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Elbert-type comparison theorems for a class of nonlinear Hamiltonian systems

The paper was written to commemorate the work of the late Professor Árpád Elbert who passed away in Budapest 15 years ago

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Abstract. Picone-type identities are established for a pair of solutions (x, y) and (X, Y) of the respective systems of the form

$$x' = r(t)x + p(t)\varphi_{1/\alpha}(y), \qquad y' = -q(t)\varphi_\alpha(x) - r(t)y, \tag{1.1}$$

and

$$X' = R(t)X + P(t)\varphi_{1/\alpha}(Y), \qquad Y' = -Q(t)\varphi_\alpha(X) - R(t)Y, \tag{1.2}$$

where α is a positive constant, p, q, r, P, Q and R are continuous functions on an interval J and $\varphi_\gamma(u)$ denotes the odd function in $u \in \mathbb{R}$ defined by

$$\varphi_\gamma(u) = |u|^\gamma \operatorname{sgn} u = |u|^{\gamma-1}u, \qquad \gamma > 0.$$

The identities are used to prove Sturmian comparison and separation results for components of solutions of systems (1.1) and (1.2).

Keywords: Hamiltonian systems, Picone's identity, Sturmian comparison.

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1 Introduction

The purpose of this paper is to present new identities of the Picone type for pairs of continuously differentiable vector functions which are solutions of the respective half-linear differential systems of the first order and use them to study the existence and location of zeros of

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components of these solutions by means of comparison. In particular, we consider differential systems of the form

$$x' = r(t)x + p(t)\varphi_{1/\alpha}(y), \qquad y' = -q(t)\varphi_\alpha(x) - r(t)y, \tag{1.1}$$

where α is a positive constant, p, q and r are continuous functions on an interval J and $\varphi_\gamma(u)$ denotes the odd function in $u \in \mathbb{R}$ defined by

$$\varphi_\gamma(u) = |u|^\gamma \operatorname{sgn} u = |u|^{\gamma-1}u, \qquad \gamma > 0.$$

System (1.1) is a nonlinear Hamiltonian system, i.e. it is of the form

$$x' = \frac{\partial H}{\partial y}, \qquad y' = -\frac{\partial H}{\partial x},$$

where

$$H(t, x, y) = \frac{1}{\alpha+1}q(t)|x|^{\alpha+1} + r(t)xy + \frac{\alpha}{\alpha+1}p(t)|y|^{1+\frac{1}{\alpha}}.$$

One of ways to gain information about zeros of solutions of system (1.1) without solving it explicitly is to compare it with another system whose solutions (or at least oscillation properties) are known. Thus, along with (1.1) we consider the differential system

$$X' = R(t)X + P(t)\varphi_{1/\alpha}(Y), \qquad Y' = -Q(t)\varphi_\alpha(X) - R(t)Y, \tag{1.2}$$

where P, Q and R are continuous functions on J.

It is well known (see Á. Elbert [2]) that if (x, y) is a nontrivial solution of system (1.1) such that its first component $x(t)$ has consecutive zeros at $t_1, t_2 \in J, t_1 < t_2$, and if

$$\big[Q(t) - q(t)\big]|\xi|^{\alpha+1} - (\alpha+1)\big[r(t) - R(t)\big]\xi\eta + \alpha\big[P(t) - p(t)\big]|\eta|^{\frac{1}{\alpha}+1} \geq 0 \tag{1.3}$$

for all $\xi, \eta \in \mathbb{R}$ and $t \in [t_1, t_2]$, then for any solution (X, Y) of system (1.2) the first component $X(t)$ has at least one zero between t_1 and t_2.

