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On the global stability of periodic Ricker maps

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

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Abstract. We find the exact region of global stability for the 2-periodic Ricker difference equation, showing that a 2-periodic solution is globally asymptotically stable whenever it is locally asymptotically stable and the equation does not have more 2-periodic solutions. We conjecture that this property holds for the general p-periodic Ricker difference equation, and in particular we prove it for $p = 3$.

Keywords: periodic Ricker map, difference equations, global stability.

2010 Mathematics Subject Classification: 39A30, 37E05, 37N25.

1 Introduction

A basic question in the qualitative theory of dynamical systems is *"under what conditions does local asymptotic stability of a fixed point imply its global asymptotic stability (LAS implies GAS)"*. One paradigmatic example from population dynamics is the Ricker map $f(x) = xe^{r-x}$, $r > 0$. It is well-known that condition $r \in (0, 2]$ is necessary and sufficient for the local asymptotic stability of the positive equilibrium $K = r$, and that this condition actually implies the global stability of K on $(0, \infty)$, that is, if $0 < r \leq 2$ then all solutions of the difference equation $x_{n+1} = f(x_n)$ starting at an initial condition $x_0 > 0$ converge to K. This statement was first established by May and Oster [7] using a graphical analysis, and an analytic proof can be derived from Singer [12]. The result has been extended in [6] to the generalized form of the Ricker map as derived in Thieme's book [13]: the positive equilibrium $K = r$ of the map $f(x) = qx + (1-q)xe^{r-x}$ is globally stable whenever it is asymptotically stable (that is, for $0 < (1-q)r \leq 2$).

Two generalizations of the one-dimensional Ricker model have already been suggested in the pioneering papers of May and co-authors.

On the one hand, Levin and May (1976) argue in their paper [5] that density-dependent mechanisms may operate with an explicit time delay, and this leads to the delayed Ricker map

$$x_{n+1} = x_n e^{r - x_{n-T}}, \tag{1.1}$$

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where $T \geq 1$ is an integer. They suggest that the folklore statement *LAS implies GAS* also holds for the positive equilibrium $K = r$ of (1.1).

On the other hand, May and Oster (1976) state in their paper [7] that their graphical method can be extended to the case where the parameter r is a periodic function: $r_n = r_{n+p}$, for an integer $p \geq 2$. This leads to the periodic Ricker map

$$x_{n+1} = x_n e^{r_n - x_n} := f_n(x_n), \qquad n = 1, 2, \ldots \tag{1.2}$$

where $r_n > 0$ and $r_{n+p} = r_n$ for all $n \geq 1$. In this case, one can consider the period map $F_p = f_p \circ f_{p-1} \circ \cdots \circ f_1$, and investigate under which conditions the local stability of a positive fixed point of F_p implies its global stability. This is equivalent to say that (1.2) has a globally asymptotically stable p-periodic solution.

For equation (1.1), the conjecture on the global stability remained as an open problem for years, and it has been recently proved in the 2-dimensional case ($T = 1$) by Bartha, Garab and Krisztin [1]. An alternative proof was later given by Franco and Perán [8]. The condition for local (and global) asymptotical stability of the positive equilibrium $K = r$ in (1.1) is $0 < r \leq 1$ in the case $T = 1$, while the case $T > 1$ is still an open problem.

In this paper, we consider the conjecture for equation (1.2). Sacker [10] proved that if $r_n \in (0, 2)$ for all $n = 1, 2, \ldots, p$, then (1.2) has a globally asymptotically stable p-periodic solution. This is a very nice result but, as noticed in [11], this condition is not sharp even for the 2-periodic case. Elaydi *et al.* [2] carried out a bifurcation analysis of (1.2) in the case $p = 2$, showing the bifurcation curves at which an equilibrium of the period map undergoes a period-doubling or a saddle-node bifurcation. For some results on a general nonautonomous Ricker map, we refer the reader to Hüls and Pötzsche [4].

