H})$ *in* L^2 *operator norm.* The corresponding distributional statement holds jointly over all a that are a.s. off the limiting spectrum (or simply all a if the distributions of the Λ_k are nonatomic).

REMARK 3.11. An operator-theoretic formulation of the theorem (which we do not develop here) would state a norm resolvent convergence: the resolvent matrices, precomposed with the finite-rank projections $L^2 \to \mathbb{F}^n$ associated with the embedding, converge to the continuum resolvent in L^2 operator norm. This mode of convergence is the strongest one can hope for in the unbounded setting [see, e.g., Section VIII.7 of Reed and Simon (1980), Weidmann (1997)].

The proof will be given over the course of the next two subsections.

3.5. *Tightness.* We will need a discrete analogue of the L^* norm and a counterpart of Lemma 3.3 with constants uniform in n. For $v \in \mathbb{F}^n \hookrightarrow L^2(\mathbb{R}_+, \mathbb{F}^r)$ as above, define the L_n^* *norm* by

$$\begin{aligned} \|v\|_{*n}^2 &= \langle v, (D_n^\dagger D_n + 1 + \overline{\eta} + E_n W_n^+) v \rangle \\ &= \int_0^\infty (|D_n v|^2 + (1+\overline{\eta})|v|^2) + v(0)^\dagger W_n^+ v(0) \end{aligned} \tag{3.19}$$

with the nonnegative part W_n^+ defined as before.

REMARK 3.12. When considering just a single W_n, W, the boundary term in (3.19) is really only required when the limit includes Dirichlet terms; it is simpler, however, not to distinguish the two cases here. More importantly, including this term clarifies the role of the boundary term in the following key bound. Note that the original case considered in RRV has $W_n = m_n$ in our notation. (The H_n form and L_n^* norm there contained a term $m_n|v_0|^2$, though it is hidden in the fact that, in our notation, they use $\triangle_n$ in place of D_n.)

LEMMA 3.13. *For every* $0 < c < 1/4\kappa$ *there is a* $C > 0$ *such that, for each* $b > 0$*, the following holds for all* n*,* $W_n \ge -b$ *and* $v \in \mathbb{F}^n$*:*

$$c\|v\|_{*n}^2 - (1+b^2)C\|v\|^2 \le \langle v, H_n v\rangle \le C\|v\|_{*n}^2. \tag{3.20}$$

PROOF. We drop the subscript n. The form associated with (3.2) is

$$\langle v, Hv\rangle = \|Dv\|^2 + \langle v, Vv\rangle + v(0)^\dagger W v(0). \tag{3.21}$$

The potential term $\langle v, Vv\rangle = \int_0^\infty v^\dagger V v$, defined in (3.1), is analyzed according to (3.5):

$$\begin{aligned} v^\dagger V v &= v^\dagger(\triangle Y_1)v + \operatorname{Re} v^\dagger(\triangle Y_2)Tv \\ &= (v^\dagger \eta_1 v + \operatorname{Re} v^\dagger \eta_2 T v) + (v^\dagger(\triangle\omega_1)v + \operatorname{Re} v^\dagger(\triangle\omega_2)Tv). \end{aligned}$$

Together with $|D_n v|^2$, the η-terms provide the structure of the bound as we now show. Afterward we will control the ω-terms and lastly deal with the boundary term.

Recall (3.6) and that $\eta_i \ge 0$. For an upper bound, rearrange $(v-Tv)^\dagger\eta_2(v-Tv) \ge 0$ to

$$\begin{aligned} \operatorname{Re} v^\dagger \eta_2 T v &\le \tfrac{1}{2} v^\dagger \eta_2 v + \tfrac{1}{2}(Tv)^\dagger \eta_2 T v \\ &\le \tfrac{1}{2}\kappa(\overline{\eta}+1)(|v|^2 + |Tv|^2). \end{aligned}$$

Now $\int \overline{\eta}|Tv|^2 = \int (T^\dagger \overline{\eta})|v|^2 \le \int \overline{\eta}|v|^2$ since $\overline{\eta}$ is nondecreasing, and we obtain

$$\|Dv\|^2 + \langle v, \eta_1 v\rangle + \operatorname{Re}\langle v, \eta_2 T v\rangle \le 2\kappa\|v\|_*^2. \tag{3.22}$$

Toward a lower bound, we use the slightly tricky rearrangement $0 \le (\frac{1}{2}v + Tv)^\dagger \eta_2(\frac{1}{2}v + Tv) = 3\operatorname{Re} v^\dagger \eta_2 T v + (Tv - v)^\dagger \eta_2 (Tv - v) - \frac{3}{4} v^\dagger \eta_2 v$. With (3.7), we get

$$\begin{aligned} \operatorname{Re} v^\dagger \eta_2 T v &\ge -\tfrac{1}{3}(Tv - v)^\dagger \eta_2 (Tv - v) + \tfrac{1}{4} v^\dagger \eta_2 v \\ &\ge -\tfrac{2}{3}|Dv|^2 + \tfrac{1}{4} v^\dagger \eta_2 v, \end{aligned}$$

so by (3.6),

$$|Dv|^2 + v^\dagger \eta_1 v + \operatorname{Re} v^\dagger \eta_2 T v \ge \tfrac{1}{3}|Dv|^2 + \tfrac{1}{4}(\overline{\eta}/\kappa - \kappa)|v|^2$$

and thus

$$\|Dv\|^2 + \langle v, \eta_1 v\rangle + \operatorname{Re}\langle v, \eta_2 T v\rangle \ge (1/4\kappa)\|v\|_*^2 - (\kappa/4)\|v\|^2. \tag{3.23}$$

We handle the ω-terms with a discrete analogue of the decomposition used in the continuum proof. Consider the moving average

$$\overline{\omega}_i = \lfloor m \rfloor^{-1} \sum_{j=1}^{\lfloor m \rfloor} T^j \omega_i$$

which has $\triangle\overline{\omega}_i = (m/\lfloor m \rfloor)(T^{\lfloor m \rfloor} - 1)\omega_i$; it is convenient to extend $\omega_i(x) = \omega_i(\lceil n/r \rceil/m_n)$ for $x > \lceil n/r \rceil/m_n$. Decompose $\omega_i = \overline{\omega}_i + (\omega_i - \overline{\omega}_i)$. For the ω_1-term,

$$v^\dagger \triangle\omega_1 v = (m/\lfloor m \rfloor) v^\dagger (T^{\lfloor m \rfloor}\omega_1 - \omega_1) v + v^\dagger \triangle(\omega_1 - \overline{\omega}_1) v.$$

By (3.8) and Cauchy–Schwarz, the first term is bounded absolutely by $(C_\varepsilon + \varepsilon\overline{\eta})|v|^2$ and its integral by $\varepsilon\|v\|_*^2 + C_\varepsilon\|v\|^2$. The second term calls for a summation by parts:

$$\begin{aligned}\langle v, \triangle(\omega_1 - \overline{\omega}_1)v\rangle &= m_n(\langle v, (\omega_1 - \overline{\omega}_1)v\rangle - \langle Tv, (\omega_1 - \overline{\omega}_1)Tv\rangle)\\ &= m_n \operatorname{Re}\langle v - Tv, (\omega_1 - \overline{\omega}_1)(v + Tv)\rangle\\ &= \operatorname{Re}\langle Dv, (\overline{\omega}_1 - \omega_1)(v + Tv)\rangle.\end{aligned}$$

The averaged bound $|\overline{\omega}_1 - \omega_1| \le (C_\varepsilon + \varepsilon\overline{\eta})^{1/2}$ and Cauchy–Schwarz bound the integrand

$$|(Dv)^\dagger(\overline{\omega}_1 - \omega_1)(v + Tv)| \le \sqrt{\varepsilon}|Dv|^2 + (1/4\sqrt{\varepsilon})(C_\varepsilon + \varepsilon\overline{\eta})(|v|^2 + |Tv|^2),$$

and its integral by $\sqrt{\varepsilon}\|v\|_*^2 + C'_\varepsilon\|v\|^2$. One thus obtains a similar bound on $|\langle v, (\triangle\omega_1)v\rangle|$.

There are corresponding bounds for the ω_2-terms. For the $\overline{\omega}_2$-term, use $2|v||Tv| \le |v|^2 + |Tv|^2$. For the $(\omega_2 - \overline{\omega}_2)$-term, modify the summation by parts:

$$\begin{aligned}&\operatorname{Re}\langle(v, \triangle(\omega_2 - \overline{\omega}_2)Tv\rangle\\ &\quad = m_n \operatorname{Re}(\langle(v - Tv), (\omega_2 - \overline{\omega}_2)Tv\rangle + \langle Tv, (\omega_2 - \overline{\omega}_2)(Tv - T^2v)\rangle)\\ &\quad = \operatorname{Re}\langle Dv + TDv, (\overline{\omega}_2 - \omega_2)Tv\rangle.\end{aligned}$$

Incorporating all the ω-terms into (3.22), (3.23) and setting ε small, we obtain (3.20) but with the boundary terms omitted (from both the form and the norm).