Elbert proved his comparison result with the help of the generalized Prüfer substitution. The purpose of this note is to give a more direct proof by employing the identities of the Picone type established in the next section and to extend Elbert's theorem to the reciprocal situation where the zeros of the second component $Y(t)$ are studied, too. For this purpose we associate with (1.1) and (1.2) the "dual" systems

$$x' = r(t)x - p(t)\varphi_{1/\alpha}(y), \qquad y' = q(t)\varphi_\alpha(x) - r(t)y, \tag{1.4}$$

and

$$X' = R(t)X - P(t)\varphi_{1/\alpha}(Y), \qquad Y' = Q(t)\varphi_\alpha(X) - R(t)Y. \tag{1.5}$$

As is easily seen, if (x, y) and (X, Y) are solutions of (1.1) and (1.2) on $[t_0, \infty)$, respectively, then $(-x, y)$ and $(x, -y)$ (resp. $(-X, Y)$ and $(X, -Y)$) are solutions of (1.4) (resp. (1.5)), and vice versa. This means that (1.1) and (1.4) (resp. (1.2) and (1.5)) are equivalent so far as the oscillatory and nonoscillatory properties of their solutions are concerned. (Notice that systems (1.1) and (1.4) (resp. (1.2) and (1.5)) are the same except that the roles of $\{x, y\}$, $\{p, q\}$, $\{r, -r\}$ and $\{\alpha, 1/\alpha\}$ are interchanged.) This simple (but very useful) relationship between (1.1) and

(1.4) (resp. (1.2) and (1.5)) is referred to as the *duality principle* and will be frequently used in the proofs of our results.

For related results concerning (1.1) and more general nonlinear differential systems based on the two-dimensional version of Picone's identity which is a special case of the formula given below see [3]. The most comprehensive development of Sturmian theory for linear differential equations can be found in the monograph of Reid [11]. Comparison and oscillation results based on identities of the Picone type for linear systems of the form (1.1) were established also in Kreith [6, 7]. Sturm's comparison theory for scalar half-linear ordinary differential equations of the second order has been developed by Li and Yeh in [8] and by the present authors in [5].

2 Main results

The following Picone-type identities concerning pairs of solutions of systems (1.1) and (1.2) will form the basis for our subsequent discussions.

To formulate the results we use $\Phi_\gamma(U,V)$ to denote the form defined for $U,V \in \mathbb{R}$ and $\gamma > 0$ by

$$\Phi_\gamma(U,V) = |U|^{\gamma+1} + \gamma|V|^{\gamma+1} - (\gamma+1)U\varphi_\gamma(V).$$

From the Young inequality it follows that $\Phi_\gamma(U,V) \geq 0$ for all $U,V \in \mathbb{R}$ and the equality holds if and only if $U = V$.

Lemma 2.1. *Let (x,y) and (X,Y) be solutions on J of systems (1.1) and (1.2), respectively.*

(i) *If $X(t) \neq 0$ in J, then*

$$\begin{aligned}\frac{d}{dt}&\left\{\frac{x}{\varphi_\alpha(X)}\left[\varphi_\alpha(X)y - \varphi_\alpha(x)Y\right]\right\}\\ &= \left[Q(t)-q(t)\right]|x|^{\alpha+1} - (\alpha+1)\left[r(t)-R(t)\right]\frac{|x|^{\alpha+1}}{\varphi_\alpha(X)}Y \\ &\quad + \alpha\left[P(t)-p(t)\right]\frac{|x|^{\alpha+1}}{|X|^{\alpha+1}}|Y|^{\frac{1}{\alpha}+1} + p(t)\Phi_\alpha\left(\varphi_{1/\alpha}(y), x\varphi_{1/\alpha}(Y)/X\right).\end{aligned} \tag{2.1}$$

(ii) *If $Y(t) \neq 0$ in J, then*

$$\begin{aligned}\frac{d}{dt}&\left\{\frac{y}{\varphi_{1/\alpha}(Y)}\left[\varphi_{1/\alpha}(y)X - \varphi_{1/\alpha}(Y)x\right]\right\}\\ &= \left[P(t)-p(t)\right]|y|^{\frac{1}{\alpha}+1} - \left(\frac{1}{\alpha}+1\right)\left[R(t)-r(t)\right]\frac{|y|^{\frac{1}{\alpha}+1}}{\varphi_{1/\alpha}(Y)}X \\ &\quad + \frac{1}{\alpha}\left[Q(t)-q(t)\right]\frac{|y|^{\frac{1}{\alpha}+1}}{|Y|^{\frac{1}{\alpha}+1}}|X|^{\alpha+1} + q(t)\Phi_{1/\alpha}\left(\varphi_\alpha(x), y\varphi_\alpha(X)/Y\right).\end{aligned} \tag{2.2}$$