We prove that a unique asymptotically stable positive fixed point of the period map F_2 for the 2-periodic Ricker equation (1.2) is globally stable, and conjecture that the same result remains true for a general $p \geq 3$. In particular, we sketch the proof for $p = 3$. In the 2-periodic case, we find the exact region of global stability in the parameter plane (r_1, r_2), which of course contains the square $(0, 2) \times (0, 2)$ given in [10].

Our main tools are a generalization of the so-called Singer's theorem [12] established by El-Morshedy and Jiménez-López [3], and some ideas from the paper by Rodriguez [9], who studied the dynamics of the composition of two Ricker maps in the context of discrete models for seasonal populations.

2 LAS implies GAS for the 2-periodic Ricker map

We recall some basic properties of the maps $f_n : [0, \infty) \to [0, \infty)$ defined in (1.2).

(I) f_n is unimodal with a unique critical point $x = 1$, at which it reaches its global maximum.

(II) $(Sf_n)(x) = (-1/2)(2 + (x-2)^2)/(x-1)^2 < 0$ for all $x \neq 1$, where $(Sf_n)(x)$ is the Schwarzian derivative of f_n.

As we have mentioned in the introduction, it is clear that p-periodic solutions of equation (1.2) correspond to fixed points of the period map $F_p = f_p \circ f_{p-1} \circ \cdots \circ f_1$. From property (II) above, it follows using the formula for the Schwarzian derivative of the composition that $(SF_p(x)) < 0$ whenever $F_p'(x) \neq 0$.

Definition 2.1. We say that a unique fixed point K of F_p is globally stable if it is locally asymptotically stable and $\lim_{n\to\infty} F_p^n(x) = K$ for all $x > 0$.

It is clear that if K is globally stable then the p-cycle $\{K, f_1(K), \ldots, f_{p-1}(K)\}$ defines a globally asymptotically stable p-periodic solution of (1.2).

It follows from Singer's results [12] that if a unimodal map with negative Schwarzian derivative has a unique fixed point K which is locally asymptotically stable, then K is globally stable. We use the following generalization of this result from [3].

Proposition 2.2 ([3, Corollary 2.9]). *Let $a \geq 0$ and $b > a$ ($b = \infty$ is allowed), and let $g : (a, b) \to [a, b]$ be a continuous map with a unique fixed point K such that $(g(x) - x)(x - K) < 0$ for all $x \neq K$. Assume that there are points $a \leq c < K < d \leq b$ such that the restriction of g to (c, d) has at most one turning point, and (whenever it makes sense) $g(x) \leq g(c)$ for every $x \leq c$, and $g(x) \geq g(d)$ for every $x \geq d$. If g is decreasing at K, assume additionally that $(Sg)(x) < 0$ for all $x \in (c, d)$ except at most one critical point of g, and $-1 \leq g'(K) < 0$. Then K is globally stable.*

Now we are in a position to state and prove our main result.

Theorem 2.3. *Assume that the period map $F_2 = f_2 \circ f_1$ has a unique fixed point K on $(0, \infty)$. Then K is globally stable if it is locally asymptotically stable, that is, if $-1 \leq F_2'(K) < 1$.*

Proof. The map F_2 can have either 1 or 3 critical points. Actually, if $r_1 \leq 1$ then F_2 is unimodal; in this case, the result follows from Singer's theorem. Thus we assume that $r_1 > 1$ and hence $f_1(1) = e^{r_1 - 1} > 1$. In this situation, there are two points $0 < q_1 < 1 < q_2$ such that $f_1(q_1) = f_1(q_2) = 1$ (see Figure 2.1 (a)). Then the map F_2 has two local maxima at q_1 and q_2, with $F_2(q_1) = F_2(q_2) = f_2(1) = e^{r_2 - 1}$, and one local minimum at 1 (see Figure 2.1 (b), (c)).

Now we use Proposition 2.2 to deal with the non-unimodal case. We can assume that $q_i \neq K$ for $i = 1, 2$, because in that case K is obviously a global attractor. There are two possibilities.