We break the boundary term in (3.21) into its positive and negative parts. A discrete analogue of a bound from Fact 3.1 will be useful:

$$|v(0)|^2 = \int_0^\infty -D|v|^2 = \int_0^\infty \operatorname{Re} m(v - Tv)^\dagger(v + Tv) \le 2\|Dv\|\|v\|.$$

It gives $|v(0)|^2 \le (\varepsilon/b)\|Dv\|^2 + (b/\varepsilon)\|v\|^2$, and then $W^- \le b$ implies that

$$0 \le v(0)^\dagger W^- v(0) \le \varepsilon\|v\|_*^2 + C''_\varepsilon b^2\|v\|^2$$

which may be subtracted from the inequality already obtained. The positive part may simply be added in using that $c \le 1 \le C$. We thus arrive at (3.20). □

REMARK 3.14. If the W_n are not bounded below then the lower bound in (3.20) breaks down: in fact, the bottom eigenvalue of H_n really goes to $-\infty$ like minus the square of the bottom eigenvalue of W_n. This is the supercritical regime.

3.6. *Convergence.* We begin with a simple lemma, a discrete-to-continuous version of Fact 3.2.

LEMMA 3.15. *Let $f_n \in \mathbb{F}^n$ with $\|f_n\|_{*n}$ uniformly bounded. Then there exist $f \in L^*$ and a subsequence along which* (i) *$f_n \to f$ uniformly on compacts,* (ii) *$f_n \to_{L^2} f$, and* (iii) *$D_n f_n \to f'$ weakly in L^2.*

PROOF. Consider $g_n(x) = f_n(0) + \int_0^x D_n f_n$, a piecewise-linear version of f_n; they coincide at points $x = i/m_n$, $i \in \mathbb{Z}_+$. One easily checks that $\|g_n\|_*^2 \le 2\|f_n\|_{*n}^2$, so some subsequence $g_n \to f \in L^*$ in all the modes of Fact 3.2; for a Dirichlet component, the boundary term in the L_n^* norm guarantees that the limit vanishes at 0. But then also $f_n \to f$ compact-uniformly by a simple argument using the uniform continuity of f, $f_n \to_{L^2} f$ because $\|f_n - g_n\|^2 \le (1/3n^2)\|D_n f_n\|^2$, and $D_n f_n \to f'$ weakly in L^2 because $D_n f_n = g_n'$ a.e. $\square$

Next, we establish a kind of weak convergence of the forms $\langle \cdot, H_n \cdot\rangle$ to $\mathcal{H}(\cdot,\cdot)$. Let $\mathcal{P}_n$ be orthogonal projection from L^2 onto $\mathbb{F}^n$ embedded as above. The following facts will be useful and are easy to check. For $f \in L^2$, $\mathcal{P}_n f \to_{L^2} f$ (the Lebesgue differentiation theorem gives pointwise convergence and we have uniform L^2-integrability); further, if $f' \in L^2$ then $D_n f \to_{L^2} f'$ ($D_n f$ is a convolution of f' with an approximate delta); for smooth φ, $\mathcal{P}_n \varphi \to \varphi$ uniformly on compacts. It is also useful to note that $\mathcal{P}_n$ commutes with R_n and with $D_n R_n$. Finally, if $f_n \to_{L^2} f$, g_n is L^2-bounded and $g_n \to g$ weakly in L^2, then $\langle f_n, g_n\rangle \to \langle f, g\rangle$.

LEMMA 3.16. *Let $f_n \to f$ be as in the hypothesis and conclusion of Lemma 3.15. Then for all $\varphi \in C_0^\infty$ we have $\langle \varphi, H_n f_n\rangle \to \mathcal{H}(\varphi, f)$. In particular, $\mathcal{P}_n \varphi \to \varphi$ in this way and so*

$$\langle \mathcal{P}_n\varphi, H_n \mathcal{P}_n \varphi\rangle = \langle \varphi, H_n \mathcal{P}_n\varphi\rangle \to \mathcal{H}(\varphi,\varphi). \tag{3.24}$$

PROOF. Since φ is compactly supported, we have $R_n\varphi = \varphi$ for n large and the R_ns may be dropped. By assumption $D_n f_n$ is L^2 bounded and $D_n f_n \to f'$ weakly in L^2, so by the preceding observations $D_n \varphi \to_{L^2} \varphi'$ and

$$\langle \varphi, D_n^\dagger D_n f_n\rangle = \langle D_n \varphi, D_n f_n\rangle \to \langle \varphi', f'\rangle.$$

For the potential term, we must verify that

$$\langle \varphi, V_n f_n\rangle = \langle \varphi, (\triangle_n Y_{n,1} + \tfrac{1}{2}((\triangle_n Y_{n,2}) T_n + T_n^\dagger(\triangle_n Y_{n,2}^\dagger))) f_n\rangle$$

converges to $-\langle \varphi', Y f\rangle - \langle \varphi, Y f'\rangle$. Recall by Assumption 1 (3.4) and (3.9) that $Y_{n,i} \to Y_i$ compact-uniformly ($i = 1, 2$) and $Y = Y_1 + \frac{1}{2}(Y_2 + Y_2^\dagger)$. Writing

$Y_n = Y_{n,1} + \frac{1}{2}(Y_{n,2} + Y_{n,2}^\dagger) \to Y$ (and disregarding the notational collision with Y_i), we first approximate V_n by $\triangle Y_n$:

$$\begin{aligned}\langle \varphi, (\triangle_n Y_n) f_n \rangle &= m_n(\langle \varphi, Y_n f_n\rangle - \langle T_n\varphi, Y_n T_n f_n\rangle) \\ &= m_n(\langle \varphi, Y_n f_n\rangle - \langle T_n\varphi, Y_n f_n\rangle + \langle T_n\varphi, Y_n f_n\rangle - \langle T_n\varphi, Y_n T_n f_n\rangle) \\ &= -\langle D_n\varphi, Y_n f_n\rangle - \langle T_n\varphi, Y_n D_n f_n\rangle,\end{aligned}$$

which converges to the desired limit by the observations preceding the lemma together with the assumptions on f_n and the fact that $T_n\varphi \to_{L^2} \varphi$ in L^2 since $m_n\|T_n\varphi - \varphi\| = \|D_n\varphi\|$ is bounded. The error in the above approximation comes as a sum of T_n and $T_n^\dagger$ terms. Consider twice the T_n term:

$$\begin{aligned}|\langle \varphi, (\triangle_n Y_{n,2})(T_n - 1) f_n\rangle| &= |\langle \varphi, (m_n^{-1}\triangle_n Y_{n,2}) D_n f_n\rangle| \\ &\le \|\varphi\| \sup_I |Y_{n,2} - T_n^\dagger Y_{n,2}| \|D_n f_n\|,\end{aligned}$$

where I is a compact interval supporting φ. (The single bars in the supremum denote the spectral or ℓ_2-operator norm, which is of course equivalent to the max norm on the entries.) Note that $D_n f_n$ is L^2-bounded because it converges weakly in L^2. Now $Y_{n,2}$ and $T_n^\dagger Y_{n,2}$ both converge to Y_2 uniformly on I, in the latter case by the uniform continuity of Y_2 on I; it follows that the supremum, and hence the whole term, vanish in the limit. The $T_n^\dagger$ term is handled similarly, the only difference being that the D_n in the estimate lands on φ instead.

Finally, for the boundary terms Assumption 3 gives

$$(\mathcal{P}_n\varphi)_i^*(0) w_{n,i} f_{n,i}(0) \to \varphi_i^*(0) w_i f_i(0),$$

where in the Dirichlet case $i > r_0$ the left-hand side vanishes for n large because φ_i is supported away from 0.

Turning to the second statement, we must verify that $\mathcal{P}_n\varphi \to \varphi$ as in Lemma 3.15. The uniform L_n^* bound on $\mathcal{P}_n\varphi$ follows from the following observations: $\|(\mathcal{P}_n\varphi)\sqrt{1+\overline{\eta}}\| = \|\mathcal{P}_n\varphi\sqrt{1+\overline{\eta}}\| \le \|\varphi\sqrt{1+\overline{\eta}}\|$; for n large enough that $R_n\varphi = \varphi$ we have $\|D_n\mathcal{P}_n\varphi\| = \|\mathcal{P}_n D_n\varphi\| \le \|D_n\varphi\| \le \|\varphi'\|$ (Young's inequality); for the boundary term note that $(\mathcal{P}_n\varphi)_i(0)$ is bounded if $i \le r_0$ and in fact vanishes for n large if $i > r_0$. The convergence is easy: $\mathcal{P}_n\varphi \to \varphi$ compact-uniformly and in L^2, and for $g \in L^2$ we have $\langle g, D_n\mathcal{P}_n\varphi\rangle = \langle \mathcal{P}_n g, D_n\varphi\rangle \to \langle g, \varphi'\rangle$. □

We finish by recalling the argument to put all the pieces together. A technical point: unlike in previous treatments we do not assume that the eigenvalues are simple.