Proof. (i) Expanding the left-hand side of (2.1) and making use of the fact that (x,y) and (X,Y)

satisfy (1.1) and (1.2), respectively, we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{x}{\varphi_\alpha(X)}\left[\varphi_\alpha(X)y - \varphi_\alpha(x)Y\right]\right\}\\
&\quad = \left[Q(t)-q(t)\right]|x|^{\alpha+1} - (\alpha+1)\left[r(t)-R(t)\right]\frac{|x|^{\alpha+1}}{\varphi_\alpha(X)}Y\\
&\qquad + \alpha P(t)\frac{|x|^{\alpha+1}}{|X|^{\alpha+1}}|Y|^{\frac{1}{\alpha}+1} + p(t)|y|^{\frac{1}{\alpha}+1} - (\alpha+1)p(t)\varphi_{1/\alpha}(y)\frac{\varphi_\alpha(x)}{\varphi_\alpha(X)}Y.
\end{aligned}$$

Finally, inserting $-\alpha p(t)|x|^{\alpha+1}|Y|^{\frac{1}{\alpha}+1}/|X|^{\alpha+1} + \alpha p(t)|x|^{\alpha+1}|Y|^{\frac{1}{\alpha}+1}/|X|^{\alpha+1}$ into the right-hand side, we obtain formula (2.1) as desired.

(ii) The truth of the second Picone-type identity (2.2) can be verified directly in a similar manner, or by making a duality conversion described in the introduction and using formula (2.1). □

Remark 2.2. Identity (2.1) is a generalization of the identity obtained by Díaz and McLaughlin [1] in the linear case, i.e. $\alpha = 1$. The reciprocal formula (2.2) seems to be new even in the linear case.

Remark 2.3. If $r(t) \equiv R(t) \equiv 0$, $p(t) > 0$ and $P(t) > 0$ on J, then systems (1.1) and (1.2) are equivalent with the second-order half-linear differential equations

$$\left(\tilde{p}(t)\varphi_\alpha(x')\right)' + q(t)\varphi_\alpha(x) = 0, \tag{2.3}$$

and

$$\left(\tilde{P}(t)\varphi_\alpha(X')\right)' + Q(t)\varphi_\alpha(X) = 0, \tag{2.4}$$

respectively, where $\tilde{p}(t) = p(t)^{-\alpha}$ and $\tilde{P}(t) = P(t)^{-\alpha}$, and identity (2.1) reduces to

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{x}{\varphi_\alpha(X)}\left[\varphi_\alpha(X)\tilde{p}(t)\varphi_\alpha(x') - \varphi_\alpha(x)\tilde{P}(t)\varphi_\alpha(X')\right]\right\}\\
&\quad = \left[Q(t)-q(t)\right]|x|^{\alpha+1} + \alpha\left[\tilde{P}(t)^{-\frac{1}{\alpha}} - \tilde{p}(t)^{-\frac{1}{\alpha}}\right]\tilde{P}(t)^{\frac{1}{\alpha}+1}\frac{|x|^{\alpha+1}}{|X|^{\alpha+1}}|X'|^{\alpha+1}\\
&\qquad + \frac{\tilde{p}(t)^{-\frac{1}{\alpha}}}{|X|^{\alpha+1}}\Phi_\alpha\left(\tilde{p}(t)^{\frac{1}{\alpha}}x'X, \tilde{P}(t)^{\frac{1}{\alpha}}xX'\right),
\end{aligned} \tag{2.5}$$

which is different from similar formula

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{x}{\varphi_\alpha(X)}\left[\varphi_\alpha(X)\tilde{p}(t)\varphi_\alpha(x') - \varphi_\alpha(x)\tilde{P}(t)\varphi_\alpha(X')\right]\right\}\\
&\quad = \left[Q(t)-q(t)\right]|x|^{\alpha+1} + \left[\tilde{p}(t)-\tilde{P}(t)\right]|x'|^{\alpha+1} + \tilde{P}(t)\Phi_\alpha\left(x', xX'/X\right)
\end{aligned} \tag{2.6}$$

established by the present authors in [4] (see also [5]).