Case 1. The map F_2 is decreasing on (K, ∞) (see Figure 2.1 (b)). Then we choose $a = 0$, $c = q_2$ and $b = d = \infty$; since the conditions of Proposition 2.2 clearly hold, K is globally stable if it is locally asymptotically stable.

Case 2. The map F_2 reaches at least one local maximum on (K, ∞) (see Figure 2.1 (c)). Then we choose $a = 0$, $c = 0$ if $q_1 > K$, $c = q_1$ if $q_1 < K$, and $b = d = q$, where q is the first point of local maximum greater than K. It is clear that the interval $I = [0, b]$ is invariant and attracting for F_2, and therefore we can restrict F_2 to I. Again, the conditions of Proposition 2.2 clearly hold, and we conclude that K is globally stable if it is locally asymptotically stable.

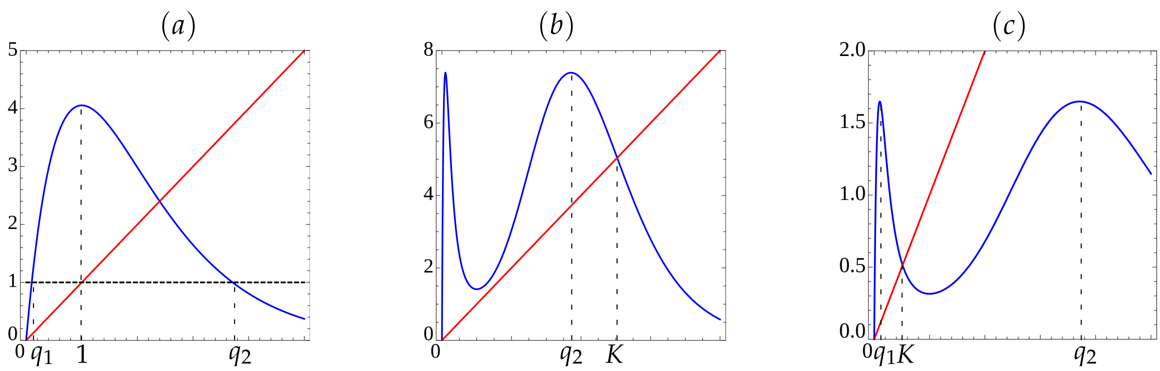

Figure 2.1: (a): Representation of the curves $y = f_1(x)$, $y = x$ and $y = 1$; (b) and (c): two possibilities for the period map F_2.

□

Remark 2.4. It is easy to verify that if $r_i \leq 1$ for $i = 1, 2, \dots, p-1$, then the period map $F_p = f_p \circ f_{p-1} \circ \cdots \circ f_1$ is unimodal and therefore the statement of Theorem 2.3 is still valid in this situation. We conjecture that Theorem 2.3 is true for an arbitrary integer $p \geq 2$. Actually, we sketch the proof for $p = 3$ in Section 4; our simulations do not show more complicated situations in the general case, but we do not have a proof of it.

3 Region of global stability for the 2-periodic Ricker map

Theorem 2.3 allows us to give a precise region for the global stability of (1.2) in the case $p = 2$. First we establish when F_2 has exactly one fixed point. The proof of the following result can be derived from the formula given for a more general case in [9, Appendix].

Proposition 3.1. *The map F_2 has more than one fixed point if and only if $r > 4$ and*

$$N_2\left(1 + e^{r_1 - N_2}\right) \leq r \leq N_1\left(1 + e^{r_1 - N_1}\right), \tag{3.1}$$

where

$$N_1 = \frac{r - \sqrt{r^2 - 4r}}{2}, \qquad N_2 = \frac{r + \sqrt{r^2 - 4r}}{2}, \qquad r = r_1 + r_2.$$

Condition (3.1) divides the set of admissible parameters $\{(r_1, r_2) : r_1 > 0, r_2 > 0\}$ into two open connected regions R_1 and R_2 represented in Figure 3.1. The set $R_1 \cup \{(2, 2)\}$ is the region where the folklore statement "LAS $\Rightarrow$ GAS" holds, that is, it contains the pairs of parameter values (r_1, r_2) for which the local asymptotic stability of the equilibrium implies its global stability.