Proof of Theorem 3.9. We first show that for all k we have $\underline{\lambda}_k = \liminf \lambda_{n,k} \ge \Lambda_k$. Assume that $\underline{\lambda}_k < \infty$. The eigenvalues of H_n are uniformly bounded below by Lemma 3.13, so there is a subsequence along which $(\lambda_{n,1},\dots,\lambda_{n,k}) \to (\xi_1,\dots,\xi_k = \underline{\lambda}_k)$. By the same lemma, corresponding orthonormal eigenvector sequences have L_n^*-norm uniformly bounded. Pass to a further subsequence so that they all converge as in Lemma 3.15. The limit functions are orthonormal; by Lemma 3.16 they are eigenfunctions with eigenvalues $\xi_j \le \underline{\lambda}_k$ and we are done.

We proceed by induction, assuming the conclusion of the theorem up to $k-1$. For $j = 0,\dots,k-1$ let $v_{n,j}$ be orthonormal eigenvectors corresponding to $\lambda_{n,j}$; for any subsequence we can pass to a further subsequence such that $v_{n,j} \to_{L^2} f_j$, eigenfunctions corresponding to Λ_j. Take an orthogonal eigenfunction f_k corresponding to Λ_k and find $f_k^\varepsilon \in C_0^\infty$ with $\|f_k^\varepsilon - f_k\|_* < \varepsilon$. Consider the vector

$$f_{n,k} = \mathcal{P}_n f_k^\varepsilon - \sum_{j=0}^{k-1} \langle v_{n,j}, \mathcal{P}_n f_k^\varepsilon\rangle v_{n,j}.$$

The L_n^*-norm of the sum term is uniformly bounded by $C\varepsilon$: indeed, the $\|v_{n,j}\|_{*n}$ are uniformly bounded by Lemma 3.13, while the coefficients satisfy $|\langle v_{n,j}, f_k^\varepsilon\rangle| \le \|f_k^\varepsilon - f_k\| + \|v_{n,j} - f_j\| < 2\varepsilon$ for large n. By the variational characterization in finite dimensions and the uniform L_n^* form bound on $\langle\cdot, H_n\cdot\rangle$ (by Lemma 3.13) together with the uniform bound on $\|\mathcal{P}_n f_k^\varepsilon\|_{*n}$ (by Lemma 3.16), we then have

$$\begin{aligned}\limsup \lambda_{n,k} &\le \limsup \frac{\langle f_{n,k}, H_n f_{n,k}\rangle}{\langle f_{n,k}, f_{n,k}\rangle} \\ &= \limsup \frac{\langle \mathcal{P}_n f_k^\varepsilon, H_n \mathcal{P}_n f_k^\varepsilon\rangle}{\langle \mathcal{P}_n f_k^\varepsilon, \mathcal{P}_n f_k^\varepsilon\rangle} + o_\varepsilon(1),\end{aligned} \tag{3.25}$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. But (3.24) of Lemma 3.16 provides $\lim\langle \mathcal{P}_n f_k^\varepsilon, H_n \mathcal{P}_n f_k^\varepsilon\rangle = \mathcal{H}(f_k^\varepsilon, f_k^\varepsilon)$, so the right-hand side of (3.25) is

$$\frac{\mathcal{H}(f_k^\varepsilon, f_k^\varepsilon)}{\langle f_k^\varepsilon, f_k^\varepsilon\rangle} + o_\varepsilon(1) = \frac{\mathcal{H}(f_k, f_k)}{\langle f_k, f_k\rangle} + o_\varepsilon(1) = \Lambda_k + o_\varepsilon(1).$$

Now letting $\varepsilon \to 0$, we conclude $\limsup \lambda_{n,k} \le \Lambda_k$.

Thus, $\lambda_{n,k} \to \Lambda_k$; Lemmas 3.13 and 3.15 imply that any subsequence of the $v_{n,k}$ has a further subsequence converging in L^2 to some $f \in L^*$; Lemma 3.16 then implies that f is an eigenfunction corresponding to Λ_k. Finally, convergence is uniform over $W_n, W \ge -b$ since the bound 3.13 is. $\square$

4. CLT and tightness for Gaussian and Wishart models. We now verify Assumptions 1–3 of Section 3 for the band Jacobi forms of Section 2, and thus prove Theorems 1.2 and 1.3 via Theorem 3.9.

We must consider the band forms as $(r\times r)$-block tridiagonal matrices. This amounts to reindexing the entries by $(k+rj, l+rj)$, where $j\in\mathbb{Z}_+$ indexes the blocks and $1\le k,l\le r$ give the index within each block. The scalar processes obtained by fixing k,l can then be analyzed jointly; finally, they can be assembled into a matrix-valued process.

The technical tool we use to establish (3.4) is a functional central limit theorem for convergence of discrete time processes with independent increments of given mean and variance (and controlled fourth moments) to Brownian motion plus a nice drift. Appearing as Corollary 6.1 in RRV, it is just a tailored version of a much more general result given as Theorem 7.4.1 in Ethier and Kurtz (1986). We record it here.

PROPOSITION 4.1. *Let $a\in\mathbb{R}$ and $h\in C^1(\mathbb{R}_+)$, and let y_n be a sequence of discrete time real-valued processes with $y_{n,0}=0$ and independent increments $\delta y_{n,j}=y_{n,j}-y_{n,j-1}=m_n^{-1}\triangle_n y_{n,j}$. Assume that $m_n\to\infty$ and*

$$m_n\mathbf{E}\delta y_{n,j}=h'(j/m_n)+o(1),\qquad m_n\mathbf{E}(\delta y_{n,j})^2=a^2+o(1),$$
$$m_n\mathbf{E}(\delta y_{n,j})^4=o(1)$$

uniformly for j/m_n on compact sets as $n\to\infty$. Then $y_n(x)=y_{n,\lfloor m_n x\rfloor}$ converges in law, with respect to the compact-uniform topology, to the process $h(x)+ab_x$ where b_x is a standard Brownian motion.

REMARK 4.2. Since the limit is a.s. continuous, Skorokhod convergence (the topology used in the references) implies uniform convergence on compact intervals [see Theorem 3.10.2 in Ethier and Kurtz (1986)] and we may as well speak in terms of the latter.

4.1. *The Gaussian case.* Take $G_n=G_{n;0}+\sqrt{n}P_n$ as in (2.4) with $G_{n;0}$ as in (2.3) and $P_n=\tilde{P}_n\oplus 0_{n-r}$. We denote upper-left $r\times r$ blocks with a tilde throughout. Set

$$m_n=n^{1/3},\qquad H_n=\frac{m_n^2}{\sqrt{n}}(2\sqrt{n}-G_n).$$

As usual, this soft-edge scaling can be predicted as follows. Centering G_n by $2\sqrt{n}$ gives, to first order, $\sqrt{n}$ times the discrete Laplacian on blocks of size r. With space scaled down by m_n, the Laplacian must be scaled up by m_n^2 to converge to the second derivative. Finally, the scaling $m_n=n^{1/3}$ is determined by convergence of the next order terms to the noise and drift parts of the limiting potential.

Decompose H_n as in (3.2), (3.3). The upper-left block is

$$\tilde{H}_n=m_n^2+m_n(W_n+Y_{n,1;0})=m_n^2(2-n^{-1/2}\tilde{G}_{n,0}-\tilde{P}_n);$$

we want the boundary term W_n to absorb the "extra" m_n^2 (the 2 in the right-hand side "should be" a 1) and the perturbation in order to make $Y_{n,1;0}$ small just like the subsequent increments of $Y_{n,i}$. We therefore set

$$W_n = m_n(1 - \tilde{P}_n).$$

With this choice Assumption 3 is an immediate consequence of the hypotheses of Theorem 1.2. The processes $Y_{n,1}, Y_{n,2}$ are determined and it remains to verify Assumptions 1 and 2.

We begin with Assumption 1, identifying the limiting integrated potential $Y : \mathbb{R}_+ \to M_r(\mathbb{F})$ as that of the multivariate stochastic Airy operator

$$Y(x) = \sqrt{2}B_x + \tfrac{1}{2}rx^2, \tag{4.1}$$

where B_x is a standard $M_r(\mathbb{F})$ Brownian motion and second term is a scalar matrix.