The next result shows that if certain Wronskian-like function is identically zero for a pair of vector solutions of the two-dimensional system of the form (1.1), then one of these solutions is a constant multiple of another in the sense specified below.

Lemma 2.4. *Let (x, y) and (X, Y) be solutions on J of the same system* (1.2) (= (1.1)).

(i) *If* $P(t) > 0$, $x(t) \neq 0$ *in* J *and*

$$x(t)\varphi_{1/\alpha}\big(Y(t)\big) - X(t)\varphi_{1/\alpha}\big(y(t)\big) \equiv 0 \quad \text{in } J, \tag{2.7}$$

then there exists a constant c *such that*

$$\big(X(t), Y(t)\big) = \big(cx(t), \varphi_\alpha(c)y(t)\big) \tag{2.8}$$

for all $t \in J$.

(ii) *If* $Q(t) > 0$, $y(t) \neq 0$ *in* J *and*

$$y(t)\varphi_\alpha\big(X(t)\big) - Y(t)\varphi_\alpha\big(x(t)\big) \equiv 0 \quad \text{in } J, \tag{2.9}$$

then there exists a constant c *such that*

$$\big(X(t), Y(t)\big) = \big(cx(t), \varphi_{1/\alpha}(c)y(t)\big) \tag{2.10}$$

for all $t \in J$.

Proof. (i) Note that

$$\begin{aligned}\left[\frac{X(t)}{x(t)}\right]' &= \frac{X'(t)x(t) - X(t)x'(t)}{x(t)^2} \\ &= \frac{x(t)\big[R(t)X(t) + P(t)\varphi_{1/\alpha}\big(Y(t)\big)\big] - \big[R(t)x(t) + P(t)\varphi_{1/\alpha}\big(y(t)\big)\big]X(t)}{x(t)^2} \\ &= P(t)\frac{x(t)\varphi_{1/\alpha}\big(Y(t)\big) - X(t)\varphi_{1/\alpha}\big(y(t)\big)}{x(t)^2} \equiv 0\end{aligned}$$

for $t \in J$. This implies that there exists a constant c such that $X(t) = cx(t)$ for $t \in J$. Substitution of this equality into the first equation of system (1.2) yields

$$\begin{aligned}P(t)\varphi_{1/\alpha}\big(Y(t)\big) &= X'(t) - R(t)X(t) = cx'(t) - cR(t)x(t) \\ &= c\big[R(t)x(t) + P(t)\varphi_{1/\alpha}\big(y(t)\big)\big] - cR(t)x(t) = cP(t)\varphi_{1/\alpha}\big(y(t)\big)\end{aligned}$$

which gives $Y(t) = \varphi_\alpha(c)y(t),\ t \in J$.

The case (ii) is true because of the duality principle. □

Theorem 2.5 (Elbert-type comparison)**.** (i) *Suppose that* (x, y) *and* (X, Y) *are solutions of* (1.1) *and* (1.2)*, respectively, satisfying* $x(b) = 0$, $x(t) \neq 0$ *for* $t \in (a, b)$ *and either* $x(a) = 0$ *or* $x(a) \neq 0$, $X(a) \neq 0$ *and*

$$\frac{y(a)}{\varphi_\alpha(x(a))} \geq \frac{Y(a)}{\varphi_\alpha(X(a))}.$$

Let $p(t) > 0$ *and*

$$\big[Q(t) - q(t)\big]|\xi|^{\alpha+1} - (\alpha+1)\big[r(t) - R(t)\big]\xi\eta + \alpha\big[P(t) - p(t)\big]|\eta|^{\frac{1}{\alpha}+1} \geq 0 \tag{2.11}$$