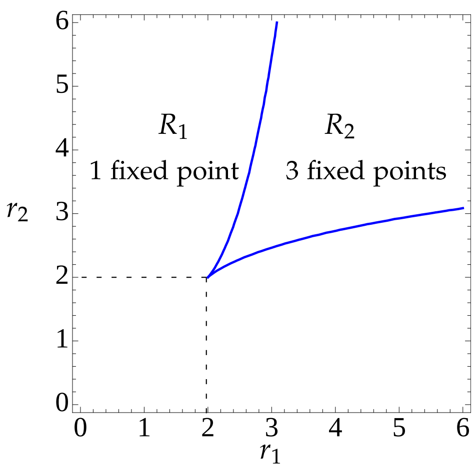

Figure 3.1: The map F_2 has more than one fixed point in the region R_2 between the two curves, including the curves but excluding the vertex $(2, 2)$.

Remark 3.2. Similar curves have been plotted numerically (but without an analytical expression) in [11]. They coincide with the bifurcation curves where a saddle-node bifurcation takes place in the 2-periodic Ricker equation (1.2); that is to say, when conditions $F_2'(x) = 1$, $F_2(x) = x$ hold simultaneously (see [2]).

Next, to find the region $R \subset R_1$ where the 2-periodic Ricker equation has a globally stable 2-periodic solution, we have to determine the curves where the equilibrium becomes unstable. These curves define a period-doubling bifurcation and are characterized by the equations $F_2'(x) = -1$, $F_2(x) = x$ (see [2]).

Proposition 3.3 ([2]). *A fixed point K of F_2 satisfies $F_2'(K) = -1$ if and only if one of the following conditions holds*

$$r = u_1 \left(1 + e^{r_1 - u_1}\right), \tag{3.2}$$

$$r = u_2 \left(1 + e^{r_1 - u_2}\right), \tag{3.3}$$

where

$$u_1 = \frac{r - \sqrt{r^2 - 4(r-2)}}{2}, \qquad u_2 = \frac{r + \sqrt{r^2 - 4(r-2)}}{2}, \qquad r = r_1 + r_2.$$

Propositions 3.1 and 3.3 allow us to represent the exact region of the plane of parameters corresponding to a globally stable 2-periodic solution of the 2-periodic Ricker map. See Figure 3.2.

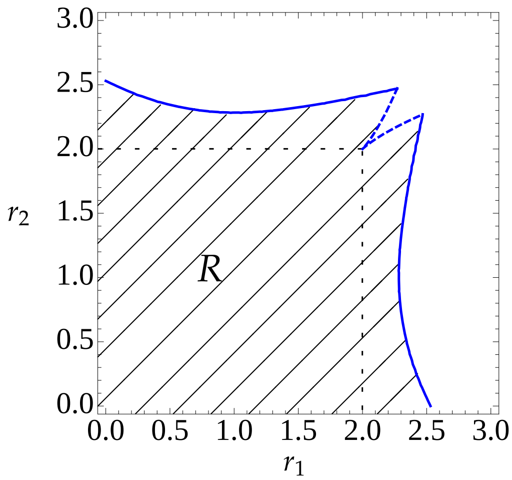

Figure 3.2: Region R where the map F_2 has a globally stable fixed point. The blue solid lines correspond to the curves where the equilibrium becomes unstable, and they are included in the global stability region. The blue dashed lines correspond to the curves where a new fixed point appears, and they are not included in the global stability region, with the only exception of the point $(2, 2)$. The square $(0, 2)^2$ is the region of global stability established in [10].