Proof of (3.4), Gaussian case. Define scalar processes $y_{k,l}$ for $1 \le l \le r$ and $l \le k \le l + r$ by

$$y_{k,l} = \begin{cases} (Y_{n,1})_{k,l}, & l \le k \le r, \\ (\tfrac{1}{2}Y_{n,2}^\dagger)_{k-r,l}, & r+1 \le k \le l+r. \end{cases} \tag{4.2}$$

(We have dropped the subscript n.) Equivalently, for $1 \le k, l \le r$,

$$(Y_{n,1})_{kl} = \begin{cases} y_{l,k}^*, & k \le l, \\ y_{k,l}, & k \ge l, \end{cases} \qquad (\tfrac{1}{2}Y_{n,2}^\dagger)_{kl} = \begin{cases} y_{k+r,l}, & k \le l, \\ 0, & k > l. \end{cases} \tag{4.3}$$

Then we have

$$\delta y_{k,l;j} = n^{-1/6} \begin{cases} -\dfrac{2}{\beta}\widetilde{g}_{k+rj}, & k = l, \\ -g_{k+rj,l+rj}, & l < k < l+r, \\ \left(\sqrt{n} - \dfrac{1}{\sqrt{\beta}}\chi_{(n-k-rj+1)\beta}\right), & k = l+r. \end{cases} \tag{4.4}$$

Note that the $y_{k,l}$ are independent increment processes that are mutually independent of one another. With the usual embedding $j = \lfloor n^{1/3}x \rfloor$, Proposition 4.1 together with standard moment computations for Gaussian and Gamma random variables—in particular

$$\mathbf{E}\chi_\alpha = \sqrt{\alpha} + O(1/\sqrt{\alpha}), \qquad \mathbf{E}(\chi_\alpha - \sqrt{\alpha})^2 = 1/2 + O(1/\alpha),$$
$$\mathbf{E}(\chi_\alpha - \sqrt{\alpha})^4 = O(1),$$

for α large [valid since we consider $j = O(n^{1/3})$ here]—leads to the convergence of processes

$$y_{k,l}(x) \Rightarrow \begin{cases} \sqrt{\dfrac{2}{\beta}}\widetilde{b}_k(x), & k = l, \\ b_{k,l}(x), & l < k < l + r, \\ \dfrac{1}{\sqrt{2\beta}} b_k(x) + \dfrac{1}{4} r x^2, & k = l + r, \end{cases}$$

where $b_k, \widetilde{b}_k$ are standard real Brownian motions and $b_{k,l}$ are standard $\mathbb{F}$ Brownian motions. By independence, the convergence occurs jointly over k, l and the limiting Brownian motions are all independent. (For the $\mathbb{F}$ Brownian motions apply Proposition 4.1 to each of the β real components, which are independent of one another.) Therefore, $Y_{n,i}$ are both tight, and using (4.3) we have, jointly for $1 \le k, l \le r$,

$$\begin{aligned} (Y_{n,1} + \tfrac{1}{2}(Y_{n,2}^{\dagger} + Y_{n,2}))_{k,l} &= \begin{cases} y_{k,k} + 2y_{k+r,k}, \\ y_{k,l} + y_{l+r,k}^{*}, \\ y_{l,k}^{*} + y_{k+r,l} \end{cases} \\ &\Rightarrow \quad \begin{cases} \sqrt{\dfrac{2}{\beta}}(\widetilde{b}_k + b_k) + \dfrac{1}{2} r x^2, & k = l, \\ b_{k,l} + b_{l+r,k}^{*}, & k > l, \\ b_{l,k}^{*} + b_{k+r,l}, & k < l. \end{cases} \end{aligned}$$

Noting that the two Brownian motions in each entry are independent and that the entries on and below the diagonal are independent of each other, we conclude that this limiting matrix process is distributed as $Y(x)$ in (4.1). □

We turn to Assumption 2. Here, we need bounds over the full range $0 \le j \le \lceil n/r \rceil - 1$. Recall that we can extend the $Y_{n,i}$ processes beyond the end of the matrix arbitrarily (R_n takes care of the truncation), and it is convenient to "continue the pattern" for an extra block or two by setting $\chi_\alpha = 0$ for $\alpha < 0$. For the decomposition (3.5), we simply take $\eta_{n,i}$ to be the expectation of $\triangle Y_{n,i}$ and $\triangle \omega_{n,i}$ to be its centered version; the components of $\eta_{n,i}$ are then easily estimated and those of $\omega_{n,i}$ become independent increment martingales. We further set $\overline{\eta}(x) = rx$.

Proof of (3.6)–(3.8), Gaussian case. From (4.4), we have $\eta_{n,1;j} = 0$ and

$$(\eta_{n,2;j})_{k,l} = \mathbf{E} 2 m_n \delta y_{k+r,l;j} = 2n^{1/6}(\sqrt{n} - \beta^{-1/2}\mathbf{E}\chi_{(n-k-r(j+1)+1)\beta}) 1_{k=l}.$$

The estimate

$$\sqrt{(\alpha-1)^+} \le \mathbf{E}\chi_\alpha = \sqrt{2}\frac{\Gamma((\alpha+1)/2)}{\Gamma(\alpha/2)} \le \sqrt{\alpha} \tag{4.5}$$

is useful. We obtain

$$2n^{1/6}\frac{rj-c}{2\sqrt{n}} \le (\eta_{n,2;j})_{k,k} \le 2n^{1/6}\frac{rj+c}{\sqrt{n}}$$

for some fixed c, which yields the matrix inequalities

$$rx - cn^{-1/3} \le \eta_{n,2}(x) \le 2rx + cn^{-1/3}$$

and verifies (3.6) with $\overline{\eta}(x) = rx$. Separately, we have the upper bound (3.7):

$$\eta_{n,2}(x) \le 2n^{2/3} = 2m_n^2.$$

The bound (3.8) may be done entry by entry, so we consider the process $\{(\omega_{i,n;j})_{k,l}\}_{j\in\mathbb{Z}_+}$ for fixed $i=1,2$ and $1\le k,l\le r$ and further omit these indices; for the $\mathbb{F}$-valued processes we restrict attention further to one of the β real-valued components, and denote the latter simply by $\omega_{n;j}$. Consider (4.4); the key points are that the increments $\delta\omega_{n;j}$ are independent and centered, and that scaled up by $n^{1/6} = m_n^{1/2}$ they have uniformly bounded fourth moments. To prove (3.8), it is enough to consider x at integer points and show that the random variables

$$\sup_{x=0,1,\dots,n/rm_n} x^{\varepsilon-1} \sup_{j=1,\dots,m_n} |\omega_{n;m_nx+j} - \omega_{n;m_nx}|^2$$

are tight over n. Squaring, bounding the outer supremum by the corresponding sum, and then taking expectations gives

$$\sum_{x=0}^{n/rm_n} \frac{\mathbf{E}\sup_{j=1,\dots,m_n}|\omega_{n;m_nx+j}-\omega_{n;m_nx}|^4}{x^{2-2\varepsilon}} \le \sum_{x=0}^{n/rm_n}\frac{16\mathbf{E}|\omega_{n;m_n(x+1)}-\omega_{n;m_nx}|^4}{x^{2-2\varepsilon}},$$

where we have used the L^p maximum inequality for martingales [see, e.g., Proposition 2.2.16 of Ethier and Kurtz (1986)]. To bound the latter expectation, expand the fourth power to obtain $O(m_n^2)$ nonzero terms that are $O(m_n^{-2})$ with constants independent of x and n. It follows that the entire sum is uniformly bounded over n, as required. □

4.2. *The Wishart case.* Take $L_{n,p} = \Sigma_{n,p}^{1/2} L_{n,p,0}$ with $L_{n,p,0}$ as in (2.5) and, denoting the upper-left $r\times r$ block with a tilde, $\Sigma_{n,p} = \tilde{\Sigma}_{n,p} \oplus I_{n\wedge p}$. Recall that $L_{n,p}$ is $((n+r)\wedge p)\times(n\wedge p)$. Put $S_{n,p} = L_{n,p}^\dagger L_{n,p}$ and similarly for $S_{n,p,0}$; these matrices are $(n\wedge p)\times(n\wedge p)$ and the latter is given explicitly in (2.6). We sometimes drop the subscripts n,p. Recall (2.7) that $S - S_0 = \tilde{L}_0^\dagger(\tilde{\Sigma}-1)\tilde{L}_0 \oplus 0$.

We set

$$m_{n,p}=\left(\frac{\sqrt{np}}{\sqrt{n}+\sqrt{p}}\right)^{2/3},\qquad H_{n,p}=\frac{m_{n,p}^2}{\sqrt{np}}((\sqrt{n}+\sqrt{p})^2-S_{n,p}).\tag{4.6}$$

See Part I for detailed heuristics behind the scaling; written in this way, it allows that $p,n\to\infty$ together arbitrarily, that is, only $n\wedge p\to\infty$. It is useful to note that

$$2^{-2/3}(n\wedge p)^{1/3}\le m_{n,p}\le(n\wedge p)^{1/3}.$$

Decompose $H_{n,p}$ as in (3.2), (3.3). The upper-left block is

$$\tilde{H}=m^2+m(W+Y_{1;0})=2m^2-\frac{m^2}{\sqrt{np}}(\tilde{S}_0-n-p+\tilde{L}_0^\dagger(\tilde{\Sigma}-1)\tilde{L}_0).$$

As before we want W to absorb the extra m^2 and the perturbation in order to make $Y_{1;0}$ small. Now the perturbation term is random, but it does not have to be fully absorbed; it is enough that $Y_{1;0}\to 0$ in probability. The reason is that the process Y_1 can absorb an overall additive random constant that tends to zero in probability, as is clear in Assumption 1 while in Assumption 2 the constant may be put into ω_1. Since $\tilde{L}_0\approx\sqrt{n}$, we set

$$W_{n,p}=m_{n,p}(1-\sqrt{n/p}(\tilde{\Sigma}_{n,p}-1)).\tag{4.7}$$

Once again, Assumption 3 follows immediately from the hypotheses of Theorem 1.3.