for all $\xi, \eta \in \mathbb{R}$ *and* $t \in [a, b]$*. If, moreover,* $X(t)^2 + Y(t)^2 > 0$ *in* $[a, b]$ *and either the strict inequality holds in* (2.11) *throughout some subinterval of* (a, b) *or*

$$x(t)\varphi_{1/\alpha}\big(Y(t)\big) - X(t)\varphi_{1/\alpha}\big(y(t)\big) \not\equiv 0 \quad \text{in } (a, b), \tag{2.12}$$

then $X(t)$ *has at least one zero in the open interval* (a,b).

(ii) *Suppose that* (x,y) *and* (X,Y) *are solutions of* (1.1) *and* (1.2), *respectively, satisfying* $y(b) = 0$, $y(t) \neq 0$ *for* $t \in (a,b)$ *and either* $y(a) = 0$ *or* $y(a) \neq 0$, $Y(a) \neq 0$ *and*

$$\frac{x(a)}{\varphi_{1/\alpha}(y(a))} \geq \frac{X(a)}{\varphi_{1/\alpha}(Y(a))}.$$

Let $q(t) > 0$ *and* (2.11) *be satisfied for all* $\xi, \eta \in \mathbb{R}$ *and* $t \in [a,b]$. *If, moreover,* $X(t)^2 + Y(t)^2 > 0$ *in* $[a,b]$ *and either the strict inequality holds in* (2.11) *throughout some subinterval of* (a,b) *or*

$$y(t)\varphi_\alpha(X(t)) - Y(t)\varphi_\alpha(x(t)) \not\equiv 0 \quad \text{in } (a,b), \tag{2.13}$$

then $Y(t)$ *has at least one zero in the open interval* (a,b).

Proof. (i) Assume that what we want to prove is false and there exists a solution (X,Y) of (1.2) with $X(t) \neq 0$ in (a,b). Then, identity (2.1) is valid. Note that the function

$$w(t) := \frac{x(t)}{\varphi_\alpha(X(t))}\left[\varphi_\alpha(X(t))y(t) - \varphi_\alpha(x(t))Y(t)\right], \qquad t \in (a,b),$$

(which will be called a *Picone's concomitant*) has limits at the endpoints a and b, so that it may be extended continuously on the closed interval $[a,b]$.

Indeed, if $X(a) \neq 0$ and $X(b) \neq 0$, then

$$\lim_{t\to a+} w(t) = \frac{x(a)}{\varphi_\alpha(X(a))}\left[\varphi_\alpha(X(a))y(a) - \varphi_\alpha(x(a))Y(a)\right] \geq 0$$

and

$$\lim_{t\to b-} w(t) = \frac{x(b)}{\varphi_\alpha(X(b))}\left[\varphi_\alpha(X(b))y(b) - \varphi_\alpha(x(b))Y(b)\right] = 0.$$

If, however, $X(a) = 0$, then $X'(a) = R(a)X(a) + P(a)\varphi_{1/\alpha}(Y(a)) = P(a)\varphi_{1/\alpha}(Y(a)) \neq 0$ and an application of L'Hôpital rule shows that the quotient $x(t)/X(t)$ has at $t = a$ a nonzero finite limit $x'(a)/X'(a)$, so that

$$\lim_{t\to a+} w(t) = \lim_{t\to a+}\left[x(t)y(t) - \frac{\varphi_\alpha(x(t))}{\varphi_\alpha(X(t))}x(t)Y(t)\right] = 0.$$

A similar argument applies at the end-point b.

Now, assumptions (2.11)–(2.12) imply that Picone's concomitant $w(t)$ is a nondecreasing not-identically constant function of t, and so

$$w(b) > w(a).$$

This contradicts the conclusion that

$$0 = w(b) \leq w(a),$$

thereby invalidating our initial hypothesis that $X(t) \neq 0$ in (a,b). Thus, the component $X(t)$ must vanish in the open interval (a,b) at least once.