We emphasize that local asymptotic stability of a fixed point of F_2 is not enough for its global stability. Indeed, in region R_2 of Figure 3.1 there are three equilibria and two of them can be locally asymptotically stable at the same time, but of course they cannot be globally stable. As an example of the possible bifurcation diagrams, we fix $r_2 = 2.2$ and use r_1 as the bifurcation parameter (Figure 3.3). For $r_1 = 0$, F_2 has a globally stable equilibrium $K_1(0) \approx 1.918$. The branch of fixed points $K_1(r_1)$ starting at $K_1(0)$ gives globally stable 2-periodic solutions of (1.2) until two new fixed points $K_2(r_1), K_3(r_1)$ of F_2 appear at $r_1 \approx 2.136$. $K_2(r_1)$ is unstable, and $K_3(r_1)$ is asymptotically stable until it becomes unstable at $r_1 \approx 2.457$. The fixed points $K_1(r_1)$ and $K_2(r_1)$ disappear at $r_1 \approx 2.32$. Thus, in the interval $(2.136, 2.32)$, equation (1.2) has two asymptotically stable 2-periodic solutions, and $K_3(r_1)$ becomes globally stable in the interval $(2.32, 2.457)$. After that, a route of period-doubling bifurcations to chaos starts.

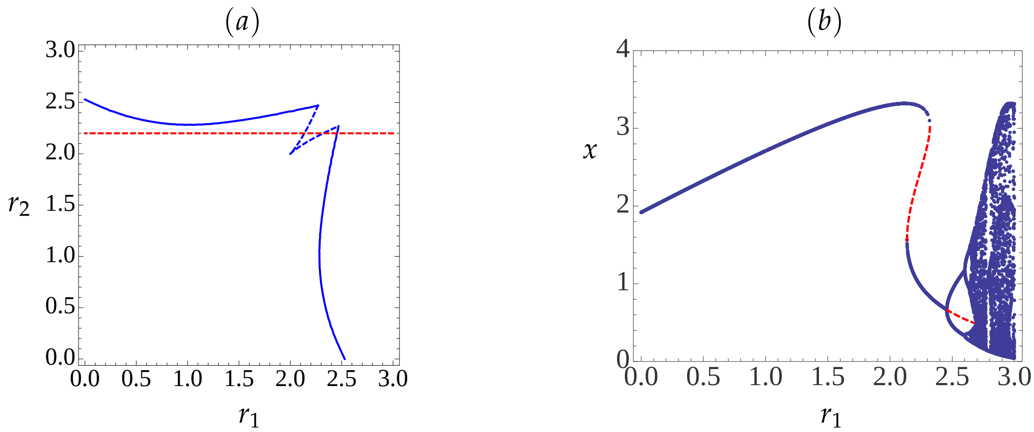

Figure 3.3: (a) We fix $r_2 = 2.2$ and use r_1 as the bifurcation parameter (red dashed line); (b) the bifurcation diagram for the period map F_2 shows regions of global stability, bistability, and chaos. A phenomenon of hysteresis is also observed. Discontinuous lines correspond to unstable equilibria.

4 The 3-periodic Ricker map

In this section, we sketch the proof of Theorem 2.3 for $p = 3$, that is, we prove the following result.

Theorem 4.1. *Assume that the period map $F_3 = f_3 \circ f_2 \circ f_1$ has a unique fixed point K on $(0, \infty)$. Then K is globally stable for F_3 if it is locally asymptotically stable.*

Actually, we conjecture that Theorem 2.3 is true for an arbitrary integer $p \geq 2$. If all local maxima of F_p have the same value, then we can easily repeat the arguments in the proof of Theorem 2.3. In particular, this is the case for F_3 if $F_2 = f_2 \circ f_1$ is unimodal or $F_2(1) \leq 1$. However, if F_2 is not unimodal and $F_2(1) > 1$ (Figure 4.1), then $F_3(1)$ becomes a local maximum in such a way that F_3 has three points of local maxima q_1, 1, q_2 (q_1 and q_2 are the preimages of 1 by F_2), and two points of local minima m_1, m_2 (corresponding to the local maxima of F_2), with $q_1 < m_1 < 1 < m_2 < q_2$, $F_3(m_1) = F_3(m_2)$, and $F_3(1) < F_3(q_1) = F_3(q_2)$ (Figures 4.1 and 4.2).