We must still deal with the perturbed term in $Y_{1;0}$ and show that

$$\frac{m}{\sqrt{np}}(n\tilde{\Sigma}-\tilde{L}_0^\dagger\tilde{\Sigma}\tilde{L}_0)\to 0\tag{4.8}$$

in probability. We defer this to the end of the proof of Assumption 1, to which we now turn. As in the Gaussian case, Y is given by (4.1).

Proof of (3.4), Wishart case. By the preceding paragraph it suffices to treat the null case $\Sigma=I$ and afterward check (4.8). Define processes $y_{k,l}$ for $1\le l\le r$ and $l\le k\le l+r$ by (4.2) as in the Gaussian case. From (2.6) with the centering and scaling of (4.6) and (3.3), we obtain

$$\delta y_{k,l;j}=\frac{m}{\sqrt{np}}\begin{cases} n+p-\frac{1}{\beta}(\widetilde{\chi}^2_{(n-k-rj+1)\beta}+\chi^2_{(p-k-r(j+1)+1)\beta})+O(1), \\ \qquad\qquad\qquad\qquad\qquad\qquad k=l, \\ -\frac{1}{\sqrt{\beta}}(\widetilde{\chi}_{(n-k-rj+1)\beta}g_{k+rj,l+rj} \\ \qquad +\chi_{(p-l-r(j+1)+1)\beta}g^*_{l+r(j+1),k+rj})+O(1), \\ \qquad\qquad\qquad\qquad\qquad\qquad l<k<l+r, \\ \sqrt{np}-\frac{1}{\beta}\widetilde{\chi}_{(n-k-rj+1)\beta}\chi_{(p-k-rj+1)\beta}, \qquad k=l+r, \end{cases}$$

where the $O(1)$ terms stand in for the interior Gaussian sums of (2.6), all of whose *moments* are bounded uniformly in n, p. Since $m^{1+k}/(np)^{k/2} \leq m^{1-2k} = o(1)$ for $k \geq 1$, these terms are negligible in the scaling of Proposition 4.1 in the sense that the associated processes converge to the zero process. Next, use that expressions of type $\chi_n - \sqrt{n}$ are $O(1)$ in the same sense, and that $\sqrt{n} - \sqrt{n-j} = O(j/\sqrt{n}) = O(m/\sqrt{n}) = o(1)$ since we consider j/m bounded here (and similarly for p), to write

$$
\delta y_{k,l;j} = \frac{m}{\sqrt{np}} \begin{cases} \dfrac{2}{\sqrt{\beta}}(\sqrt{n}(\sqrt{\beta n} - \widetilde{\chi}_{(n-k-rj+1)\beta}) \\ \qquad + \sqrt{p}(\sqrt{\beta p} - \chi_{(p-k-r(j+1)+1)\beta})) + O(1), \\ \qquad k = l, \\ -\sqrt{n} g_{k+rj,l+rj} - \sqrt{p} g^*_{l+r(j+1),k+rj} + O(1), \\ \qquad l < k < l + r, \\ \dfrac{1}{\sqrt{\beta}}(\sqrt{p}(\sqrt{\beta n} - \widetilde{\chi}_{(n-k-rj+1)\beta}) \\ \qquad + \sqrt{n}(\sqrt{\beta p} - \chi_{(p-k-rj+1)\beta})) + O(1), \\ \qquad k = l + r. \end{cases} \tag{4.9}
$$

It suffices to prove tightness and convergence in distribution along a subsequence of any given subsequence, and we may therefore assume that $p/n \to \gamma^2 \in [0, \infty]$. Each case of (4.9) contains two terms, and each one of these terms forms an independent increment process to which Proposition 4.1 may be applied. (Break the $\mathbb{F}$-valued terms up further into their real-valued parts.) Standard moment computations as in the Gaussian case, together with independence, then lead to the joint convergence of processes

$$
y_{k,l}(x) \Rightarrow \begin{cases} \sqrt{\dfrac{2}{\beta}}\left(\dfrac{1}{1+\gamma}\widetilde{b}_k(x) + \dfrac{\gamma}{1+\gamma} b_k(x)\right) + \dfrac{\gamma}{(1+\gamma)^2} r x^2, \\ \qquad\qquad k = l, \\ \dfrac{1}{1+\gamma} b_{k,l}(x) + \dfrac{\gamma}{1+\gamma} b^*_{l+r,k}(x), \qquad l < k < l + r, \\ \dfrac{1}{\sqrt{2\beta}}\left(\dfrac{\gamma}{1+\gamma}\widetilde{b}_k(x) + \dfrac{1}{1+\gamma} b_k(x)\right) + \dfrac{1+\gamma^2}{4(1+\gamma)^2} r x^2, \\ \qquad\qquad k = l + r, \end{cases}
$$

where $b_k, \widetilde{b}_k$ are standard real Brownian motions and $b_{k,l}$ are standard $\mathbb{F}$ Brownian motions, all independent except that $b_{k+r,l+r}$ and $b_{k,l}$ are identified. Therefore, $Y_{n,i}$ are both tight. Furthermore, using (4.3) we have

$$
(Y_{n,1} + \tfrac{1}{2}(Y^\dagger_{n,2} + Y_{n,2}))_{k,l} = \begin{cases} y_{k,k} + 2 y_{k+r,k}, \\ y_{k,l} + y^*_{l+r,k}, \\ y^*_{l,k} + y_{k+r,l} \end{cases}
$$

$$\Rightarrow \quad \begin{cases} \sqrt{\dfrac{2}{\beta}}(\widetilde{b}_k + b_k) + \dfrac{1}{2}rx^2, & k = l, \\ b_{k,l} + b^*_{l+r,k}, & k > l, \\ b^*_{l,k} + b_{k+r,l}, & k < l \end{cases}$$

jointly for $1 \le k, l \le r$. After the dust clears, we thus arrive at exactly the same limiting process as in the Gaussian case, namely (4.1).

We now address (4.8). Here, we can replace $\tilde{L}_0$ with $\sqrt{n}I_r$ at the cost of an error that has uniformly bounded second and fourth moments. Now (4.7) and the assumed lower bound on $W_{n,p}$ give that $\tilde{\Sigma} \le 1 + 2\sqrt{p/n}$ for n, p large; this matrix inequality holds entrywise in the diagonal basis for $\tilde{\Sigma}$ (which was fixed over n, p). One therefore obtains error terms with mean square $O(m^2/n + m^2/p) = O(m^{-1})$ which is $o(1)$ as required. $\square$

Turning to Assumption 2, we may continue the processes $Y_{n,i}$ past the end of the matrix for convenience just as in the Gaussian case. The Wishart case presents an additional issue at the "end" of the matrix: recall that the final r rows and columns of S in (2.6) may have some apparently nonzero terms set to zero. However, these changes are easily absorbed into the bounds that follow. For (3.5), we once again take $\eta_{n,i}$ to be the expectation of $\triangle Y_{n,i}$ and $\triangle \omega_{n,i}$ to be its centered version. We also set $\overline{\eta}(x) = rx$ as before.

PROOF OF (3.6)–(3.8), WISHART CASE. This time we have

$$\begin{aligned} (\eta_{n,1;j})_{k,l} &= \mathbf{E} m\delta y_{k,l;j} = m^2(np)^{-1/2}(2rj - r + 1)1_{k=l}, \\ (\eta_{n,2;j})_{k,l} &= \mathbf{E} 2m\delta y_{k+r,l;j} \\ &= 2m^2(1 - \beta^{-1}(np)^{-1/2}\mathbf{E}\widetilde{\chi}_{(n-k-rj+1)\beta}\chi_{(p-k-rj+1)\beta})1_{k=l}. \end{aligned}$$

Using (4.5) one finds, for some constant c, that

$$m^{-1}(rj + c) \le (\eta_{n,1;j} + \eta_{n,2;j})_{k,k} \le m^{-1}(2rj + c)$$

which yields (3.6) with $\overline{\eta}(x) = rx$. Separately, we have the upper bound (3.7). The oscillation bound (3.8) may be proved exactly as in the Gaussian case: we have once again that $\{\sqrt{m_n}(\omega_{n,i;j})_{k,l}\}_{j \in \mathbb{Z}_+}$ are martingales with independent increments whose fourth moments are uniformly bounded. $\square$

5. Alternative characterizations of the laws.

In this section, we derive the SDE and PDE characterizations, proving Theorems 1.5 and 1.6.

5.1. *First-order linear ODE.* For each noise path B_x, the eigenvalue equation $\mathcal{H}_{\beta,W} f = \lambda f$ can be rewritten as a first-order linear ODE with continuous coefficients. We begin with the formal second-order linear differential equation

$$f''(x) = (x - \lambda + \sqrt{2}B'_x)f(x), \tag{5.1}$$

where $f : \mathbb{R}_+ \to \mathbb{F}^r$, with initial condition

$$f'(0) = Wf(0). \tag{5.2}$$

As usual, we allow $W \in M_r^*(\mathbb{F})$ and interpret (5.2) via (3.13). Rewrite (5.1) in the form

$$(f' - \sqrt{2}Bf)' = (x - \lambda)f - \sqrt{2}Bf'.$$

Now let $g = f' - \sqrt{2}Bf$. The equation becomes

$$\begin{aligned} g' &= (x - \lambda)f - \sqrt{2}Bf' \\ &= (x - \lambda - 2B^2)f - \sqrt{2}Bg. \end{aligned}$$

In other words, the pair $(f(x), g(x))$ formally satisfies the first-order linear system

$$\begin{bmatrix} f' \\ g' \end{bmatrix} = \begin{bmatrix} \sqrt{2}B & 1 \\ x - \lambda - 2B^2 & -\sqrt{2}B \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}. \tag{5.3}$$

Since $B_0 = 0$, g simply replaces f' in the initial condition (5.2). If one prefers, this condition can be written in the standard form

$$-\tilde{W}f(0) + \tilde{I}g(0) = 0, \tag{5.4}$$

where $\tilde{W} = \sum_{i \le r_0} w_i u_i u_i^\dagger + \sum_{i > r_0} u_i u_i^\dagger$ and $\tilde{I} = \sum_{i \le r_0} u_i u_i^\dagger$.