(ii) Similar reasoning shows that assuming $Y(t) \neq 0$ in (a,b) and employing Picone's identity (2.2) lead to a contradiction with the hypotheses of the theorem, so that $Y(t)$ must have at least one zero between a and b. □

Remark 2.6. A crucial role in the Elbert-type comparison result established in Theorem 2.5 is played by the positive semi-definiteness of the form on the left-hand side of (2.11). Thus, it is desirable to find conditions which would guarantee that the function $f(\xi,\eta)$ of the form

$$f(\xi,\eta) = A|\xi|^{\alpha+1} - (\alpha+1)B\xi\eta + \alpha C|\eta|^{\frac{1}{\alpha}+1}, \qquad \xi,\eta \in \mathbb{R},$$

where the coefficients A, B and C are real numbers, is positive semi-definite. It is easy to see that for satisfaction of $f(\xi,\eta) \geq 0$ it is necessary that $A \geq 0$ and $C \geq 0$. If both $A = 0$ and $C = 0$, then also B must be zero, and so $f(\xi,\eta) \equiv 0$ trivially for all $\xi,\eta \in \mathbb{R}$ in this case. Suppose that $A > 0$ and $C \geq 0$. Then

$$\begin{aligned} &A|\xi|^{\alpha+1} - (\alpha+1)B\xi\eta + \alpha C|\eta|^{\frac{1}{\alpha}+1} \\ &\quad = \left|A^{\frac{1}{\alpha+1}}\xi\right|^{\alpha+1} - (\alpha+1)A^{\frac{1}{\alpha+1}}\xi A^{-\frac{1}{\alpha+1}}B\eta + \alpha\left|A^{-\frac{1}{\alpha+1}}B\eta\right|^{\frac{1}{\alpha}+1} - \alpha\left|A^{-\frac{1}{\alpha+1}}B\eta\right|^{\frac{1}{\alpha}+1} + \alpha C|\eta|^{\frac{1}{\alpha}+1} \\ &\quad \geq \alpha|\eta|^{\frac{1}{\alpha}+1}\left(C - A^{-\frac{1}{\alpha}}|B|^{\frac{\alpha+1}{\alpha}}\right), \end{aligned}$$

where the Young inequality has been used in the last step. Consequently, the positive semi-definiteness of $f(\xi,\eta)$ is guaranteed by

$$C - A^{-\frac{1}{\alpha}}|B|^{\frac{1}{\alpha}+1} \geq 0,$$

or, equivalently, by

$$A^{\frac{1}{\alpha}}C - |B|^{\frac{1}{\alpha}+1} \geq 0.$$

Similarly, if $C > 0$ and $A \geq 0$, then $f(\xi,\eta) \geq 0$ for all $\xi,\eta \in \mathbb{R}$, if

$$AC^{\alpha} - |B|^{\alpha+1} \geq 0.$$

Thus, the following corollary of Theorem 2.5 is true.

Corollary 2.7. (i) *Let the conditions of Theorem 2.5 (i) be satisfied with (2.11) replaced by*

$$0 < p(t) \leq P(t), \qquad q(t) \leq Q(t) \qquad \text{in } [a,b] \tag{2.14}$$

and

$$\left[Q(t) - q(t)\right]^{\frac{1}{\alpha}}\left[P(t) - p(t)\right] - |R(t) - r(t)|^{\frac{1}{\alpha}+1} \geq 0, \qquad t \in [a,b]. \tag{2.15}$$

Then the assertion of Theorem 2.5 (i) is true.

(ii) *Let the conditions of Theorem 2.5 (ii) be satisfied with (2.11) replaced by*

$$0 < q(t) \leq Q(t), \qquad p(t) \leq P(t) \qquad \text{in } [a,b], \tag{2.16}$$

and (2.15). Then the assertion of Theorem 2.5 (ii) is true.