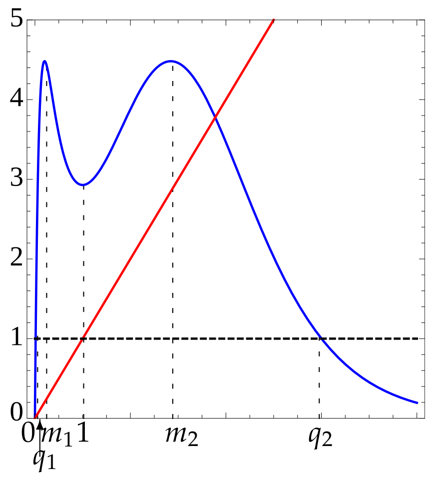

Figure 4.1: Representation of the curves $y = F_2(x)$, $y = x$ and $y = 1$ when F_2 is not unimodal and $F_2(1) > 1$.

We only consider the cases where $F_3'(K) < 0$, since the others are easier to address.

- Case (a) in Figure 4.2 occurs if F_3 is decreasing on (K, ∞). Then we just choose $a = 0$, $c = q_2$, and $b = d = \infty$ to apply Proposition 2.2.
- Case (b) in Figure 4.2 occurs if the fixed point of F_3 lies between the first local maximum q_1 and the first local minimum m_1. Then we choose $a = 0$, $c = q_1$, $d = m_1$, and $b = q_2$ (it is clear that the interval $I = [0, q_2]$ is invariant and attracting for F_3).
- Case (c) in Figure 4.2 occurs if the fixed point of F_3 lies between 1 and the second local minimum m_2. In this case the interval $I = [m_1, q_2]$ is invariant and attracting for F_3. Then we choose $a = c = m_1$, $b = q_2$, and $d = m_2$ to apply Proposition 2.2 again.

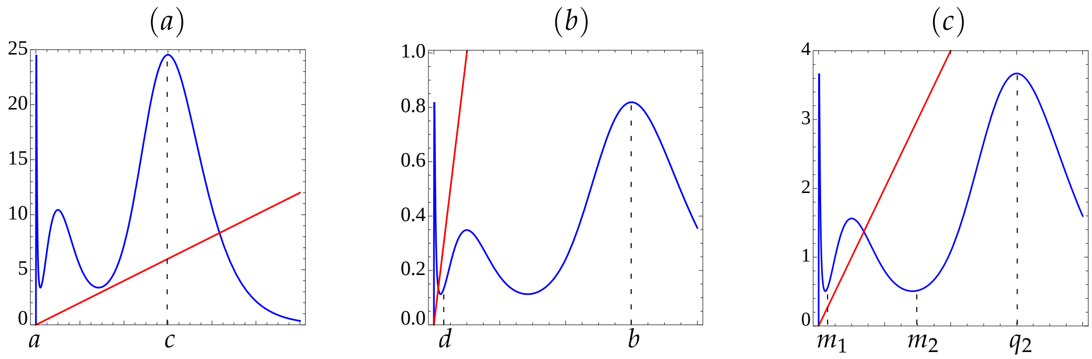

Figure 4.2: Representation of the curves $y = F_3(x)$ and $y = x$ in three different situations.

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Dedication

Dedicated to Tibor Krisztin on the occasion of his 60th birthday. He has greatly contributed to the understanding of functional differential equations, but also of difference equations. In this field of research, he solved (with his co-authors) a conjecture concerning the global stability of the 2-dimensional Ricker map. Their contribution [1] was recognized as the best paper published in the Journal of Difference Equations and Applications in 2013.

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