One could allow general measurable coefficients and define a solution to be a pair of absolutely continuous functions (f, g) satisfying (5.3) Lebesgue a.e. This definition, equivalent to writing (5.3) in an integrated form, is easily seen to coincide with (3.18). As in Remark 3.5, however, we note the coefficients are continuous; solutions may therefore be taken to satisfy (5.3) everywhere and are in fact continuously differentiable. It is classical that the initial value problem has a unique solution which exists for all $x \in \mathbb{R}_+$ (and further depends continuously on the parameter λ and the initial condition W).

5.2. *Matrix oscillation theory.* The matrix generalization of Sturm oscillation theory goes back to the classic work of Morse Morse (1932) [see also Morse (1973)]. Textbook treatments of self-adjoint differential systems include that of Reid (1971). Our reference will be the paper of Baur and Kratz (1989), which allows sufficiently general boundary conditions.

We first consider the eigenvalue problem on a finite interval $[0,L]$ with Dirichlet boundary condition $f(L)=0$ at the right endpoint. In the scalar-valued setting, the number of eigenvalues below λ is found to coincide with the number of zeros of f (the solution of the initial value problem) that lie in $(0,L)$. The correct generalization to the matrix-valued setting involves tracking a matrix whose columns form a basis of solutions, and counting the so-called "focal points".

We need a little terminology and a few facts from Baur and Kratz (1989), especially Definition 1 on page 338 there and the points that follow. A *matrix solution* of (5.3) is a pair $F,G:\mathbb{R}_+\to\mathbb{F}^{r\times r}$ such that each column of $\left[\begin{smallmatrix}F\\G\end{smallmatrix}\right]$ is a solution. A *conjoined basis* for (5.3) is a matrix solution (F,G) with the additional properties that $F^\dagger G=G^\dagger F$ and $\operatorname{rank}\left[\begin{smallmatrix}F\\G\end{smallmatrix}\right]=r$. The latter properties hold identically on $\mathbb{R}_+$ as soon as they do at a single point; in particular, we may set $F(0)=\tilde I$ and $G(0)=\tilde W$ to obtain a conjoined basis for the initial condition (5.4). A point $x\in\mathbb{R}_+$ is called a *focal point* if $F(x)$ is singular, of *multiplicity* nullity $F(x)$. The following proposition summarizes what we need from the more general results of Baur and Kratz (1989).

Proposition 5.1. *Consider the differential system*

$$\begin{bmatrix}f'\\g'\end{bmatrix}=\begin{bmatrix}A & B\\ C-C_0\lambda & -A^\dagger\end{bmatrix}\begin{bmatrix}f\\g\end{bmatrix}$$

with real parameter λ, where $A(x),B(x),C(x),C_0(x)$ are $n\times n$ matrices depending continuously on $x\in\mathbb{R}$ with B,C,C_0 Hermitian and $B,C_0>0$. For each $\lambda\in\mathbb{R}$, let (F,G) be a conjoined basis with some fixed initial condition at 0. Consider also the associated eigenvalue problem on $[0,L]$ with the same boundary condition at 0 and Dirichlet condition $f=0$ at L. Then, for all $\lambda\in\mathbb{R}$, the number of focal points of (F,G) in $(0,L)$ equals the number of eigenvalues below λ. Furthermore, the spectrum is purely discrete and bounded below with eigenvalues tending to infinity.

Proof. The idea is that focal points are isolated and move continuously to the left as λ increases. For sufficiently negative λ, there are no focal points on $(0,L]$; each time λ passes an eigenvalue, a new focal point is introduced at L.

We indicate how the proposition follows from the results of Baur and Kratz (1989). Note that Conditions (A1), (A2) on page 337 are satisfied by our coefficients, and that (A3) on page 340 is satisfied by our boundary conditions. Theorem 1 on page 345 thus applies. See (3.5) on page 341 for the definition of $\Lambda(\lambda)$; the Dirichlet condition at L gives the particularly simple result that the right-hand side of (4.1) vanishes, so the quantity $n_2(\lambda)$ is constant. Theorem 2 applies as well, and we obtain $n_1(\lambda)-n_1=n_3(\lambda)$.

Here, $n_1(\lambda)$ is the number of focal points in $[0, L)$, $n_1 = \lim_{\lambda \to -\infty} n_1(\lambda)$ and $n_3(\lambda)$ is the number of eigenvalues below λ. To finish, we consult Theorem 3 on page 353; noting that (A4′) is satisfied by Section 7.2, page 365, to find that n_1 is simply the multiplicity of the focal point at 0. The oscillation result follows. For the assertion about the spectrum, we apply Theorem 4, noting that (A5), page 358 holds by (i) there, and (A6), page 359 also holds. □

We conclude the following for our matrix system.

LEMMA 5.2. *Consider the eigenvalue problem (5.3) on $[0, L]$ with boundary conditions (5.4) and $f(L) = 0$. For each $\lambda \in \mathbb{R}$, let (F, G) be the conjoined basis initialized by $F(0) = \tilde{I}$ and $G(0) = \tilde{W}$; then the number of focal points in the interval $(0, L)$, counting multiplicity, equals the number of eigenvalues below λ. Furthermore, the spectrum is purely discrete and bounded below with eigenvalues tending to infinity.*

A soft argument now recovers an oscillation theorem for the original half-line problem.

THEOREM 5.3. *Consider the eigenvalue problem (5.3), (5.4) on $L^2(\mathbb{R}_+)$. For each $\lambda \in \mathbb{R}$, let (F, G) be the conjoined basis as above; then the number of focal points in $(0, \infty)$ equals the number of eigenvalues strictly below λ.*

PROOF. Let $\Lambda_{L,k}, \Lambda_k$, $k = 0, 1, \ldots$ denote the lowest eigenvalues of the truncated and half-line operators $\mathcal{H}_L, \mathcal{H}$, respectively; it suffices to show that $\lim_{L\to\infty} \Lambda_{L,k} = \Lambda_k$ for each k. Indeed, taking $L \to \infty$ in Lemma 5.2 then yields the conclusion for each $\lambda \in \mathbb{R} \setminus \{\Lambda_0, \Lambda_1, \ldots\}$. Letting $\lambda \searrow \Lambda_k$, the right-most focal point must tend to ∞ by monotonicity and continuity, so the claim actually holds for all $\lambda \in \mathbb{R}$.

The variational problem for $\mathcal{H}_L$ simply minimizes over the subset of L^* functions that vanish on $[L, \infty)$; the Dirichlet condition is important here. It follows immediately that $\Lambda_{L,k} \geq \Lambda_k$, using the min–max formulation of the variational characterization. Proceed by induction, assuming that $\Lambda_{L,j} \to \Lambda_L$ for $j = 0, \ldots, k-1$.

Let $f_{L,j}$ be orthonormal eigenvectors corresponding to $\Lambda_{L,j}$. By the induction hypothesis, the variational characterization for $\mathcal{H}$ and the finite-dimensionality of its eigenspaces, every subsequence has a further subsequence such that $f_{L,j} \to_{L^2} f_j$, eigenvectors corresponding to Λ_j. Let f_k be an orthogonal eigenvector corresponding to Λ_k and take f_k^ε compactly supported with $\|f_k^\varepsilon - f_k\|_* < \varepsilon$. Let

$$g_L = f_k^\varepsilon - \sum_{j=0}^{k-1} \langle f_k^\varepsilon, f_{L,j} \rangle f_{L,j}.$$

For large L, the inner products are at most 2ε, so $\|g_L - f_k\|_* \le c\varepsilon$. Noting that g_L is eventually supported on $[0, L]$, the variational characterization gives

$$\limsup_{L\to\infty} \Lambda_{L,k} \le \limsup_{L\to\infty} \frac{\mathcal{H}(g_L, g_L)}{\langle g_L, g_L\rangle}$$

and the right-hand side tends to $\mathcal{H}(f_k, f_k)/\langle f_k, f_k\rangle = \Lambda_k$ as $\varepsilon \to 0$. □

5.3. *Riccati SDE: Stochastic airy meets dyson.* Let (F, G) be a conjoined basis for (5.3) as defined in the previous subsection. Then, on any interval with no focal points, the matrix $Q = GF^{-1}$ is self-adjoint and satisfies the *matrix Riccati equation*

$$Q' = rx - \lambda - (Q + \sqrt{2}B)^2 \tag{5.5}$$

[see page 338 of Baur and Kratz (1989)].