If, in particular, $r(t) \equiv R(t) \equiv 0$ in $[a,b]$, then the above criterion reduces to the comparison result by Mirzov [9] (see also [10]).

Example 2.8. As an elementary but instructive example of application of Corollary 2.7 in the case where $r(t) \equiv R(t) \equiv 0$ consider the systems

$$x' - k^{\alpha+1}\varphi_{1/\alpha}(y) = 0, \qquad y' + m^{\alpha+1}\varphi_{\alpha}(x) = 0, \tag{2.17}$$

and

$$X' - K^{\alpha+1}\varphi_{1/\alpha}(Y) = 0, \qquad Y' + M^{\alpha+1}\varphi_{\alpha}(X) = 0, \tag{2.18}$$

where $0 < k < K$ and $0 < m < M$ are constants. Systems (2.17) and (2.18) have the oscillatory solutions $\big(k\sin_\alpha(k^\alpha mt), m^\alpha\cos_\alpha(k^\alpha mt)\big)$ and $\big(K\sin_\alpha(K^\alpha Mt), M^\alpha\cos_\alpha(K^\alpha Mt)\big)$, respectively, where $\sin_\alpha$ (resp. $\cos_\alpha$) denotes the generalized sine function (resp. generalized cosine function) defined to be the first (resp. the second) component of the solution of the system

$$u' - \varphi_{1/\alpha}(v) = 0, \qquad v' + \varphi_\alpha(u) = 0, \tag{2.19}$$

determined by the initial condition

$$u(0) = 0, \qquad v(0) = \left(\frac{2}{\alpha+1}\right)^{\frac{\alpha}{\alpha+1}}. \tag{2.20}$$

Components of the above oscillatory solutions have infinitely many zeros which are regularly spaced at the distance $\pi_\alpha/(k^\alpha m)$ (resp. $\pi_\alpha/(K^\alpha M)$) where

$$\pi_\alpha = \frac{2\alpha^{\frac{1}{\alpha+1}}\pi}{(\alpha+1)\sin\frac{\pi}{\alpha+1}}.$$

Thus, for any solution (X, Y) of (2.18) satisfying $X(0) = 0$ its first component $X(t)$ must have a zero in the open interval $\big(0, \frac{\pi_\alpha}{k^\alpha m}\big)$ as was to have been anticipated.

Remark 2.9. Theorem 2.5 applies also to the case where systems (1.1) and (1.2) coincide, i.e., $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ and $r(t) \equiv R(t)$ in $[a, b]$. An important consequence of the interlacing phenomenon described in the following theorem is the fact that either all solutions of system (1.1) are oscillatory, or all of them are nonoscillatory.

Theorem 2.10 (Separation). (i) *Suppose that (x, y) and (X, Y) are solutions of the same system (1.2), with $P(t) > 0$ in $[a, b]$, satisfying $x(b) = 0$, $x(t) \neq 0$ for $t \in (a, b)$ and either $x(a) = 0$ or $x(a) \neq 0$, $X(a) \neq 0$ and*

$$\frac{y(a)}{\varphi_\alpha(x(a))} \geq \frac{Y(a)}{\varphi_\alpha(X(a))}.$$

Let

$$\big(X(t), Y(t)\big) \not\equiv \big(cx(t), \varphi_\alpha(c)y(t)\big)$$

in (a, b) for any constant c. Then $X(t)$ has exactly one zero in the open interval (a, b).

(ii) *Suppose that (x, y) and (X, Y) are solutions of the same system (1.2), with $Q(t) > 0$ in $[a, b]$, satisfying $y(b) = 0$, $y(t) \neq 0$ for $t \in (a, b)$ and either $y(a) = 0$ or $y(a) \neq 0$, $Y(a) \neq 0$ and*

$$\frac{x(a)}{\varphi_{1/\alpha}(y(a))} \geq \frac{X(a)}{\varphi_{1/\alpha}(Y(a))}.$$

If

$$\big(X(t), Y(t)\big) \not\equiv \big(cx(t), \varphi_{1/\alpha}(c)y(t)\big)$$

in (a, b) for any constant c, then $Y(t)$ has exactly one zero in the open interval (a, b).

Proof. From Theorem 2.5 and Lemma 2.4 it follows that $X(t)$ (resp. $Y(t)$) must have at least one zero in (a, b). We claim that $X(t)$ cannot vanish twice between a and b. If it did and there would exist at least two zeros a_1 and b_1, $a_1 < b_1$, of $X(t)$ in (a, b), then by previous arguments with the roles of x and X interchanged, $x(t)$ will have a zero in (a_1, b_1) which contradicts the assumption that a and b are consecutive zeros of x. Similarly for $Y(t)$. □

We now apply Theorem 2.5 to get information on the arrangement of zeros of oscillatory solutions of system (1.1) in the special case where $r(t) \equiv 0$.

Let (x, y) be an oscillatory solution of system (1.1). Denote by $\{\sigma_k\}_{k=1}^{\infty}$ and $\{\tau\}_{k=1}^{\infty}$ the sequences of zeros of $x(t)$ and $y(t)$, respectively. It is easy to see that $\{\sigma_k\}_{k=1}^{\infty}$ and $\{\tau\}_{k=1}^{\infty}$ have the interlacing property.

Theorem 2.11. *Assume that the functions $p(t)$ and $q(t)$ are increasing (or decreasing) on $[0, \infty)$. Let (x, y) be an oscillatory solution of system (1.1) and let $\{\sigma_k\}_{k=1}^{\infty}$ and $\{\tau\}_{k=1}^{\infty}$ denote the respective sequences of zeros of $x(t)$ and $y(t)$. Then, the sequences $\{\sigma_{k+1} - \sigma_k\}$ and $\{\tau_{k+1} - \tau_k\}$ are decreasing (or increasing).*

Proof. It suffices to deal with the case where $p(t)$ and $q(t)$ are increasing. Let $k \in \mathbb{N}$ be fixed and consider the half-linear differential systems

$$x' - p(t + \sigma_k)\varphi_{1/\alpha}(y) = 0, \qquad y' + q(t + \sigma_k)\varphi_\alpha(x) = 0 \tag{2.21}$$

and

$$X' - p(t + \sigma_{k+1})\varphi_{1/\alpha}(Y) = 0, \qquad Y' + q(t + \sigma_{k+1})\varphi_\alpha(X) = 0 \tag{2.22}$$

on $[0, \infty)$. Clearly, $\big(x(t+\sigma_k), y(t+\sigma_k)\big)$ and $\big(x(t+\sigma_{k+1}), y(t+\sigma_{k+1})\big)$ are solutions of (2.21) and (2.22), respectively. Note that $x(t + \sigma_k)$ has consecutive zeros 0 and $\sigma_{k+1} - \sigma_k$, and $x(t + \sigma_{k+1})$ has consecutive zeros 0 and $\sigma_{k+2} - \sigma_{k+1}$. Since

$$p(t + \sigma_k) \le p(t + \sigma_{k+1}), \qquad q(t + \sigma_k) \le q(t + \sigma_{k+1}),$$

applying Theorem 2.5 to (2.21) and (2.22), we see that $\sigma_{k+1} - \sigma_k \le \sigma_{k+2} - \sigma_{k+1}$.

In order to show that $\tau_{k+1} - \tau_k \le \tau_{k+2} - \tau_{k+1}$, we need only to repeat the same comparison arguments as above to y-components of systems

$$x' - p(t + \tau_k)\varphi_{1/\alpha}(y) = 0, \qquad y' + q(t + \tau_k)\varphi_\alpha(x) = 0,$$

and

$$X' - p(t + \tau_k)\varphi_{1/\alpha}(Y) = 0, \qquad Y' + q(t + \tau_k)\varphi_\alpha(X) = 0,$$

having, respectively, the solutions $\big(x(t + \tau_k), y(t + \tau_k)\big)$ and $\big(x(t + \tau_{k+1}), y(t + \tau_{k+1})\big)$. This completes the proof. □

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