As x passes through a focal point x_0, an eigenvalue q of Q "explodes to $-\infty$ and restarts at $+\infty$". The precise evolution of Q near x_0 can be seen by choosing $a \in \mathbb{R}$ so that $\tilde{Q} = (Q - a)^{-1} = F(G - aF)^{-1}$ is defined; then $\tilde{Q}$ satisfies

$$\tilde{Q}' = (1 + \tilde{Q}(\sqrt{2}B + a))(1 + (\sqrt{2}B + a)\tilde{Q}) - (x - \lambda)\tilde{Q}^2. \tag{5.6}$$

Writing $\tilde{q} = 1/(q - a)$ and v for the corresponding eigenvector, notice how

$$\tilde{q}'(x_0) = v(x_0)^\dagger \tilde{Q}'(x_0) v(x_0) = 1.$$

Thus, $\tilde{q}$ is "pushed up through zero", corresponding to the explosion/restart in $q = 1/\tilde{q} + a$. In this way, we may consider $Q(x) \in M_r^*(\mathbb{F})$ to be defined for all x. The initial condition is then simply $Q(0) = W$.

Now let $P = F'F^{-1}$. While $P = Q + \sqrt{2}B$ is not differentiable, by (5.5) it certainly satisfies the integral equation

$$P_{x_2} - P_{x_1} = \sqrt{2}(B_{x_2} - B_{x_1}) + \int_{x_1}^{x_2} (ry - \lambda - P_y^2)\, dy$$

if $[x_1, x_2]$ is free of focal points. In other words, *P is a strong solution of the Itô equation*

$$dP_x = \sqrt{2} dB_x + (rx - \lambda - P_x^2)\, dx \tag{5.7}$$

off the focal points. The evolution of P through a focal point can be described in the coordinate $\tilde{P} = (P - a)^{-1} = F(F' - aF)^{-1}$. Using (5.6) and Itô's lemma, one could write down an SDE for $\tilde{P} = \tilde{Q}(1 + \sqrt{2}B\tilde{Q})^{-1}$. The initial condition here is also $P(0) = W$.

Consider the eigenvalues $p_1, \dots, p_r$ of P. The main point is that the drift term in (5.7) is unitarily equivariant and passes through the usual derivation of Dyson's Brownian motion [Dyson (1962)]. The eigenvalues therefore evolve as an autonomous Markov process.

To describe the law on paths we need a space, and there are two issues: it will be necessary to keep the eigenvalues ordered but also allow for explosions/restarts. We therefore define a sequence of *Weyl chambers* $C_k \subset (-\infty, \infty]^r$ by

$$\begin{aligned} C_0 &= \{p_1 < \cdots < p_r\}, \\ C_1 &= \{p_2 < \cdots < p_r < p_1\}, \\ C_2 &= \{p_3 < \cdots < p_r < p_1 < p_2\} \end{aligned}$$

and so on, permuting cyclically. We glue successive adjacent chambers together at infinity in the natural way to make the disjoint union $\mathcal{C} = C_0 \cup C_1 \cup \ldots$ into a connected smooth manifold. That is, taking $p_1 \to -\infty$ in C_0 puts you at $p_1 = +\infty$ in C_1; the smooth structure is defined by the coordinate $\tilde{p}_1 = 1/p_1$, which vanishes along the seam. Glue C_{k-1} to C_k similarly along $\{p_{k \bmod r} = \infty\}$. We also define $\overline{C}_k$, $\overline{\mathcal{C}}$ in which some coordinates may be equal, and $\partial C_k = \overline{C}_k \setminus C_k$, $\partial \mathcal{C} = \overline{\mathcal{C}} \setminus \mathcal{C}$ in which some coordinates are equal.

THEOREM 5.4. *Represent the eigenvalues of $W \in M_r^*(\mathbb{F})$ as $\mathbf{w} = (w_1 \ldots, w_r) \in \overline{C}_0$. The eigenvalues $\mathbf{p} = (p_1, \ldots, p_r)$ of P evolve as an autonomous Markov process whose law on paths $\mathbb{R}_+ \to \overline{\mathcal{C}}$ is the unique weak solution of the SDE system*

$$dp_i = \frac{2}{\sqrt{\beta}}\, db_i + \left(rx - \lambda - p_i^2 + \sum_{j \neq i} \frac{2}{p_i - p_j} \right) dx \tag{5.8}$$

with initial condition $\mathbf{p}(0) = \mathbf{w}$, where $b_1, \ldots, b_r$ are independent standard real Brownian motions. An eigenvalue p_i can explode to $-\infty$ and restart at $+\infty$, meaning $\mathbf{p}$ crosses from C_k to C_{k+1}; the evolution through an explosion is described in the coordinate $\tilde{p}_i = 1/p_i$, which satisfies

$$d\tilde{p}_i = -\frac{2}{\sqrt{\beta}} \tilde{p}_i^2\, db_i + \left(1 + \left(\lambda - rx + \sum_{j \neq i} \frac{2\tilde{p}_i \tilde{p}_j}{\tilde{p}_i - \tilde{p}_j} \right) \tilde{p}_i^2 + \frac{4}{\beta} \tilde{p}_i^3 \right) dx. \tag{5.9}$$

PROOF. Deriving (5.8) from (5.7) is simply a matter of applying Itô's lemma, at least in $\mathcal{C}$ where the eigenvalues are distinct. One needs to differentiate an eigenvalue with respect to a matrix, and this information is given by Hadamard's variation formulas. In detail, let $A \in M_r(\mathbb{F})$ vary smoothly in time and suppose $A(0)$ has distinct spectrum. Then eigenvalues $\lambda_1, \ldots, \lambda_r$ of A and corresponding eigenvectors $v_1, \ldots, v_r$ vary smoothly near 0 by the implicit function theorem. Differentiating $Av_i = \lambda_i v_i$ and $v_i^\dagger v_i = 1$ lead to the formulas

$$\dot{\lambda}_i = v_i^\dagger \dot{A} v_i, \qquad \ddot{\lambda}_i = v_i^\dagger \ddot{A} v_i + 2 \sum_{j \neq i} \frac{|v_i^\dagger \dot{A} v_j|^2}{\lambda_i - \lambda_j}.$$

Writing $X=\dot{A}(0)$ and ∇_X for the directional derivative, and taking $v_1(0),\dots,v_r(0)$ to be the standard basis, we find

$$\nabla_X\lambda_i = X_{ii}, \qquad \nabla_X^2\lambda_i = 2\sum_{j\neq i}\frac{|X_{ij}|^2}{\lambda_i-\lambda_j}.$$

Returning to (5.7), at each fixed time x we can change to the diagonal basis for P_x because the noise term is invariant in distribution and the drift term is equivariant. Itô's lemma amounts to formally writing $dp_i=\nabla_{dP}p_i+\frac{1}{2}\nabla_{dP}^2p_i$ and using that dB_{ii} are jointly distributed as $\sqrt{2/\beta}\,db_i$ for $i=1,\dots,r$ while $|dB_{ij}|^2=dt$ for $j\neq i$. We thus arrive at (5.8).

Recall that the evolution of P through a focal point is still described by an SDE, after changing coordinates. The same is therefore true of $\mathbf{p}$ through an explosion; the form (5.9) is obtained from (5.8) by an application of Itô's lemma.

Just as with the usual Dyson's Brownian motion, the p_i are almost surely distinct at all positive times: $\mathbf{p}(x)\in\mathcal{C}$ for all $x>0$. One can show this "no collision property" holds for any solution of (5.8), (5.9), even with an initial condition $\mathbf{p}(0)\in\partial C_0$. (Technically, one defines an entrance law from $\partial\mathcal{C}$ by a limiting procedure.) Since the coefficients are regular inside $\mathcal{C}$, this suffices to prove uniqueness of the law. See Anderson, Guionnet and Zeitouni (2010), Section 4.3.1 for a detailed proof in the driftless case. $\square$

Proof of Theorem 1.5. Explosions of $\mathbf{p}$ as in Theorem 5.4 correspond to focal points of F for each λ. By Theorem 5.3, the total number of explosions K is equal to the number of eigenvalues strictly below λ. (Notice that $\mathbf{p}$ ends up in C_K.) For a *fixed* λ, translation invariance of the driving Brownian motions b_i allows one to shift time $x\mapsto x-\lambda/r$ and use (1.3) started at $x_0=-\lambda/r$. Putting $a=-\lambda$ we have $\mathbf{P}(-\Lambda_k\le a)=\mathbf{P}(\Lambda_k\ge\lambda)=\mathbf{P}_{a/r,\mathbf{w}}(K\le k)$ as required. $\square$

5.4. *PDE and boundary value problem.* We now prove the PDE characterization, Theorem 1.6. We will need two properties of the eigenvalue diffusion.

Lemma 5.5. *Let $\mathbf{p}:[x_0,\infty)\to\overline{\mathcal{C}}$ have law $\mathbf{P}_{x_0,\mathbf{w}}$ as in (1.3) and let K be the number of explosions. Then the following hold:*

(i) *Given x_0,k, $\mathbf{P}_{x_0,\mathbf{w}}(K\le k)$ is increasing in $\mathbf{w}$ with respect to the partial order $\mathbf{w}\le\mathbf{w}'$ given by $w_i\le w_i'$, $i=1,\dots,r$.*
(ii) *$\mathbf{P}_{x_0,\mathbf{w}}$-almost surely, $p_1,\dots,p_r$ remain bounded below in C_K (after the last explosion), or equivalently in C_0 on the event $\{K=0\}$.*

Proof. Part (i) is a consequence Theorem 1.5 and Remark 1.1, the pathwise monotonicity of the eigenvalues Λ_k as a function of the boundary parameter W with respect to the usual matrix partial order. It can also be seen from the related fact that the matrix partial order is preserved pathwise by the matrix Riccati equation (5.7), which implies that a solution started from W explodes no later than one started from $W' \geq W$. This fact holds for the P evolution if it holds for the Q evolution (5.5), and for the latter it is Theorem IV.4.1 of Reid (1972).

Part (ii) follows from the stronger assertion that $p_i \sim \sqrt{rx}$ as $x \to \infty$. In the $r = 1$ case, this is Proposition 3.7 of RRV. Heuristically, the single particle drift linearizes at the stable equilibrium $\sqrt{rx}$ to $2\sqrt{rx}(\sqrt{rx} - p_i)$; even with the repulsion terms one expects fluctuations of variance only $C/\sqrt{x}$. We omit the proof. $\square$

Proof of Theorem 1.6. Assume the diffusion representation of Theorem 1.5 for $F_\beta(x;\mathbf{w}) = \mathbf{P}(-\Lambda_0 \leq x)$ on $\mathbb{R} \times \overline{C}_0$. We first show $F = F_\beta$ has the asserted properties and afterward argue uniqueness. Writing L for the space-time generator of (1.3), the PDE (1.6) is simply the equation $LF = 0$ after replacing x with x/r. In other words, it is the Kolmogorov backward equation for the hitting probability (1.4) (more precisely, the probability of never hitting $\{w_1 = -\infty\}$), which is L-harmonic. This extends to $w_r = +\infty$ by using the local coordinate there; from (5.9) one sees that the coefficients remain regular. Although the diffusivity vanishes at $w_r = +\infty$, the drift does not, and it follows that F is continuous up to $w_r = +\infty$. The PDE holds even at points $\mathbf{w} \in \partial C_0$ with appropriate one-sided derivatives; notice that the apparent singularity in the "Dyson term" of the PDE is in fact removable for F regular and symmetric in the w_i. [For a toy version, consider a function $f : \mathbb{R} \to \mathbb{R}$ that is twice differentiable and even; then f' is odd and $f'(w)/w$ is continuous with value $f''(0)$ at $w = 0$. These functions form the domain of the generator of the Bessel process on the half-line $\{w \geq 0\}$ in the same way that symmetric functions form the domain of the generator of Dyson's Brownian motion on a Weyl chamber.] Finally, the picture can be copied to $\mathbf{w} \in (-\infty, \infty]^r$ by symmetry, permuting the w_i.

The boundary condition (1.7) follows from the monotonicity property of Lemma 5.5(i). For fixed $\mathbf{w}$, $F(x;\mathbf{w}) \to 1$ as $x \to \infty$ because it is a distribution function in x; by monotonicity in $\mathbf{w}$, the convergence is uniform over a set of $\mathbf{w}$ bounded below. To understand the boundary condition (1.8) (using w_1 in $\overline{C}_0$), change to the coordinate $\tilde{w}_1 = 1/w_1$ and close the domain to include the "bottom boundary" $\{w_1 = -\infty\}$. Then (1.8) becomes an ordinary Dirichlet condition. While the diffusivity vanishes on this boundary, the drift is nonzero into the boundary. The hitting probability is therefore continuous up to the boundary.

For F^k, there is the following more general picture. Consider the PDE in $\overline{C}_0 \cup \cdots \cup \overline{C}_k$, defined across the seams by changing coordinates as in (5.9). Put the boundary condition (1.7) on all the chambers and (1.8) on the bottom of $\overline{C}_k$. Then the solution is F^k in $\overline{C}_0$; the reason is the same as for $F = F^0$, but now using (1.5) and the hitting event "at most k explosions". Similarly, the solution is F^{k-1} in $\overline{C}_1$ and so on down to F^0 in $\overline{C}_k$. Continuity holds across the seams and (1.9) follows after permuting coordinates.

Toward uniqueness, suppose $\tilde{F}$ is another bounded solution of the boundary value problem (1.6)–(1.8) on $\mathbb{R} \times \overline{C}_0$. With the notation of Theorem 1.5, $\tilde{F}(rx; \mathbf{p}_x)$ is a local martingale under $\mathbf{P}_{x_0,\mathbf{w}}$ by the PDE (1.6). It is therefore a bounded martingale. Let $\zeta \in (x_0, \infty]$ be the time of the first explosion; optional stopping gives $\tilde{F}(rx_0; \mathbf{w}) = \mathbf{E}_{x_0,\mathbf{w}} \tilde{F}(r(\zeta \wedge x); \mathbf{p}_{\zeta\wedge x})$ for all $x \geq x_0$. Taking $x \to \infty$, we conclude by bounded convergence, the boundary behaviour (1.7), (1.8) of $\tilde{F}$ and Lemma 5.5(ii) that $\tilde{F}(rx_0, \mathbf{w}) = \mathbf{P}_{x_0,\mathbf{w}}(\zeta = \infty)$. By Theorem 1.5, this probability is $F_\beta(rx_0, \mathbf{w})$. One argues similarly for the higher eigenvalues. $\square$

6. Connection with Painlevé II. In Part I, we used the PDE characterization to give new proofs of certain Painlevé II formulas for the single-parameter (rank one deformed) distribution functions $F_\beta(x; w)$ in the cases $\beta = 2, 4$, in particular recovering the Painlevé II representations for the corresponding undeformed Tracy–Widom distributions by taking $w \to \infty$. The Painlevé formulas appeared originally in Baik and Rains (2000, 2001) in a different context; in the random matrix theory setting, Baik (2006) derived them from the BBP result in the case $\beta = 2$ but they are new for $\beta = 4$ when $w \neq 0$ [see Wang (2008)].

Baik (2006) also derives a Painlevé II formula for the multi-parameter distribution function $F_2(x; w_1, \ldots, w_r)$. While we do not have a full independent proof at present, we used the computer algebra system Maple to verify symbolically that it does indeed satisfy our PDE (1.6) at $\beta = 2$ for $r = 2, 3, 4, 5$. Since this article was first posted, a pencil-and-paper proof for all r was found [Bloemendal and Baik (2013)]. We first state Baik's formula and then briefly describe the symbolic computation.

Let $u(x)$ be the Hastings–McLeod solution of the homogeneous Painlevé II equation

$$u'' = 2u^3 + xu, \tag{6.1}$$

characterized by

$$u(x) \sim \operatorname{Ai}(x) \qquad \text{as } x \to +\infty,$$

where $\operatorname{Ai}(x)$ is the Airy function. Put

$$v(x)=\int_x^\infty u^2, \tag{6.2}$$

$$E(x)=\exp\biggl(-\int_x^\infty u\biggr), \qquad F(x)=\exp\biggl(-\int_x^\infty v\biggr). \tag{6.3}$$

Next, define two functions $f(x,w)$, $g(x,w)$ on $\mathbb{R}^2$, analytic in w for each fixed x, by the first- order linear ODEs

$$\frac{\partial}{\partial w}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} u^2 & -wu-u' \\ -wu+u' & w^2-x-u^2 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} \tag{6.4}$$

and the initial conditions

$$f(x,0)=E(x)=g(x,0).$$

Equation (6.4) is one member of the Lax pair for the Painlevé II equation. The other member of the pair is

$$\frac{\partial}{\partial x}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & u(x) \\ u(x) & -w \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}, \tag{6.5}$$

which holds for each fixed $w\in\mathbb{R}$. The consistency condition for the overdetermined system (6.4), (6.5) (i.e., that the partials commute) is the Painlevé II equation (6.1). The functions f,g can also be defined in terms of an associated Riemann–Hilbert problem [see, e.g., Baik (2006)].

Baik's formula is

$$F_2(x;w_1,\dots,w_r)=F(x)\frac{\det((w_i+\partial/(\partial x))^{j-1}f(x,w_i))_{1\le i,j\le r}}{\prod_{1\le i<j\le r}(w_j-w_i)}. \tag{6.6}$$

Our symbolic verification for small values of r consisted of the following steps. The differential relations given by (6.1)–(6.5) were encoded as formal substitution rules. The determinant in (6.6) was expanded (this step becomes problematic for larger $r\dots$!) and the result plugged into our PDE (1.6). The substitution rules were then applied repeatedly. Finally, the result was factored using Maple's built-in command. Each time, the output contained the factor

$$v+u^4-(u')^2+xu^2,$$

which vanishes identically: differentiate and apply (6.1) to see it is constant, and take $x\to\infty$ to see the constant is zero.

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